Bottom frictional effects on periodic long wave propagation

A. Orfila [a], G. Simarro [b], P. L.-F. Liu [c,*]

[a] *IMEDEA(CSIC-UIB). Miquel Marques, 21. 07190 Esporles. Spain*

[b] *ETS Caminos Canales y Puertos. UCLM, 13071 Ciudad Real, Spain*

[c] *School of Civil and Env. Eng. Cornell University. 14853 Ithaca, NY*

Abstract

A new Boussinesq-type model describing periodic wave propagation over a constant depth has been developed for the cases where the effects of a turbulent boundary layer are significant. In this paper, the eddy viscosity model is employed in the turbulent boundary layer and is further approximated as a linear function of the distance measured from the seafloor. The boundary-layer velocities are coupled with the irrotational velocity in the core region through the boundary conditions. The leading order effects of the boundary layer on wave propagation appear in the depth-integrated continuity equation to account for the velocity deficit inside the boundary layer. The bottom stress, the boundary layer thickness and the magnitude of the turbulent eddy viscosity are part of the solutions. An iterative scheme is introduced to find them. A numerical example for the evolution of periodic waves propagating in a one-dimensional channel is discussed to illustrate the numerical procedure and the physics involved.

Key words: Turbulent boundary layer, Boussinesq approximation, bottom friction, eddy viscosity

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1 Introduction

For long water waves traveling over a long distance or for wind waves propagating into shallow water, the bottom frictional effects become increasingly important. The turbulence generated inside the bottom boundary layer will not only attenuate wave energy, but also modify the wave form and the wave

* Corresponding author. Phone(+1) 6072555090
Email address: pll3@cornell.edu (P. L.-F. Liu).

speed. Recently, Liu and Orfila (2004) (this paper will be referred as LO thereafter) derived a set of depth-integrated continuity equation and momentum equations with boundary-layer effects considered for long-wave propagation. LO showed that the leading-order boundary-layer effects appear in the continuity equation as a correction term representing the velocity deficit in the boundary layer. The correction term is expressed as a convolution integral and therefore, the bottom frictional effects have a memory in time. More recently, Liu *et al.* (2005) extended LO's work to examine the damping and shoaling of solitary wave. Laboratory experiments in a wave flume were conducted and experimental data confirmed that LO's formulation is accurate as long as both the sidewall and bottom boundary layers are considered. We remark here that in the laboratory experiments (Liu *et al.*, 2005) the boundary layers are laminar. However, the theoretical model developed by LO can be applied to a turbulent boundary layer with the limitation of a constant eddy viscosity.

In nature the roughness on seafloor enhances turbulence. For a fully developed turbulent boundary layer, many researchers have suggested that the eddy viscosity can be modeled as (Kaijura, 1968; Jonsson and Carlsen, 1976; Grant and Madsen, 1979)

$$\nu_t'(\zeta') = \kappa\,|\,\mathbf{u}_*'|\,\zeta', \tag{1}$$

where ζ' is the local coordinate normal to the sea bottom, κ the von Krmn constant (~ 0.40), and $\mathbf{u}_*'$ the frictional velocity, which is related to the bottom stress. In this paper we shall extend LO's model to include the effects of a fully developed turbulent bottom boundary layer, in which the eddy viscosity model, (1), can be used. Furhtermore, we shall focus only on periodic waves.

The paper is structured in the following way. For completeness we first summarize the governing equations and boundary conditions for the long wave propagation in Section 2. The Boussinesq-type equations are derived in Section 3. These equations are expressed in terms of the Fourier (harmonic) components of the free surface displacement and the depth-averaged horizontal velocity. The effects of the boundary layer are described. Section 4 presents the analysis for turbulent boundary layer under an oscillatory flow. In particular, the expressions for boundary layer thickness and bottom stress are given. We need to point out that the boundary layer thickness is a part of solution. Once the relationship between the the boundary-layer solution and the depth-averaged velocity is found, the Boussinesq-type equations presented in Section 3 can be combined into one set of equations in terms of the free surface displacement. The final equations are presented in Section 5. Numerical results for a sinusoidal wave propagating in a numerical tank are shown in Section 6. The effects of turbulent boundary layer on the evolution of different harmonics are discussed. Finally, Section 7 concludes the paper.

2 Governing Equations and Boundary Conditions

In this paper, we consider a periodic wave train with the surface displacement $\eta'(x',y',t')$ propagating in a constant water depth, h_0'. The wave train is characterized by a typical wave amplitude, a_0', its fundamental frequency, ω_0' and the corresponding wave number, $k_0' = \omega_0'/\sqrt{g'h_0'}$. The following dimensionless variables are introduced:

$$(x,y) = k_0'\,(x',y')\,, \quad z = z'/h_0', \quad t = \omega_0' t'$$

$$\eta = \eta'/a_0', \quad p = p'/\rho' g' a_0',$$

$$(u,v) = (u',v')\,/\epsilon\sqrt{g'h_0'}, \quad w = \mu w'/\epsilon\sqrt{g'h_0'}, \tag{2}$$

in which p' denotes the pressure, (u',v') the horizontal velocity components in the (x',y') directions, w' the vertical velocity component in the z' direction, ρ' the fluid density, and g' the gravitational acceleration. Two dimensionless parameters have been introduced in the dimensionless variables:

