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Sonic Analog of Gravitational Black Holes in Bose-Einstein Condensates

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It is shown that, in dilute-gas Bose-Einstein condensates, there exist both dynamically stable and unstable configurations which, in the hydrodynamic limit, exhibit a behavior resembling that of gravitational black holes. The dynamical instabilities involve creation of quasiparticle pairs in positive and negative energy states, as in the well-known suggested mechanism for black-hole evaporation. We propose a scheme to generate a stable sonic black hole in a ring trap.

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Many investigations of dilute gas Bose-Einstein condensates are directed towards experimentally creating nontrivial configurations of the semiclassical mean field, or to predicting the properties of such configurations in the presence of quantum fluctuations. Such problems are hardly peculiar to condensates, but ultracold dilute gases are so easy to manipulate and control, both experimentally [1] and theoretically [2], that they may allow us to analyze less amenable systems by analogy. As an essay in such an application of condensates, in this paper we discuss the theoretical framework and propose an experiment to create the analog of a black hole in the laboratory and simulate its radiative instabilities.

The hydrodynamic analog of an event horizon [3] was suggested originally by Unruh [4] as a more accessible phenomenon which might shed some light on the Hawking effect [5] (thermal radiation from black holes, stationary insofar as back reaction is negligible) and, in particular, on the role of ultrahigh frequencies [6–8]. An event horizon for sound waves appears in principle wherever there is a closed surface through which a fluid flows inwards at the speed of sound, the flow being subsonic on one side of the surface and supersonic on the other. There is a close analogy between sound propagation on a background hydrodynamic flow, and field propagation in a curved spacetime; and although hydrodynamics is only a long-wavelength effective theory for physical (super)fluids, so also field theory in curved spacetime is to be considered a long-wavelength approximation to quantum gravity [7,9]. Determining whether and how sonic black holes radiate sound, in a full calculation beyond the hydrodynamic approximation or in an actual experiment, can thus offer some suggestions about black-hole radiance and its sensitivity to high frequency physics.

The basic challenge of our proposal is to keep the trapped Bose-Einstein gas sufficiently cold and well isolated to maintain a locally supersonic flow long enough to observe its intrinsic dynamics. Detecting thermal phonons radiating from the horizons would obviously be a difficult additional problem, since such radiation would be indistinguishable from many other possible heating effects. This further difficulty does not arise in our proposal, however, because the black-hole radiation we predict is, unlike Hawking radiation, not quasistationary, but grows exponentially under appropriate conditions. It should therefore be observable in the next generation of atom traps.

A Bose-Einstein condensate is the ground state of a second quantized many body Hamiltonian for N interacting bosons trapped by an external potential $V_{\text{ext}}(\mathbf{x})$ [2]. At zero temperature, when the number of atoms is large and the atomic interactions are sufficiently small, almost all the atoms are in the same single-particle quantum state $\Psi(\mathbf{x}, t)$, even if the system is slightly perturbed. The evolution of Ψ is then given by the well-known Gross-Pitaevskii

equation

$$i\hbar\partial_t\Psi = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\rm ext} + \frac{4\pi a\hbar^2}{m}|\Psi|^2\right)\Psi,$$

where m is the mass of the atoms, a is the scattering length, and we normalize to the total number of atoms $\int d^3\mathbf{x}\,|\Psi(\mathbf{x},t)|^2 = N$.

Our purposes do not require solving the Gross-Pitaevskii equation with some given external potential $V_{\rm ext}(\mathbf{x})$; our concern is the propagation of small collective perturbations of the condensate, around a background stationary state $\Psi_s(\mathbf{x},t) = \sqrt{\rho(\mathbf{x})}\,e^{i\vartheta(\mathbf{x})}e^{-i\mu t/\hbar}$, where μ is the chemical potential. Thus it is only necessary that it be possible, in any external potential that can be generated, to create a condensate in this state. Many realistic techniques for "quantum state engineering," to create designer potentials and bring condensates into specific states, have been proposed, and even implemented successfully [10]; our simulations indicate that currently known techniques should suffice to generate the condensate states that we propose.

