

On the problem of data assimilation by means of synchronization

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[1] The potential use of synchronization as a method for data assimilation is investigated in a Lorenz96 model. Data representing the reality are obtained from a Lorenz96 model with added noise. We study the assimilation scheme by means of synchronization for different noise intensities. We use a novel plot representation of the synchronization error in a phase diagram consisting of two variables: the amplitude and the width of the error after a suitable logarithmic transformation (the so-called mean-variance of logarithms diagram). Our main result concerns the existence of an “optimal” coupling for which the synchronization is maximal. We finally show how this allows us to quantify the degree of assimilation, providing a criterion for the selection of optimal couplings and validity of models.

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1. Introduction

[2] A difficult aspect of modeling and forecasting spatially extended chaotic systems is the problem of data assimilation [Kalnay, 2002]. We have physical observations of the state (hereafter *reality*) of a given system that we want to describe by a mathematical/computer model so that we can make predictions on the future states of the system. The physical system under consideration is very complicated, nonlinear, and has many degrees of freedom, like for instance the Earth’s atmosphere. Data assimilation refers to the process by which the representation of the atmosphere in the computer model is periodically adjusted to attempt to make it consistent with current observations of the atmospheric state.

[3] There are many difficulties when trying to assimilate real observations in spatially extended dynamical systems. Typically, observational data are collected at discrete positions and a few time instants. This leaves much uncertainty about the actual structure of the physical state below some spatial resolution or/and above a certain frequency range. Furthermore, the mathematical model is often an imperfect description of the real system because it is affected by parameter mismatches and uncertainties, it describes the physical system at a certain scale so that many spatial degrees of freedom are excluded, and simplifications are often used. Also, there are experimental errors in every observation of the real system, which would make it impossible to find an error free forecasting from the mathematical model, even under a “perfect model” hypothesis. Prediction is further complicated by the existence of relevant nonlinearities and the corresponding chaotic nature

of the system evolution. All these difficulties make the problem of forecasting spatially extended nonlinear systems a great challenge.

[4] While there is probably no perfect solution to the assimilation problem, much effort has been invested in the last few years in obtaining forecasts that evolve as close as possible to the true system for some reasonable time. In practice, this is usually done by means of variational methods (e.g., 4dVar [Kalnay, 2002]), in which an optimization with respect to the compatibility of the solution and the model on the one hand, and with respect to the difference with the observations on the other hand, is carried out. Such an optimization is numerically very expensive and has the unavoidable drawback that there might be many local minima of the functional that one needs to minimize. This may hinder the numerical solution from converging to the optimal solution in a finite time. Furthermore, since in a variational approach the problem is posed as a boundary value problem, one cannot be sure that the solutions have the same properties at the borders of the assimilated interval as in the middle. Actually, this might be a great problem since in prediction one is usually interested in the solution at the end of the time interval. Therefore studying new alternative methods for assimilating observational data into our simplified mathematical models is nowadays of great interest.

[5] Recently, there have been studies proposing a dynamical approach, rather than a variational approach, to assimilate and predict chaotic systems [Yang *et al.*, 2006; Duane *et al.*, 2006; Duane and Tribbia, 2007; Cohen *et al.*, 2008]. The basic idea of this novel approach is to somehow synchronize the mathematical model to the real system. Synchronization refers to the common tendency of identical systems to behave in a similar way when coupled [Boccaletti *et al.*, 2002; Pikovsky *et al.*, 2003]. The dynamical synchronization approach to assimilation (in contrast to the variational ones) has the immediate advantage of being computationally much less expensive. Furthermore, the

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differential role of the boundary versus the interior of the assimilated interval does not exist in synchronization methods. This suggests that assimilation by means of synchronization may be a promising alternative to variational methods.