$$\epsilon = a_0'/h_0', \quad \mu = k_0' h_0'. \tag{3}$$

As explained in LO, the dynamics of the problem can be viewed in the following manner. The flow motions are essentially irrotational except in the boundary layer adjacent to the bottom, $z = -h_0$. In order to satisfy the no-slip boundary condition on the bottom, the leading order of magnitude of the tangential (to the seafloor) rotational velocity component inside the bottom boundary layer is the same as that of the irrotational velocity, i.e., $O(1)$. From the continuity equation, a normal component of the rotational velocity is generated inside the bottom boundary layer, which is no longer zero at the bottom. Therefore, to satisfy the no flux boundary condition on the seafloor, the irrotational flow in the core region must be modified, feeling the effects of bottom boundary layer.

In the core region where the flow is assumed to be irrotational, a velocity potential Φ is introduced. The continuity equation in dimensionless form becomes,

$$\mu^2\nabla^2\Phi + \frac{\partial^2\Phi}{\partial z^2} = 0, \qquad -1 < z < \epsilon\eta, \tag{4}$$

and the free surface kinematic and dynamic boundary conditions are,

$$\frac{\partial\eta}{\partial t} + \epsilon\nabla\eta\cdot\nabla\Phi = \frac{1}{\mu^2}\frac{\partial\Phi}{\partial z}, \qquad z = \epsilon\eta. \tag{5}$$

$$\frac{\partial\Phi}{\partial t} + \frac{\epsilon}{2}\left\{|\nabla\Phi|^2 + \frac{1}{\mu^2}\left(\frac{\partial\Phi}{\partial z}\right)^2\right\} + \eta = 0, \qquad z = \epsilon\eta. \tag{6}$$

At the sea bottom the no-slip and no-flux boundary conditions are required. Defining the tangential and normal components of the rotational velocity inside the bottom boundary layer as $\mathbf{u}_\xi$ and u_ζ, respectively, the no-flux boundary condition can be expressed as

$$\frac{\partial \Phi}{\partial z} = -u_\zeta, \qquad z = -1. \tag{7}$$

We remark here that the normal component of the rotational velocity at the seafloor can be related to the irrotational velocity in the core region through a boundary-layer analysis, which will be shown in Section 4.

3 Boussinesq-type equations

In this section, we shall present the simplified governing equations for the irrotational flow by adopting the Boussinesq approximation, i.e., $O(\epsilon) \sim O(\mu^2)$. Moreover, for simplicity we further assume that the water depth is constant. The primary difference between the traditional Boussinesq equations and the present problem is that the normal component of the irrotational velocity at the bottom is not zero in the present situation. Following LO's approach, we expand the potential function as a power series in the vertical coordinate,

$$\Phi(\mathbf{x}, z, t) = \sum_{n=0}^{\infty} (z+1)^n \phi_n(\mathbf{x}, t). \tag{8}$$

Substituting the expansion into the Laplace equation, (4), and the bottom boundary condition, (7), we obtain the following recursive relation:

$$\phi_{n+2} = \frac{-\mu^2 \nabla^2 \phi_n}{(n+1)(n+2)}, \tag{9}$$

with

$$\phi_1 = -u_\zeta. \tag{10}$$

where the boundary layer effects are introduced through the velocity potential ϕ_1. Thus, using the recursive relation in the expansion, we get the potential function truncated up to $O(\mu^5)$

$$\Phi = \phi_0 + (z+1)\phi_1 - \frac{\mu^2}{2}(z+1)^2 \nabla^2 \phi_0 + \frac{\mu^4}{24}(z+1)^4 \nabla^2 \nabla^2 \phi_0 + O(\mu^6). \tag{11}$$

We reiterate here that the rotational velocity, u_ζ, which appears in ϕ_1 in (10), is expected to be greater than $O(\mu^5)$ and is to be determined from the boundary-layer analysis in Section 4.

Defining the depth-averaged velocity as,

$$\overline{\mathbf{u}} = \frac{1}{1+\epsilon\eta}\int_{-1}^{\epsilon\eta} \nabla\Phi \; dz, \tag{12}$$

the depth-integrated momentum and continuity equations for slowly varying depth can be derived following LO, from the free surface boundary conditions, (5) and (6),

$$\frac{\partial \overline{\mathbf{u}}}{\partial t} + \frac{\epsilon}{2}\nabla\left(\overline{\mathbf{u}}\overline{\mathbf{u}}\right) + \nabla\eta - \frac{\mu^2}{3}\nabla\left(\nabla\cdot\frac{\partial \overline{\mathbf{u}}}{\partial t}\right) = O(\mu^4), \tag{13}$$

$$\nabla\cdot\left\{\left(1+\epsilon\eta\right)\overline{\mathbf{u}}\right\} + \frac{\partial\eta}{\partial t} + \frac{u_\zeta}{\mu^2} = O(\mu^4). \tag{14}$$