Perturbations about the stationary state $\Psi_s(\mathbf{x},t)$ obey the Bogoliubov system of two coupled second order differential equations. Within the regime of validity of the hydrodynamic (Thomas-Fermi) approximation [2], these two equations for the density perturbation ϱ and the phase perturbation ϕ in terms of the local speed of sound $c(\mathbf{x}) \equiv \frac{\hbar}{m}\sqrt{4\pi a\rho(\mathbf{x})}$, and the background stationary velocity $\mathbf{v} \equiv \frac{\hbar}{m}\nabla\vartheta$, read

$$\dot{\varrho} = -\nabla\left(\frac{m}{4\pi a\hbar}c^2\nabla\phi + \mathbf{v}\varrho\right),$$

$$\dot{\phi} = -\mathbf{v}\nabla\phi - \frac{4\pi a\hbar}{m}\varrho\,.$$

Furthermore, low frequency perturbations are essentially just waves of (zero) sound. Indeed, the Bogoliubov equations may be reduced to a single second order equation for the condensate phase perturbation ϕ. This differential equation has the form of a relativistic wave equation $\partial_\mu(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\phi) = 0$, with $g = \det g_{\mu\nu}$, in an effective curved spacetime with the metric $g_{\mu\nu}$ being entirely determined by the local speed of sound c and the background stationary velocity $\mathbf{v}$. Up to a conformal factor, this effective metric has the form

$$(g_{\mu\nu}) = \begin{pmatrix} -(c^2 - \mathbf{v}^2) & -\mathbf{v}^{\rm T} \\ -\mathbf{v} & \mathbf{1} \end{pmatrix}.$$

This class of metrics can possess event horizons. For instance, if an effective sink for atoms is generated at the center of a spherical trap (such as by an atom laser outcoupling technique [11]), and if the radial potential profile is suitably arranged, we can produce densities $\rho(r)$ and flow velocities $\mathbf{v}(\mathbf{x}) = -v(r)\mathbf{r}/r$ such that the quantity $c^2 - \mathbf{v}^2$ vanishes at a radius $r = r_h$, being negative inside and positive outside. The sphere at radius r_h is a sonic event horizon completely analogous to those appearing in gravitational black holes, in the sense that sonic perturbations cannot propagate through this surface in the outward direction [4,7,9]. The physical mechanism of the sonic black hole is quite simple: inside the horizon, the background flow speed v is larger than the local speed of sound c, and so sound waves are dragged inwards.

In fact, there are two conditions which must hold for this dragged sound picture to be accurate. Wavelengths larger than the black hole itself will, of course, not be dragged in, but merely diffracted around it. And perturbations must have wavelengths $\lambda \gg 2\pi\xi,\ 2\pi\xi/\sqrt{|1 - v/c|}$, where $\xi(\mathbf{x}) \equiv \hbar/[mc(\mathbf{x})]$ is the local healing length. Otherwise they do not behave as sound waves since they lie outside the regime of validity of the hydrodynamic approximation. These short-wavelength modes must be described by the full Bogoliubov equations, which allow signals to propagate faster than the local sound speed, and thus permit escape from sonic black holes. Even if such an intermediate range of wavelengths does exist, the modes outside it may still affect the stability of the black hole as discussed below.

As it stands, this description is incomplete. The condensate flows continually inwards, and therefore at $r = 0$ there must be a sink that takes atoms out of the condensate. Otherwise, the continuity equation $\nabla(\rho\mathbf{v}) = 0$, which must hold for stationary configurations, will be violated. We have analyzed several specific systems which may be suitable theoretical models for future experiments, and have found that the qualitative behavior is analogous in all of them. Black holes which require atom sinks are both theoretically and experimentally more involved, however; moreover, maintaining a steady transonic flow into a sink may require either a very large condensate or some means of replenishment. We will therefore discuss here an alternative configuration which may be experimentally more accessible and whose description is particularly simple: a condensate in a very thin ring that effectively behaves as a periodic one-dimensional system. Under conditions that we will discuss, the supersonic region in a ring may be bounded by two horizons: a black-hole horizon through which phonons cannot exit, and a "white hole" horizon through which they cannot enter.