[6] However, criteria to evaluate the quality of the dynamical synchronization approach have yet to be firmly established. Recently, in the work of *Yang et al.* [2006] the authors presented some new ideas to solve this issue for the case of a low-dimensional chaotic system. By contrast, the problem in high-dimensional chaos is much more complicated and it is far from clear how one could proceed to extend the tools that work for low-dimensional systems. This is particularly problematic in systems with spatial degrees of freedom, like atmosphere evolution models. Spatially extended dynamical systems have an arbitrary large number of degrees of freedom and often exhibit spatiotemporal chaos. A full understanding of the effect of spatial correlations in assimilation and prediction has long remained elusive.

[7] In this paper we propose a new technique to solve the problem of evaluating the quality of assimilation via dynamical synchronization in spatially extended systems. We exploit the information provided by the recently introduced mean-variance diagram [*Primo et al.*, 2005, 2007; *Gutiérrez et al.*, 2008; *Fernández et al.*, 2008]. Our results show that there exists an optimal coupling for which the partial synchronization of the model to the reality is largely enhanced. We find that the extent and magnitude of spatial correlations are the quantities that determine the quality of the assimilation in spatially extended systems. For this optimal coupling the spatial extent of connected synchronized regions becomes maximal and the predictive power of the model is thus the best we can achieve with the synchronization setting in the presence of noise in the dynamics of the reality.

[8] As an application example of our proposed technique to analyze the performance of synchronization as a method to assimilate a highly dimensional model to observations we study an ideal test bed system consisting of a Lorenz96 model with random noise representing the “real data”. In the case studied here no observational noise is considered, i.e., the observations coincide with the reality. We try to describe the observations with a noiseless Lorenz96 system, which is then coupled to observations in order to attain a synchronized state. We discuss both the case of “perfect” and “imperfect” model. Numerical estimators to quantify the degree or goodness of the assimilation via synchronization are then analyzed in both cases. The optimal coupling is studied and an ansatz describing the ability to compare the dynamics of models by synchronization is proposed.

[9] The paper is organized as follows. In section 2 we introduce the system we use, required definitions, and describe the synchronization scheme. Section 3 is devoted to introduce the quantities that will be needed to characterize the synchronization error, as well as the mean-variance plot. In section 4 we discuss assimilation by means of synchronization in the limit case of a perfect model, where the computer model is assumed to be identical to the observation. Section 5 is devoted to the most interesting case in which we try to assimilate a noisy reality into our

computer model. Finally, some concluding remarks are provided in section 6.

2. The Model

[10] In this paper we do not aim at studying very intricate models, but rather focus on a minimal model exhibiting spatiotemporal chaotic dynamics that can be used to test the use of synchronization as a tool to assimilate observations. We assume our mathematical model is reasonably good at describing the physical system, otherwise it would have absolutely no predictive power. In other words, we assume our computer model is able to capture well the slow varying degrees of freedom (low frequencies) of the physical system, while rapidly changing degrees of freedom escape our mathematical description. In the model example we consider here we achieve this in a simple way by using a deterministic dynamical model with random noise added to the dynamics. The noise term represents fast degrees of freedom that our computer model will not be able to capture. A given trajectory of this noisy model constitutes our reality.

[11] As a prototypical example of a spatially extended chaotic system we study a noisy Lorenz96 model [*Lorenz*, 1996]:

$$\partial_t u_n = -u_{n-1}(u_{n-2} - u_{n+1}) - u_n + F + \eta_n(t), \quad (1)$$

where $n = 1 \dots L$, $L = 256$ is the system size, $F = 8$, and $\eta_n(t)$ is a Gaussian distributed noise with delta correlations $\langle \eta_n(t_1) \eta_m(t_2) \rangle = 2D \delta_{nm} \delta(t_1 - t_2)$, where D is the noise intensity. The Lorenz96 system was integrated using an Euler algorithm with time step $\Delta t = 10^{-4}$. Periodic boundary conditions are applied in all our simulations. This toy model mimics some aspects of the dynamics of the atmosphere such as advection, constant forcing, and linear damping. In our simulations we have studied system sizes of $L = 128, 256$, and 512 . The idea is to explore large enough systems, where the properties we are interested in exhibit universality. Although traces of this universal behavior are expected to be recognizable also in smaller systems. All our plots correspond to $L = 256$, but similar results are found for the other system sizes studied.