In this paper we shall focus on the periodic wave propagation with fundamental frequency ω_0'. The free surface displacement and the velocity field can be expressed as a Fourier series in time,

$$\eta = \frac{1}{2}\sum_n \eta_n \exp\left(int\right), \qquad n = \pm 1, \pm 2, ... \tag{15}$$

$$\overline{\mathbf{u}} = \frac{1}{2}\sum_n \overline{\mathbf{u}}_n \exp\left(int\right), \qquad n = \pm 1, \pm 2, ... \tag{16}$$

$$u_\zeta = \frac{1}{2}\sum_n u_{\zeta,n} \exp\left(int\right), \qquad n = \pm 1, \pm 2, ... \tag{17}$$

where $(\eta_{-n}, \overline{\mathbf{u}}_{-n}, u_{\zeta,-n})$ are the complex conjugates of $(\eta_n, \overline{\mathbf{u}}_n, u_{\zeta,n})$. We note that $\mathbf{u}_0$ and η_0 represent the mean velocity and the mean free surface elevation, respectively. In the present study, we assume that the mean flow field is generated only through the nonlinearity.

Introducing (15), (16) and (17) into (13) and (14) and collecting the different Fourier components, we have

$$in\overline{\mathbf{u}}_n + \frac{\epsilon}{4}\sum_{\substack{s\neq n\\ s\neq 0}} \nabla(\overline{\mathbf{u}}_s\cdot\overline{\mathbf{u}}_{n-s}) + \nabla\eta_n - \frac{in\mu^2}{3}\nabla(\nabla\cdot\overline{\mathbf{u}}_n) = O(\mu^4), \tag{18}$$

$$in\eta_n + \frac{\epsilon}{2}\sum_{\substack{s\neq n\\ s\neq 0}} \nabla\cdot(\eta_s\overline{\mathbf{u}}_{n-s}) + \nabla\cdot\overline{\mathbf{u}}_n + \frac{u_{\zeta,n}}{\mu^2} = O(\mu^4), \tag{19}$$

for $n \neq 0$. The mean velocity and mean free surface elevation are of order of magnitude of $O(\epsilon)$ and if the setdown at deep water is zero, can be calculated as:

$$\eta_0 = -\frac{\epsilon}{4}\sum_{s\neq 0}(\overline{\mathbf{u}}_s\cdot\overline{\mathbf{u}}_{-s}) + O(\mu^4), \tag{20}$$

$$\overline{\mathbf{u}}_0 = -\frac{\epsilon}{2}\sum_{s\neq 0}(\eta_s\overline{\mathbf{u}}_{-s}) + O(\mu^4). \tag{21}$$

Thus, (18) and (19) constitute the governing equations for $\overline{\mathbf{u}}_n$ and η_n $(n \neq 0)$ and (20) and (21) are used to calculated the mean flow field. However, in this paper we will assume that both the mean free surface elevation as well as the mean velocity component, η_0 and $\mathbf{u}_0$ respectively, are strictly zero.

For future use, we also note that for $n \neq 0$

$$\overline{\mathbf{u}}_n = \frac{i}{n}\nabla\eta_n + O(\mu^2), \tag{22}$$

$$\nabla \cdot \overline{\mathbf{u}}_n = -in\,\eta_n + O(\mu^2), \tag{23}$$

However, before (18) and (19) can be solved, the boundary-layer flow must be analyzed so as to find the expression for the normal component of rotational velocity evaluated at the sea bottom, $u_{\zeta,n}$.

4 Turbulent bottom boundary layer under an oscillatory flow

As we have mentioned in the Introduction, in the turbulent bottom boundary layer, the eddy viscosity can be modeled as a function linearly proportional to the distance from the seafloor, i.e., (1). However, since the frictional velocity, $\mathbf{u}'_*$, depends on time, it makes the problem not tractable. Here, we further simplify the eddy viscosity model as

$$\nu'_t\,(\zeta') = \kappa\langle|\,\mathbf{u}'_*|\rangle\zeta', \tag{24}$$

in which $\langle\ \ \rangle$ denotes the average over the fundamental wave period. Introducing a stretched coordinate in the boundary layer

$$\widetilde{\zeta} \equiv \frac{\zeta'}{\widetilde{\delta}'}, \tag{25}$$

where $\widetilde{\delta}' = \langle\delta'\,(t')\rangle$ is the averaged boundary-layer thickness, (24) can be expressed as

$$\nu'_t(\zeta') = \widetilde{\zeta}\,\widetilde{\nu}'_0, \tag{26}$$

with

$$\widetilde{\nu}'_0 = \kappa\langle|\mathbf{u}'_*|\rangle\widetilde{\delta}', \tag{27}$$

being the order of magnitude of the eddy viscosity. Since we anticipate that the order of magnitude of the boundary layer thickness is

$$\widetilde{\delta}' = \sqrt{\frac{\widetilde{\nu}'_0}{\omega'_0}}, \tag{28}$$

the scales of eddy viscosity and the boundary layer thickness can be expressed in terms of the averaged bottom stress, $\langle|\tau_b'|\rangle = \rho\langle|\mathbf{u}_*'|\rangle^2$, as

$$\widetilde{\nu}_0' = \frac{\kappa^2 \langle|\tau_b'|\rangle}{\rho'\omega_0'} \qquad \text{and} \qquad \widetilde{\delta}'^2 = \frac{\kappa^2\langle|\tau_b'|\rangle}{\rho'\omega_0'^2}. \tag{29}$$

Therefore, even with the simplifications adopted for the eddy viscosity, the boundary layer thickness as well as the eddy viscosity, can only be precisely determined, when the bottom shear stress is solved.