In a sufficiently tight ring-shaped external potential of radius R, motion in radial (r) and axial (z) cylindrical coordinates is effectively frozen. We can then write the wave function as $\Psi(z,r,\theta,\tau) = f(z,r)\Phi(\theta,\tau)$ and normalize Φ to the number of atoms in the condensate $\int_0^{2\pi} d\theta\,|\Phi(\theta)|^2 = N$, where with the azimuthal coordinate θ we have introduced the dimensionless time $\tau = \frac{\hbar}{mR^2}t$. The Gross-Pitaevskii equation thus becomes effectively one-dimensional:

$$i\partial_\tau\Phi = \left(-\frac{1}{2}\partial_\theta^2 + \mathcal{V}_{\rm ext} + \frac{\mathcal{U}}{N}|\Phi|^2\right)\Phi, \qquad (1)$$

where $\mathcal{U} \equiv 4\pi aNR^2\int dz\,dr\,r|f(z,r)|^4$ and $\mathcal{V}_{\rm ext}(\theta)$ is the dimensionless effective potential (in which we have already included the chemical potential) that results from the dimensional reduction. The stationary solution can

then be written as $\Phi_s(\theta,\tau) = \sqrt{\rho(\theta)}\, e^{i\int d\theta\, v(\theta)}$ and the local dimensionless angular speed of sound as $c(\theta) = \sqrt{\mathcal{U}\rho(\theta)/N}$. Periodic boundary conditions around the ring require the "winding number" $w \equiv \frac{1}{2\pi}\int_0^{2\pi} d\theta\, v(\theta)$ to be an integer.

The qualitative behavior of horizons in a ring is well represented by the two-parameter family of condensate densities

$$\rho(\theta) = \frac{N}{2\pi}(1 + b\cos\theta),$$

where $b \in [0,1]$. Continuity, $\partial_\theta(\rho v) = 0$, then determines the dimensionless flow-velocity field

$$v(\theta) = \frac{\mathcal{U} w\sqrt{1-b^2}}{2\pi c(\theta)^2},$$

which depends on w as a third discrete independent parameter. Requiring that $\Phi_s(\theta,\tau)$ be a stationary solution to the Gross-Pitaevskii equation then determines how the trapping potential must be modulated as a function of θ. All the properties of the condensate, including whether and where it has sonic horizons, and whether or not they are stable, are thus functions of $\mathcal{U}$, b, and w. For instance, if we require that the horizons be located at $\theta_h = \pm\pi/2$, which imposes the relation $\mathcal{U} = 2\pi w^2(1-b^2)$, then we must have $c^2 - v^2$ positive for $\theta \in (-\pi/2, \pi/2)$, zero at $\theta_h = \pm\pi/2$, and negative otherwise, provided that $\mathcal{U} < 2\pi w^2$. The further requirement that perturbations on wavelengths shorter than the inner and the outer regions are indeed phononic implies $\mathcal{U} \gg 2\pi$, which in turn requires $w \gg 1$ and $1 \gg b \gg 1/w^2$. In fact, detailed analysis shows that $w \gtrsim 5$ is sufficient.

A black-hole solution should also be stable over sufficiently long time scales in order to be physically realizable. Since stability must be checked for perturbations on all wavelengths, the full Bogoliubov [2] spectrum must be determined. For large black holes within infinite condensates, this Bogoliubov problem may be solved using WKB methods that closely resemble those used for solving relativistic field theories in true black-hole spacetimes [8]. The results are also qualitatively similar to those we have found for black holes in finite traps, where we have resorted to numerical methods because, in these cases, WKB techniques may fail for just those modes which threaten to be unstable.

Our numerical approach for our three-parameter family of black/white holes in the ring-shaped condensate has been to write the Bogoliubov equations in discrete Fourier space, and then truncate the resulting infinite-dimensional eigenvalue problem. Writing the wave function as $\Phi = \Phi_s + \varphi e^{i\int d\theta\, v(\theta)}$, decomposing the perturbation φ in discrete modes

$$\varphi(\theta,\tau) = \sum_{\omega,n} e^{-i\omega\tau} e^{in\theta} A_{\omega,n} u_{\omega,n}(\theta) + e^{i\omega^*\tau} e^{-in\theta} A^*_{\omega,n} v^*_{\omega,n}(\theta),$$

and substituting into the Gross-Pitaevskii equation, we obtain the following equation for the modes $u_{\omega,n}$ and $v_{\omega,n}$:

$$\omega\begin{pmatrix} u_{\omega,n} \\ v_{\omega,n}\end{pmatrix} = \sum_p \begin{pmatrix} h^+_{np} & f_{np} \\ -f_{np} & h^-_{np}\end{pmatrix}\begin{pmatrix} u_{\omega,p} \\ v_{\omega,p}\end{pmatrix}.$$