[12] The trajectories generated by equation (1) represent the “reality”, i.e., the real observations of the physical system. These data have to be assimilated in the computer model of the physical system. The computer model we use to describe the observations is simply the deterministic Lorenz96 model, i.e., equation (1) without noise. Note that we want to describe situations in which reality and model can be made gradually compatible. This can be easily done by varying the noise intensity for instance: the larger the noise intensity, the further away will our model be from the data we want to assimilate. Mismatches between model and reality parameters (like physical constants, system size, . . .), although likely to be present in realistic situations, will not be considered at this stage. Let us remark that our preliminary study focuses on situations in which our model can capture quite well reality apart from the very short scale and fast varying degrees of freedom, which we assume cannot be resolved.

[13] In our synchronization scheme trajectories of equation (1) will play the role of observations. Herein we choose the simplest coupling and write the equation of the response system as

$$\partial_t v_n = -v_{n-1}(v_{n-2} - v_{n+1}) - v_n + F + \kappa(u_n - v_n), \quad (2)$$

where the term $\kappa(u_n - v_n)$ couples the model $v_n(t)$ with the observation $u_n(t)$, where κ is the strength of the coupling.

[14] The case of zero noise intensity, $D = 0$, has been studied in great detail in recent years in the context of chaos synchronization of spatially extended dynamical systems. In this case, the coupled system (1)–(2) shows a critical phase transition, from an unsynchronized to a synchronized state, for κ above some critical threshold [Ahlers and Pikovsky, 2002]. This synchronization transition belongs to the universality class of the bounded Kardar-Parisi-Zhang equation (bKPZ [Ahlers and Pikovsky, 2002; Bagnoli and Rechtman, 2006]). For couplings below the critical value the synchronization error, $\mathbf{w} = \mathbf{u} - \mathbf{v}$ (we use bold letters to denote a vector, $\mathbf{u}(t) = \{u_n(t)\}_{n=1}^L$), evolves toward some asymptotic state characterized by a typical amplitude and (finite) correlation length [Szendro et al., 2009]. This correlation length corresponds to the typical extent of highly synchronized regions. By contrast, in the noisy case, $D \neq 0$, there is no longer a sharp phase transition – namely, no matter how strong the coupling is, perfect synchronization is never reached.

[15] Note that in realistic situations synchronization may be strongly affected by parameter mismatches or structural differences of the model equations with respect to the equations governing the reality. Therefore the situation described in this article should be considered as a starting point for further, more realistic, studies. Interestingly, synchronization can also be used as a tool to obtain better models. For example, the degree of synchronization of model and reality can be used to obtain better estimates for model parameters [Parlitz et al., 1996].

3. Surface Growth Picture and Mean-Variance Diagram

[16] For the analysis of the synchronization errors $\mathbf{w}(t) = \mathbf{u}(t) - \mathbf{v}(t)$ we find very convenient to make use of a surface growth picture [Pikovsky and Kurths, 1994; Pikovsky and Politi, 1998]. This mapping allows one to generically interpret the spatial structure and growth of the logarithm of perturbations in terms of the Kardar-Parisi-Zhang (KPZ) equation [Kardar et al., 1986; Barabási and Stanley, 1995] describing the evolution of stochastically driven surfaces. This framework is very powerful for the analysis of both generic infinitesimal [Pikovsky and Kurths, 1994; Pikovsky and Politi, 1998; Szendro et al., 2007; Pazó et al., 2008, 2009] and finite perturbations [López et al., 2004; Primo et al., 2006, 2007; Gutiérrez et al., 2008; Fernández et al., 2008] as well as synchronization errors [Ahlers and Pikovsky, 2002; Szendro et al., 2009].