Following LO, the continuity and linearized momentum equations for the rotational velocity components in the boundary layer, expressed in a local coordinate system to the sea bottom, are respectively,

$$\nabla_\xi \cdot \mathbf{u}_\xi + \frac{1}{\widetilde{\alpha}\mu}\frac{\partial u_\zeta}{\partial\widetilde{\zeta}} = 0, \tag{30}$$

and

$$\frac{\partial \mathbf{u}_\xi}{\partial t} = \frac{\partial}{\partial\widetilde{\zeta}}\left(\widetilde{\zeta}\,\frac{\partial \mathbf{u}_\xi}{\partial\widetilde{\zeta}}\right) \tag{31}$$

with $\widetilde{\alpha} \equiv \widetilde{\delta}'k_0'$. We reiterate here that the tangential and normal components of the rotational velocity are, respectively, $\mathbf{u}_\xi$ and u_ζ. In the present formulation we have adopted the Boussinesq hypothesis $O(\epsilon) \approx O(\mu^2)$ and $O(\widetilde{\alpha}) \approx O(\mu^4)$ so as to linearize the boundary-layer equations. Hereafter, it is assumed $O(\widetilde{\alpha}) \approx O(\epsilon^2) \approx O(\mu^4)$ in the remainder of the paper.

Since only the periodic motions are considered, we shall also express the rotational velocity inside the boundary layer as a Fourier series in time

$$\mathbf{u}_\xi\left(\xi,\widetilde{\zeta},t\right) = \frac{1}{2}\sum_{n=-\infty}^{\infty}\mathbf{u}_{\xi,n}\left(\xi,\widetilde{\zeta}\right)\exp\left(int\right), \tag{32}$$

where $\mathbf{u}_{\xi,-n}$ is the complex conjugate of $\mathbf{u}_{\xi,n}$. Similarly, the vertical rotational velocity u_ζ and the velocity potential Φ can also be expanded as

$$u_\zeta\left(\xi,\widetilde{\zeta},t\right) = \frac{1}{2}\sum_{n=-\infty}^{\infty}u_{\zeta,n}\left(\xi,\widetilde{\zeta}\right)\exp\left(int\right), \tag{33}$$

and

$$\Phi_n\left(\xi,\widetilde{\zeta},t\right) = \frac{1}{2}\sum_{n=-\infty}^{\infty}\Phi_n\left(\xi,\widetilde{\zeta}\right)\exp\left(int\right). \tag{34}$$

The momentum equation for $\mathbf{u}_{\xi,n}$ is obtained by introducing (32) into (31),

$$in\mathbf{u}_{\xi,n} = \frac{\partial}{\partial\widetilde{\zeta}}\left(\widetilde{\zeta}\,\frac{\partial \mathbf{u}_{\xi,n}}{\partial\widetilde{\zeta}}\right). \tag{35}$$

The no-slip at the bottom requires that,

$$\mathbf{u}_{\xi,n} = -\nabla\Phi_n \qquad \text{at} \qquad \widetilde{\zeta} = \frac{\zeta_0'}{\widetilde{\delta}'} \; \left[\equiv \widetilde{\zeta}_0\right] \tag{36}$$

with $\zeta_0' \equiv k_s'/30$ and k_s' the equivalent bottom roughness. For hydraulically smooth bottom, the above boundary condition is applied at $\widetilde{\zeta} = 0$. At the outer edge of the boundary layer, the rotational velocity must vanish, i.e.,

$$\mathbf{u}_{\xi,n} \to 0 \qquad \text{at} \qquad \widetilde{\zeta} \to \infty. \tag{37}$$

The solution of the two-point boundary-value problem described by (35)–(37) can be readily obtained (Grant and Madsen, 1979)

$$\mathbf{u}_{\xi,n} = -\nabla\Phi_n \, \Xi_n^0\left(\widetilde{\zeta}\right) \quad \text{for} \quad n > 0, \tag{38}$$

with

$$\Xi_n^j\left(\widetilde{\zeta}\right) \equiv \frac{\mathrm{K}_j\left(2\sqrt{in\widetilde{\zeta}}\right)}{\mathrm{K}_0\left(2\sqrt{in\widetilde{\zeta}_0}\right)}, \tag{39}$$

where K_j is the modified Bessel function of second kind and order j.