In this equation,

$$f_{np} = \frac{\mathcal{U}}{2\pi}\left(\delta_{n,p} + \frac{b}{2}\,\delta_{n,p+1} + \frac{b}{2}\,\delta_{n,p-1}\right),$$

$$h^\pm_{np} = \frac{1}{2}(n+p)w\sqrt{1-b^2}\,\alpha_{n-p} \pm \left(f_{np} + \frac{4n^2-1}{8}\,\delta_{n,p} + \frac{1-b^2}{8}\,\beta_{n-p}\right),$$

$$\alpha_i = \sum_{j\geq|i|, i+j\,\text{even}}^{\infty} \left(\frac{-b}{2}\right)^j \binom{j}{(i+j)/2},$$

$$\beta_i = \sum_{j\geq|i|, i+j\,\text{even}}^{\infty} \left(\frac{-b}{2}\right)^j \binom{j}{(i+j)/2}(j+1).$$

Eliminating Fourier components above a sufficiently high cutoff Q has negligible effect on possible instabilities, which can be shown to occur at relatively long wavelengths. The numerical solution to this eigenvalue equation, together with the normalization condition $\int d\theta\,(u^*_{\omega^*,n}u_{\omega',n'} - v^*_{\omega^*,n}v_{\omega',n'}) = \delta_{nn'}\delta_{\omega\omega'}$, provides the allowed frequencies. Real negative eigenfrequencies for modes of positive norm are always present, which means that black-hole configurations are energetically unstable, as expected. This feature is inherent in supersonic flow, since the speed of sound is also the Landau critical velocity. In a sufficiently cold and dilute condensate, however, the time scale for dissipation may in principle be made very long, and so these energetic instabilities need not be problematic [12].

More serious are dynamical instabilities, which occur for modes with complex eigenfrequencies and are genuine physical phenomena. For sufficiently high values of the cutoff (e.g., $Q \geq 25$ in our calculations), the complex eigenfrequencies obtained from the truncated eigenvalue problem become independent of the cutoff within the numerical error. The existence and rapidity of dynamical instabilities depend sensitively on $(\mathcal{U}, b, w)$. For instance, see Fig. 1 for a contour plot of the maximum of the absolute values of the imaginary parts of all eigenfrequencies for $w = 7$, showing that the regions of instability are long, thin fingers in the $(\mathcal{U}, b)$ plane. Not shown in the figure is the important fact that the size of the imaginary parts, which gives the rate of the instabilities, increases starting from zero, quite rapidly with b, although they remain small as compared with the real parts.

The stability diagram of Fig. 1 suggests a strategy for creating a sonic black hole from an initial stable state. Within the upper subsonic region, the vertical axis $b = 0$ corresponds to a homogeneous persistent current in a ring, which can in principle be created using different techniques [13]. Gradually changing $\mathcal{U}$ and b, it is possible to

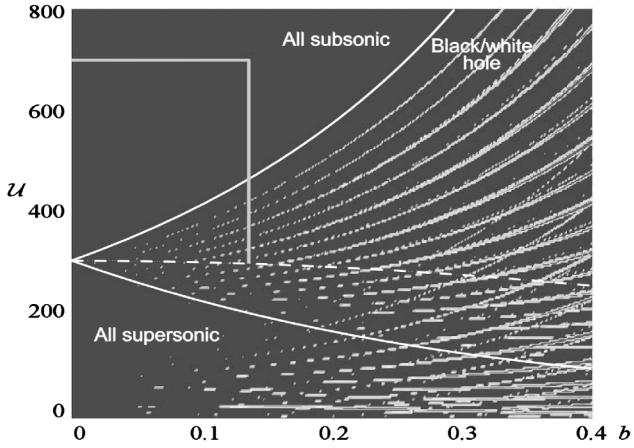

FIG. 1. Stability diagram for winding number $w = 7$. Solid dark-grey areas represent the regions of stability. Smaller plots at higher resolution confirm that the unstable "fingers" are actually smooth and unbroken. Points on the dashed curve are states with horizons at $\pm\pi/2$, so that the black/white hole fills half the ring.

move from such an initial state to a black/white hole state, along a path lying almost entirely within the stable region, and passing only briefly through instabilities where they are sufficiently small to cause no difficulty.