[17] Let us now define the “surface” $h_n(t) = \ln|w_n(t)| = \ln|u_n(t) - v_n(t)|$. The spatial structure of the surface h_n can be analyzed by computing the power spectral density (PSD) or structure factor $S_q(t) = \langle \hat{h}_q(t) \hat{h}_{-q}(t) \rangle$, where $\hat{h}_q(t) = (1/L) \sum_{n=1}^L [h_n(t) - \bar{h}(t)] \exp(iqn/L)$ is the Fourier transform

of $h_n - \bar{h}$, and $\bar{h}(t) = (1/L) \sum_{n=1}^L h_n(t)$ is the average surface position. For a generic perturbation in a (homogeneous) spatially extended chaotic system the surface constructed in this way is expected to exhibit scale-invariant correlations below the growing correlation length $\ell_\times(t)$. In other words, we have a stationary PSD that decays with two asymptotes:

$$S_q(t) \sim \begin{cases} q^{-(2\alpha+1)} & \text{if } q \gg q_c(t) \\ a(t) & \text{if } q \ll q_c(t) \end{cases} \quad (3)$$

where $q_c(t) \propto \ell_\times^{-1}(t)$, and function $a(t)$ does not depend on q . Then we say that h_n is a rough curve with a roughness exponent α and a correlation length $\ell_\times(t)$. Rough interfaces in the KPZ universality class, which is the case relevant for our study, invariably have a roughness exponent $\alpha = 1/2$.

[18] One of the main advantages of this surface picture is that many important quantities can be mapped into well-known magnitudes in terms of the surface. For instance, an important quantity is the squared surface width $W^2(t) = \langle (1/L) \sum_{n=1}^L [h_n(t) - \bar{h}]^2 \rangle$, which gives useful information about the spatial structure of the perturbation surface. One immediately obtains that the width W^2 is directly connected with the correlation length ℓ_\times since, by the Parzival relation, we can relate the surface width and the area below the PSD curve:

$$W^2(t) \propto \int S_q(t) dq \sim \ell_\times(t)^{2\alpha}. \quad (4)$$

[19] This is a very simple but important result that allows us to obtain the extent of the spatial correlations from the simple measurement of the surface width.

[20] On the other hand, the error amplitude, can also be immediately obtained as the exponential of the average surface height, $\varepsilon(t) = \langle |\mathbf{w}(t)| \rangle = \langle \exp[\bar{h}(t)] \rangle$, and informs about the typical size of the perturbation at any time. We have $\varepsilon(t) \sim \exp(\lambda t)$, where λ is just the average surface velocity ($\bar{h} = \lambda t$) and also corresponds to the first Lyapunov exponent.

[21] Therefore the dynamics of the perturbations can be analyzed in a nice and intuitive way by two quantities – namely, average height and surface width – that inform about the size and spatial correlation length of the perturbation, respectively. These two quantities can be conveniently plotted in a graph $W^2(t)$ versus $\ln \varepsilon(t)$, the so-called mean-variance of the logarithm of perturbations (MVL) diagram, to analyze perturbation growth. This plot is very simple to obtain and inexpensive in computational terms, even for complicated models. The MVL diagram has already proven useful in the analysis of the impact of initial conditions on the dynamics of free finite perturbations in operative weather models [Primo et al., 2005, 2007; Gutiérrez et al., 2008; Fernández et al., 2008].

4. Perfect Model Case $D = 0$

[22] It is instructive to first analyze the limiting case of a perfect model, i.e., a zero noise strength $D = 0$. In this case our computer model, $v_n(t)$, describes perfectly well the reality, $u_n(t)$, and the “assimilation” procedure in equation (2) simply leads to standard chaotic synchronization. In other