Finally, integrating continuity equation (30) inside the boundary layer for the n^{th} harmonic, we obtain the leading order vertical rotational velocity at the sea bottom,

$$u_{\zeta,n}\left(\widetilde{\zeta}_0\right) = -\widetilde{\alpha}\mu\nabla^2\Phi_n \frac{1-i}{\sqrt{2n}} \sqrt{\widetilde{\zeta}_0}\, \Xi_n^1\left(\widetilde{\zeta}_0\right) + O(\mu^6) \tag{40}$$

for $n > 0$. We remark that since $\nabla^2\Phi_n = \nabla \cdot \overline{\mathbf{u}}_n + O(\mu^2)$, the above equation can be re-written as

$$u_{\zeta,n}\left(\widetilde{\zeta}_0\right) = -\widetilde{\alpha}\mu\nabla \cdot \overline{\mathbf{u}}_n \frac{1-i}{\sqrt{2n}} \sqrt{\widetilde{\zeta}_0}\, \Xi_n^1\left(\widetilde{\zeta}_0\right) + O(\mu^6) \tag{41}$$

for $n > 0$. This normal velocity induced by the boundary layer can be used in the depth-integrated continuity equation for irrotational flows in the core region, (19). The effects of the boundary layer appear to be of the order of $O(\mu^3)$. We note that $\widetilde{\alpha}(\equiv \widetilde{\delta}' k_0')$ and $\widetilde{\delta}'$ depend on the bottom stress as shown in (29). In the following section, we present the formulae for bottom stress, τ_b', and the boundary layer thickness, $\widetilde{\delta}'$, so as to close the problem.

4.1 Bottom shear stress

The bottom shear stress in dimensional form at ζ_0' is

$$\tau_b' = \rho' \nu_t' \left. \frac{\partial \mathbf{u}_\xi'}{\partial \zeta'} \right|_{\widetilde{\zeta}'=k_s'/30}. \tag{42}$$

The corresponding dimensionless form can be written as

$$\tau_b = \widetilde{\zeta} \left. \frac{\partial \mathbf{u}_\xi}{\partial \widetilde{\zeta}} \right|_{\widetilde{\zeta}=\widetilde{\zeta}_0} \quad \text{with} \quad \tau_b \equiv \frac{\tau_b'}{\widetilde{\alpha}\epsilon\rho' g' h_0'}. \tag{43}$$

Introducing (32) and (38) into (43), the dimensionless bottom stress can be related to the depth-integrated velocity, $\overline{\mathbf{u}}_n$, as

$$\tau_b = \sqrt{\frac{k_s'}{30\widetilde{\delta}'}} \sum_{n>0}^{\infty} \Re\left\{ \overline{\mathbf{u}}_n \sqrt{in}\, \Xi_n^1 \left(\widetilde{\zeta}_0\right) \exp\left(int\right) \right\}, \tag{44}$$

in which $\Re\{\}$ denotes that only the real part is considered. Recalling (29), we can obtain the boundary layer thickness $\widetilde{\delta}'$ as

$$\widetilde{\delta}'^{3/2} = \frac{\epsilon\kappa^2}{k_0'} \sqrt{\frac{k_s'}{30}} \left\langle \left| \sum_{n>0}^{\infty} \Re\left\{ \overline{\mathbf{u}}_n \sqrt{in}\, \Xi_n^1 \left(\widetilde{\zeta}_0\right) \exp\left(int\right) \right\} \right| \right\rangle. \tag{45}$$

The equation above is a transcendental equation for $\widetilde{\delta}'$ since $\widetilde{\zeta}_0$ is a function of $\widetilde{\delta}'$, i.e. (36).

5 Governing equation for the free surface displacement

Once the boundary-layer rotational velocity is related to the depth-integrated velocity in the core region, we can present the Boussinesq-type equation in terms of a single unknown variable, the free surface displacement. Following the procedure presented in Liu *et al.* (1985), we combine (18) and (19) into a single equation to describe the free surface evolution,

$$\varphi_{1,n}\nabla^2\eta_n + \varphi_{2,n} n^2 \eta_n = \frac{\epsilon}{2}\sum_{\substack{s\neq n\\ s\neq 0}} (n^2 - s^2)\eta_s\eta_{n-s} - \frac{\epsilon}{2}\sum_{\substack{s\neq n\\ s\neq 0}} \frac{n+s}{n-s}\nabla\eta_s\cdot\nabla\eta_{n-s} - \\ - \epsilon \sum_{\substack{s\neq n\\ s\neq 0}} \frac{1}{s(n-s)} \left(\frac{\partial^2\eta_s}{\partial x^2}\frac{\partial^2\eta_{n-s}}{\partial y^2} - \frac{\partial^2\eta_s}{\partial x\partial y}\frac{\partial^2\eta_{n-s}}{\partial x\partial y} \right) + O(\mu^4), \tag{46}$$

where

$$\varphi_{1,n} \equiv 1 - \frac{n^2\mu^2}{3}, \tag{47}$$

and

$$\varphi_{2,n} \equiv 1 + \frac{\widetilde{\alpha}}{\mu}\frac{(1-i)}{\sqrt{2n}}\sqrt{\widetilde{\zeta}_0}\,\Xi_n^1\left(\widetilde{\zeta}_0\right). \tag{48}$$

We note that (22) and (23) have been employed in deriving (46). If the viscous term is neglected (i.e., $\widetilde{\alpha} = 0$), (46) reduces to that obtained in Liu *et al.* (1985). Equation (46) is a system of non-linear equations for $\eta_n, (n = 1, 2, ...)$. Once η_n is obtained, the depth-integrated velocity $\mathbf{u}_n$, can be found by using (22).