Indeed, we have simulated this process of adiabatic creation of a sonic black/white hole by solving numerically (using the split operator method) the time-dependent Gross-Pitaevskii equation (1) that provides the evolution of the condensate when the parameters of the trapping potential change so as to move the condensate state along various paths in parameter space. One of these paths is shown in Fig. 1 (light-grey solid line): we start with a current at $w = 7$, $b = 0$, and sufficiently high $\mathcal{U}$; we then increase b adiabatically keeping $\mathcal{U}$ fixed until an appropriate value is reached; finally, keeping b constant, we decrease $\mathcal{U}$ adiabatically (which can be physically implemented by decreasing the radius of the ring trap), until we meet the dashed contour for black holes of comfortable size. Our simulations confirm that the small instabilities which briefly appear in the process of creation do not disrupt the adiabatic evolution. The final quantum state of the condensate, obtained by this procedure, indeed represents a stable black/white hole. We have further checked the stability of this final configuration by numerically solving the Gross-Pitaevskii equation (1) for very long periods of time (as compared with any characteristic time scale of the condensate) and for fixed values of the trap parameters. This evolution reflects the fact that no complex frequencies are present, as predicted from the mode analysis, and that the final state is stationary.

Once the black/white hole has been created, one could further change the parameters $(\mathcal{U}, b)$ so as to move between the unstable "fingers" into a stable region of higher b (a deeper hole) or one could deliberately enter an unstable region. In the latter case, the black hole should disappear in an explosion of phonons, which may be easy to detect experimentally. Such an event might be related to the evaporation process suggested for real black holes in the sense that pairs of quasiparticles are created near the horizon in both positive and negative energy modes. The hermiticity of the Bogoliubov Hamiltonian implies that eigenmodes with complex frequencies appear always in dual pairs, whose frequencies are complex conjugate. In the language of second quantization, the linearized Hamiltonian for each such pair has the form

$$H = \sum_n (\omega A^\dagger_{\omega^*,n} A_{\omega,n} + \omega^* A^\dagger_{\omega,n} A_{\omega^*,n}),$$

and the only nonvanishing commutators among these operators are $[A_{\omega,n}, A^\dagger_{\omega^*,n'}] = \delta_{nn'}$. It is then clear that none of these operators is actually a harmonic oscillator creation or annihilation operator in the usual sense. However, the linear combinations (note that $A^\dagger_{\omega^*,n} \neq A^\dagger_{\omega,n}$)

$$a_n = \frac{1}{\sqrt{2}}(A_{\omega,n} + A_{\omega^*,n}), \qquad b_n = \frac{i}{\sqrt{2}}(A^\dagger_{\omega,n} + A^\dagger_{\omega^*,n})$$

and their Hermitian conjugates are true annihilation and creation operators, with the standard commutation relations, and in terms of these the Bogoliubov Hamiltonian becomes

$$H = \sum_n [\mathrm{Re}(\omega)(a^\dagger_n a_n - b^\dagger_n b_n) - \mathrm{Im}(\omega)(a^\dagger_n b^\dagger_n + a_n b_n)],$$

which obviously leads to self-amplifying creation of positive and negative frequency pairs. Evaporation through an exponentially self-amplifying instability is not equivalent, however, to the usual kind of Hawking radiation [8]; this issue will be discussed in detail elsewhere.

Trapped bosons at ultralow temperature can provide an analog to a black-hole spacetime. Similar analogs have been proposed in other contexts, such as superfluid helium [14], solid state physics [15], and optics [16]; but the outstanding recent experimental progress in cooling, manipulating, and controlling atoms [10] makes Bose-Einstein condensates an especially powerful tool for this kind of investigation. We have analyzed in detail the case of a condensate in a ring trap, and proposed a realistic scheme for adiabatically creating stable sonic black/white holes.

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Note added.—Further details as well as the study of cigar shaped condensates with atom sinks at the center will appear in Ref. [17].

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