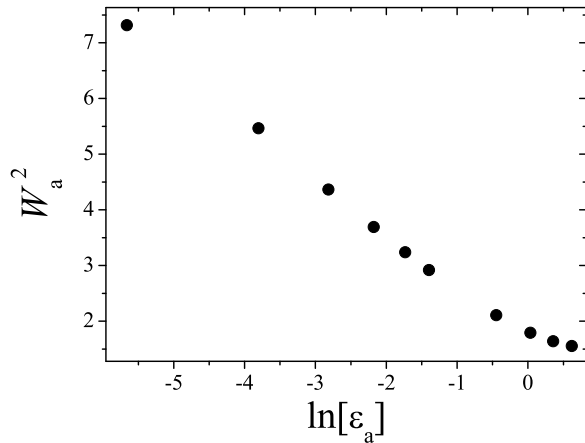


Figure 1. The squared width is plotted versus amplitude for $D = 0$ and different coupling strengths below the threshold for synchronization, $0.5 \leq \kappa \leq 1.75 < \kappa_c$. Points correspond to the asymptotic stationary values (fixed points) of the pair (ε_a, W_a) for increasing values of the coupling strength, as we get closer to the critical value κ_c from right to left.

words, for a strong enough coupling, $\kappa > \kappa_c$, both systems, reality and model synchronize up to any given precision Δ in a finite time t_Δ , so that $|u_n(t) - v_n(t)| < \Delta$ for times $t > t_\Delta$, no matter their respective initial conditions. Therefore the predictive power of our computer model is only limited by a forecasting horizon time $\tau \sim \lambda^{-1}$, where λ is the main Lyapunov exponent of the uncoupled system. Analyzing the idealized perfect model limit will help us to understand the more general situation with noise $D \neq 0$ discussed in next section. A quite exhaustive study of this synchronization transition in terms of the MVL diagram can be found in the work of Szendro *et al.* [2009] for a different model system.

[23] Here we focus on the aspects that are relevant for assimilation. We have computed the error \mathbf{w} between observation and model for different choices of the coupling κ below the synchronization threshold. We present numerical simulations for a system of size $L = 256$ and the averages are taken over 10^3 independent initial conditions. For any coupling the PSD of the associated error surface $h_n(t) = \ln |w_n(t)|$ scales according to equation (3) with $S_q \sim q^{-2}$ for length scales $q < q_c \sim 1/\ell_\times(t)$, as corresponds to a KPZ surface (see discussion in section 3). In the stationary state, the characteristic wave number is expected to tend to zero, $q_c \sim |\kappa - \kappa_c|^\nu$, as we approach the synchronization transition, $\kappa \rightarrow \kappa_c^-$. The exponent $\nu \approx 0.85$ is universal (i.e., fairly model independent), as expected for systems near a phase transition [Ahlers and Pikovsky, 2002; Bagnoli and Rechtman, 2006; Szendro *et al.*, 2009].

[24] At long times the synchronization error becomes stationary and both the amplitude and the surface width reach a fixed point. This fixed point is fully characterized by the asymptotic values $W_a^2(\kappa) = \lim_{t \rightarrow \infty} W(t)^2$ and $\varepsilon_a(\kappa) = \lim_{t \rightarrow \infty} \varepsilon(t)$, which describe the asymptotic state of the synchronization error for a given coupling strength κ [Szendro *et al.*, 2009].

[25] We now focus on this asymptotic state in the perfect model case. For couplings below the threshold for complete

synchronization, $\kappa < \kappa_c$, the error $w_n(t)$ between reality and model tends toward an asymptotic state of partial synchronization. This means that the two systems are synchronizing at length scales below the characteristic length $l_c \sim q_c^{-1}$, where $l_c = \lim_{t \rightarrow \infty} \ell_\times(t)$. It is interesting to notice that the characteristic length can be identified with the typical size of regions on which the synchronization error has the same structure as the first Lyapunov vector [Szendro *et al.*, 2009], i.e., on these regions the synchronization error behaves like an infinitesimal perturbation to the trajectory of the reality. For the purpose of assimilation the typical extent of synchronizing regions (and therefore the size of the regions in which the synchronization error behaves as an infinitesimal perturbation) is important because only below this length scale the structure of the solution of the model can be expected to be compatible with the dynamics of the reality.