6 Viscous damping of a modulation wave train in a long channel

Now, we shall examine the viscous effects on the evolution of periodic waves propagating in a long channel with a constant depth. The free-surface displacement for the n-th harmonic, can be written as,

$$\eta_n(x) = A_n(x)\exp(-inx), \tag{49}$$

where $A_n(x)$ is the complex amplitude function. It is well known that due to nonlinear interactions among different harmonics the amplitude functions modulate periodically along the channel. Mei and Unluata (1972), investigated this phenomena as a second-harmonic generation problem. Here, we re-examine the problem with the additional consideration of the boundary-layer effects.

Substituting (49) into (46) and considering that the water depth is a constant, we obtain:

$$\varphi_{1,n}\left(\frac{d^2A_n}{dx^2} - 2in\frac{dA_n}{dx}\right) + \beta_n n^2 A_n = \frac{\epsilon n}{2}\sum_{\substack{s\neq n\\ s\neq 0}}(n+s)A_sA_{n-s} -$$
$$-\frac{\epsilon}{2}\sum_{\substack{s\neq n\\ s\neq 0}}\frac{n+s}{n-s}\left(\frac{dA_s}{dx}\frac{dA_{n-s}}{dx} - isA_s\frac{dA_{n-s}}{dx} - i(n-s)A_{n-s}\frac{dA_s}{dx}\right) + O(\mu^4), \tag{50}$$

where

$$\beta_n \equiv (\varphi_{2,n} - \varphi_{1,n}) = \frac{n^2\mu^2}{3} + \frac{\widetilde{\alpha}}{\mu}\frac{(1-i)}{\sqrt{2n}}\sqrt{\widetilde{\zeta}_0}\Xi_n^1\left(\widetilde{\zeta}_0\right). \tag{51}$$

Moreover, assuming weak amplitude variations in the x-direction, i.e.,

$$\frac{\partial A_n}{\partial x} = O(\mu^2), \qquad \frac{\partial^2 A_n}{\partial x^2} = O(\mu^4), \tag{52}$$

equation (50) can be simplified to

$$\frac{dA_n}{dx} = -\frac{in}{2}\,\beta_n A_n + \frac{i\epsilon}{4}\sum_{\substack{s\neq n\\ s\neq 0}} (n+s)\,A_s A_{n-s} + O(\mu^4). \tag{53}$$

which is similar to the system of nonlinearly coupled equations obtained by Grataloup and Mei (2003) for the damping of of weakly nonlinear waves due to multiple scattering by a randomly rough seabed.

Considering the first 5 harmonics in the wave propagation, the governing equations for the amplitude functions become,

$$\frac{dA_1}{dx} = -\frac{i}{2}\,\beta_1 A_1 + \frac{3i\epsilon}{4}\left(A_{-1}A_2 + A_{-2}A_3 + A_{-3}A_4 + A_{-4}A_5\right), \tag{54}$$

$$\frac{dA_2}{dx} = -\frac{2i}{2}\,\beta_2 A_2 + \frac{6i\epsilon}{4}\left(\frac{1}{2}A_1^2 + A_{-1}A_3 + A_{-2}A_4 + A_{-3}A_5\right), \tag{55}$$

$$\frac{dA_3}{dx} = -\frac{3i}{2}\,\beta_3 A_3 + \frac{9i\epsilon}{4}\left(A_1A_2 + A_{-1}A_4 + A_{-2}A_5\right), \tag{56}$$

$$\frac{dA_4}{dx} = -\frac{4i}{2}\,\beta_4 A_4 + \frac{12i\epsilon}{4}\left(\frac{1}{2}A_2^2 + A_1A_3 + A_{-1}A_5\right), \tag{57}$$

$$\frac{dA_5}{dx} = -\frac{5i}{2}\,\beta_5 A_5 + \frac{15i\epsilon}{4}\left(A_2A_3 + A_1A_4\right), \tag{58}$$

which are accurate up to $O(\mu^4)$.

6.1 Boundary layer thickness and an iterative procedure

The damping term, $\varphi_{2,n}$, in (54)–(58) is a function of the non-dimensional parameters $\widetilde{\zeta}_0 = k_s'/30\widetilde{\delta}'$ and $\widetilde{\alpha} \equiv \widetilde{\delta}' k_0'$, which are part of solutions. An iterative procedure is adopted to find the solutions. Initial guesses (constant values) for $\widetilde{\zeta}_0$ and $\widetilde{\alpha}$ are made so as to calculate $\Xi_n^1(\widetilde{\zeta}_0)$ in (48). The system of nonlinear ordinary differential equations, (54)–(58), is solved numerically with an implicit finite difference scheme. Once the free surface elevation for the n^{th} harmonic, η_n, is obtained, using the relation (22), the leading order of the depth averaged velocity $\bar{\mathbf{u}}_n$ can be computed. The non-dimensional parameter $\widetilde{\zeta}_0$ is then updated applying (45), which can also be expressed as

$$\widetilde{\zeta}_0 = \left\{\frac{30\epsilon\kappa^2}{\varrho}\left\langle\left|\sum_{n>0}^{\infty}\Re\left\{\bar{\mathbf{u}}_n\sqrt{in}\,\Xi_n^1\left(\widetilde{\zeta}_0\right)\,\exp\left(int\right)\right\}\right|\right\rangle\right\}^{-2/3}. \tag{59}$$

with $\varrho \equiv k_0' k_s'$. The new value $\widetilde{\zeta}_0$ is used to update $\widetilde{\alpha}$, for $\widetilde{\alpha} = \varrho/30\widetilde{\zeta}_0$. The procedure described above is repeated until a converge criteria between two successive iterations is reached (the relative error between two successive iterations is smaller than 10^{-3} for all x).