[26] In Figure 1 we plot the squared width W_a^2 versus the amplitude ε_a in the asymptotic partially synchronized state for different choices of the coupling strength. Here just as reported by Szendro *et al.* [2009] for a different system, we observe that the closer to the critical coupling strength, the smaller the asymptotic value of the amplitude and the larger the asymptotic value of the width. This behavior actually reflects the approach to the complete synchronization state as one gets closer to κ_c . The increase of the width follows from equation (4) and it is due to the growth of the correlation length $l_c \sim |\kappa - \kappa_c|^{-\nu}$ when the coupling approaches the critical coupling κ_c . Therefore the coupling parameter κ determines the degree up to which the computer model v_n converges to the observation u_n . For couplings above κ_c the system is fully synchronized with an error $|\mathbf{w}| \rightarrow 0$.

5. Non-Perfect Model Case $D \neq 0$ and the Assimilation Problem

[27] We now study the case in which our model is not a perfect copy of the reality. As discussed above, within our approach this corresponds to having a noise $D \neq 0$ in equation (1). Since in this case the reality and the computer model are actually different systems, synchronization is always partial no matter how strong the coupling is. We want now to quantify up to what extent observations and model are synchronized because this gives the measure of the quality of the assimilation.

[28] We first compute the synchronization error \mathbf{w} in the asymptotic state for different coupling strengths and noise intensities. As we did in the preceding section for the perfect model case, we study the time evolution of the synchronization error in the MVL diagram. The pair of variables $[\varepsilon(t), W^2(t)]$ also converges with probability one to an asymptotic fixed point $[\varepsilon_a(\kappa, D), W_a^2(\kappa, D)]$ that depends on both noise intensity and coupling strength. In the following we focus on analyzing the properties of these asymptotic states of partial synchronization as the parameters D and κ are varied.

[29] The effects of assimilating the reality by our synchronization scheme can be readily seen in Figure 2, where we plot the asymptotic values of the error surface width versus amplitude for different values of the noise intensity D and various choices of the coupling parameter within the range $\kappa \in [0.5, 3.0]$. The data on this plot were averaged

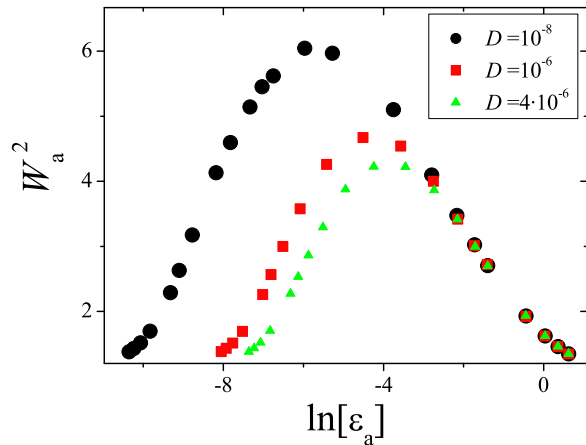


Figure 2. The squared width at the asymptotic state is plotted against the amplitude for various choices of the noise and coupling strengths, $0.5 \leq \kappa \leq 3$. As the coupling is increased, the amplitude still decreases monotonically but there appears a maximum for the asymptotic width, which depends on the coupling strength. For larger noise strength, all asymptotic values are shifted to larger values of ε and smaller values of W^2 .

over 10^3 independent trajectories for good quality statistics. Note that the shape of the curves in Figure 2 is similar for different noise intensities, so that the properties of the error surface in the asymptotic limit are qualitatively independent of the noise strength.