In the numerical example, a wave maker, located at $x = 0$, generates a sinusoidal wave train with $A_1(0) = 1$ and $A_2(0) = A_3(0) = A_4(0) = A_5(0) = 0$. The values for the nonlinear parameter and the frequency dispersion are $\epsilon = 0.1$, and $\mu^2 = 0.1216$, respectively. The bottom roughness is specified by $\varrho \equiv k'_0 k'_s = 10^{-4}$. The following initial guesses are made for $\widetilde{\zeta}_0 = 0.01$ and $\widetilde{\alpha} = 3.3 * 10^{-4}$ for all x positions.

The amplitude variation for the first five harmonics is shown in Figure 1. The grey line correspond to the classical solution where the viscous terms are neglected (i.e., $\widetilde{\alpha} = 0$). The solutions with the boundary layer effects considered are displayed as black lines. In both situations, as the wave train propagates into the channel, wave energy is transferred from the first to higher harmonics as the result of the nonlinearity in (54)–(58). However, when the turbulent boundary layer effects are included, the viscous damping reduces the amplitudes for all harmonics. As shown in Figure 2, the depth averaged velocity $< \overline{u} >$ along the channel is reduced and the boundary layer thickness $\widetilde{\delta}'$ decreases from 19 mm to 14 mm as a result of the decrease of wave amplitude due to damping.

The time histories of the free-surface elevation at several stations (for $x = 30, 60, 90$ and 120) are shown in Figure 3. The grey lines correspond to the inviscid solutions where $\widetilde{\alpha} = 0$ and the black lines denote the solutions with the effects of the boundary layer included. Since the boundary-layer dissipative effect is accumulative, near the wavemaker (Figure 3a and 3b), the wave train has not been significantly influenced by the boundary layer and two solutions are almost identical. However, as the wave train propagates farther down the channel (Figure 3c), the dissipative effects reduce the wave amplitude. At $x = 120$ (Figure 3d), the damping is around 28%.

7 Concluding remarks

In the present paper a new Boussinesq-type model for periodic wave propagation has been developed when the effects of a turbulent boundary layer are significant. In this model the eddy viscosity model is used in the turbulent boundary layer and is further approximated as a linear function of the distance measured from the seafloor. The analytical expressions for the boundary-layer velocity can be found in terms of the irrotational velocity in the core region and the effects of the boundary layer appear in the depth-integrated continuity equation because of the velocity deficit. The bottom stress, the boundary layer thickness and the magnitude of the turbulent eddy viscosity are part of solutions. An iterative scheme is introduced to solve the system of equations. Numerical solutions for the evolution of periodic waves propagating in a one-dimensional channel are discussed. The model can be extended and implemented straightforwardly for two-dimensional problems.

8 Acknowledgments

AO would like to thank financial support from MEC trough the postdoctoral grant at Cornell University (EX2003-0430) and Govern Balear, UGIZC project. GS was supported by the visiting professor program from UCLM. PLFL would like to acknowledge the support from National Science Foundation through various grants to Cornell University.

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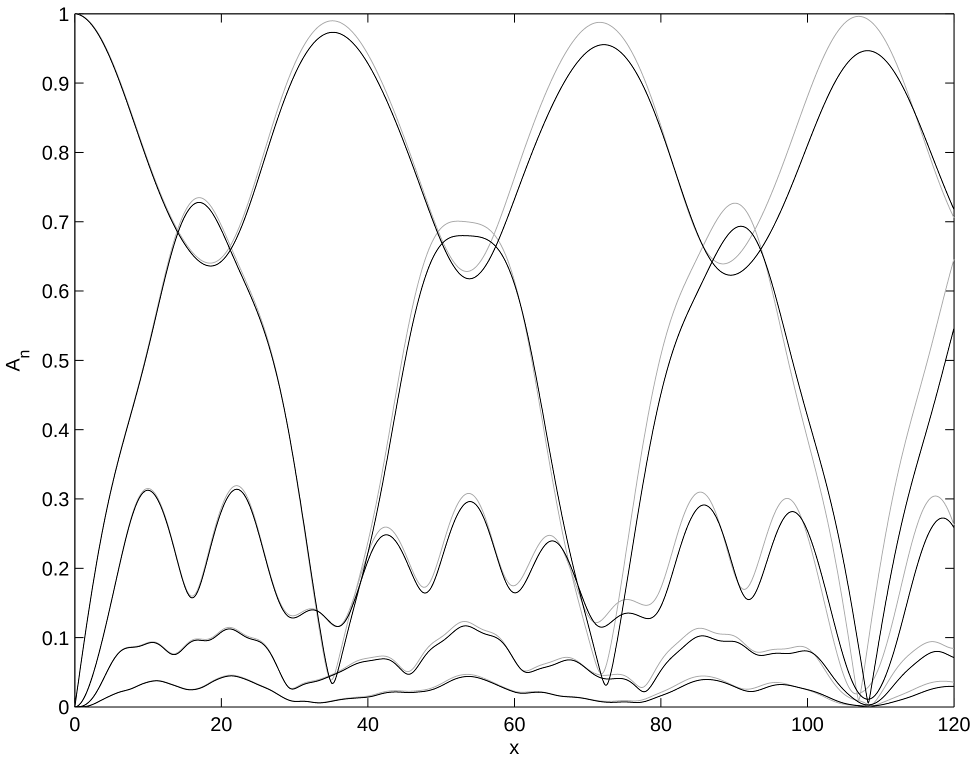
A Figure captions

Figure 1. Amplitudes of the modulated wave train, using 5 harmonics. The grey lines correspond to the simulations without the viscous effects, black lines with the boundary layer effects included.