[30] Figure 2 shows important differences as compared with the equivalent plot for the noiseless case in Figure 1. We observe that the asymptotic error amplitude $\varepsilon_a(\kappa, D)$ decreases when the coupling strength increases, as expected. However, for any noise strength, the error surface width $W_a^2(\kappa, D)$ exhibits a maximum. This is a remarkable finding because it immediately implies the existence of an “optimal coupling” for which synchronization of observation and model is the best in a sense we will now enunciate. For any given noise intensity D there is a value for the coupling, $\kappa_{\text{op}}(D)$, at which the error surface width peaks, while for larger values of the coupling parameter the width decreases. Since the surface width is mathematically connected by equation (4) with the correlation extent, a decrease of the width for strong couplings indicates a destruction of spatial correlation, i.e., a reduction of the typical spatial extent of connected regions over which observation and model synchronize. In other words, assimilation (via synchronization) of the observations leads to an asymptotic state that is partially in the synchronizing state up to a length scale l_c , which becomes maximal for certain coupling $\kappa = \kappa_{\text{op}}$. However, further increasing the coupling strength leads to a destruction of correlations and, therefore, a loss of predictive power.

[31] The reduction of the correlation length can also be made apparent by studying the PSD, S_q , of the error surface. In Figures 3a and 3b we plot the PSD of the asymptotic state for a noise strength $D = 10^{-8}$ and various values of the coupling, $\kappa \leq \kappa_{\text{op}}$ (a), and $\kappa \geq \kappa_{\text{op}}$ (b). As can be immediately observed, the PSD shows a power law behavior over wave numbers q above a cutoff $q_c(\kappa)$. For couplings

below $\kappa_{\text{op}}(D = 10^{-8}) \simeq 1.77$, the power law has a slope of -2 , as expected (see discussion in section 3), and the cutoff length $l_c(\kappa) \sim 1/q_c(\kappa)$ increases as the coupling approaches κ_{op} . The spatial extent of synchronized regions is maximal at κ_{op} , where the q^{-2} power law decay of S_q extends over up to very small wave numbers q . However, when the coupling is increased beyond κ_{op} , not only the cutoff length but also the slope of the power law is decreased, demonstrating a progressive destruction of correlations on all length scales. Identical behavior is observed for other noise intensities (not shown) with the PSD showing maximal correlation extent for an optimal coupling $\kappa_{\text{op}}(D)$ that varies with the noise intensity D in the same way as can be seen in Figure 2.

[32] The existence of an optimal coupling can be seen as a fingerprint of the critical phase transition discussed in section 4 for the zero noise case. To show this, in Figure 4 we plot the asymptotic error amplitude and squared width versus the coupling strength for various choices of the noise intensity. Note that, although the amplitude is monotonically decreasing for increasing coupling strength, there is a certain value of the coupling strength $\kappa \approx \kappa_{\text{op}}$ where the curves become steeper. The behavior of the amplitude resembles that of an order parameter at a phase transition. In contrast with a true phase transition, though, the order parameter (error amplitude) does not drop to zero but to a value that depends on the noise strength. As the noise strength becomes smaller the amplitude values after the drop become closer to zero. This hints at the existence of a true phase transition in the limit $D \rightarrow 0$, as expected from the discussion in section 4.

[33] Moreover, the squared width shows a very sharp peak at $\kappa \approx \kappa_{\text{op}}$ (see Figure 4b). This behavior is also highly reminiscent of a critical phase transition, where the correlation length diverges at the critical point. According to equation (4) we have $W_a^2(D, \kappa) \sim l_c(D, \kappa)^{2\alpha}$. Therefore a maximal width at $\kappa \approx \kappa_{\text{op}}(D)$ immediately implies that the largest correlation length occurs at that optimal coupling. Again, a true phase transition is only found in the limit $D \rightarrow 0$, as can be seen in Figure 4b, where the peak becomes higher as the noise intensity is lowered. In the zero noise limit we expect $W_a^2(\kappa_{\text{op}}) \sim L^{2\alpha}$ to diverge with the system size with exponent $\alpha = 1/2$, and $\kappa_{\text{op}}(D \rightarrow 0) = \kappa_c$. Thus as