Figure 2. a) The depth-averaged velocity $\bar{\mathbf{u}}$ along the wave channel. b) Distribution of $\widetilde{\delta}' = k'_s/30\zeta_0$. The characteristics of the wave motion are defined by, $\mu^2 = 0.1216, \epsilon = 0.1$ and $\varrho = 10^{-4}$

Figure 3. Free surface evolution at (a) $x = 20$, (b) $x = 40$, (c) $x = 60$, and (d) $x = 80$ for a wave train propagating with no damping (grey) and with damping (black)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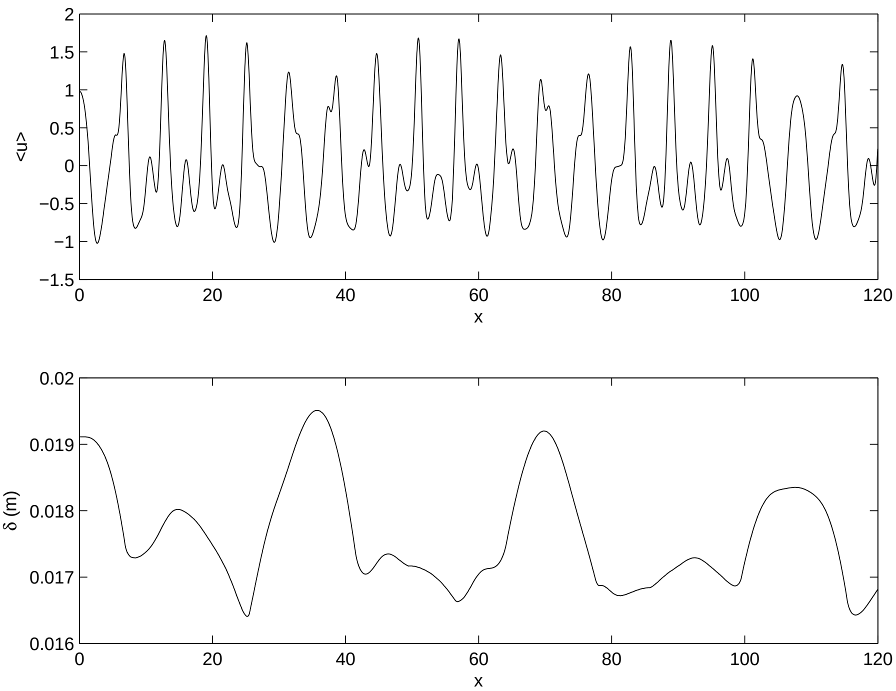
1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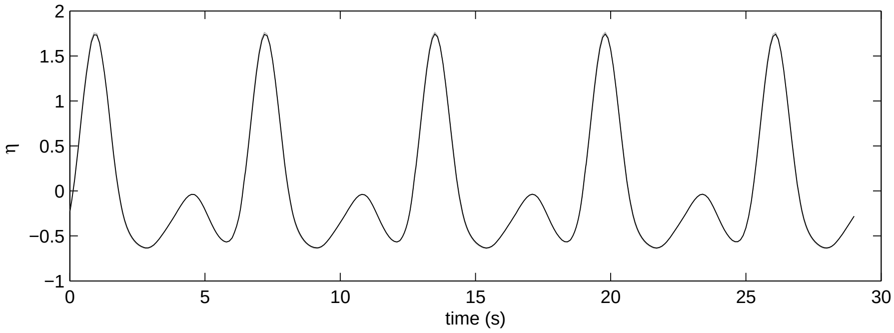

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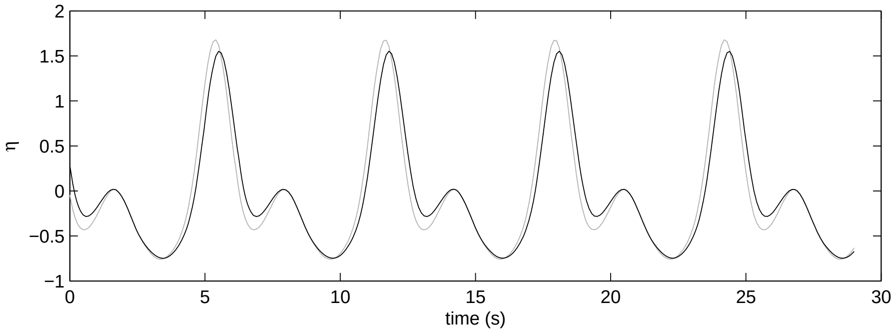
30
time (s)



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