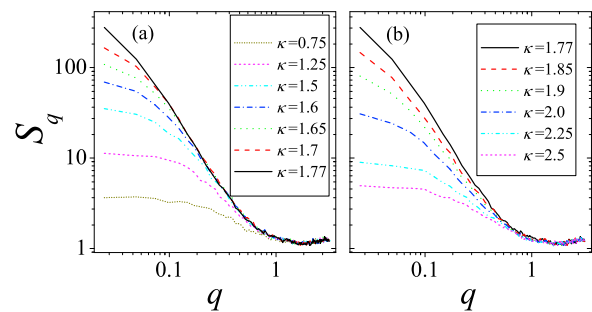


Figure 3. We plot the power spectra of the asymptotic interfaces corresponding to $D = 10^{-8}$ and couplings $\kappa \leq \kappa_m$ and $\kappa \geq \kappa_m$, respectively. (a) For $\kappa \leq \kappa_m$ the correlation length increases as κ is increased. (b) As κ is increased beyond κ_m , correlations are destroyed progressively on all length scales.

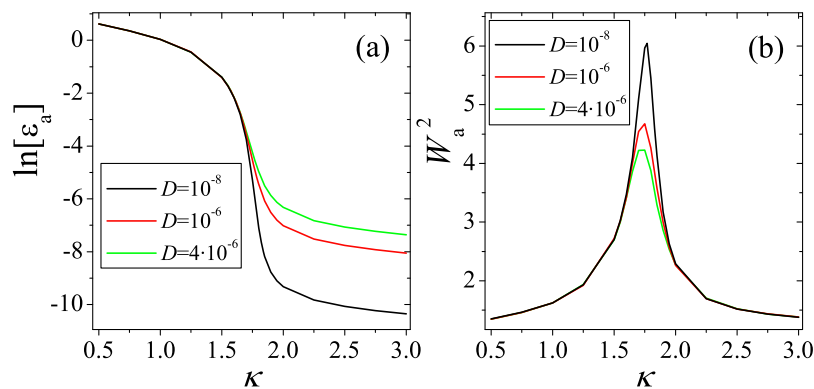


Figure 4. We plot (a) the amplitude and (b) the squared width against the coupling strength for various choices of the noise strength. Note the steep decay of the amplitude and the sharp peak of the squared width at κ_m .

$D \rightarrow 0$ we would have $\kappa_{\text{op}} \rightarrow \kappa_c$ and $l_c(D \rightarrow 0, \kappa_{\text{op}}) \rightarrow L$, where the two systems would become completely synchronized at κ_c . The phase transition is therefore blurred by the presence of the noise. It is worth to mention that κ_{op} is fairly independent of system size for large enough systems, $L > 128$. However, the width at the maximum increases with L , as expected, since in a larger system the correlation length can reach larger distances.

[34] Let us briefly summarize in words the mathematical results we obtained up to now in this section. It becomes apparent that the destruction of the synchronized state for too large couplings in the noisy case can be somehow seen as an *overreaction* of the model with respect to the observations that we want to assimilate. A very large coupling certainly makes the model \mathbf{v} to be more similar to the observation \mathbf{u} on average, as it is reflected by the fact that the error amplitude can be made very small indeed. However, using very large couplings has a drawback – all information encoded in the spatial structure of the error \mathbf{w} is progressively lost, as reflected by the sharp drop of the spatial extent of the correlation length. In other words, the spatial structure of the error progressively deviates from that of the desired synchronized state. Therefore if the coupling is too large, we end up with an error that may be small on average, but also a model that is performing badly as synchronization with observation is concerned. This is somehow similar to the phenomenon of overfitting. Our main finding is that, for any noise intensity, there exists an optimal coupling $\kappa_{\text{op}}(D)$ that provides the best compromise by maximizing spatial extent of the synchronized regions between reality and model.

[35] Note that the optimum state we have found here is essentially different from the one described by *Yang et al.* [2006] for a low-dimensional system. In that case, the noise is not included in the dynamics of the reality but is added subsequently to the signal, mimicking observational noise. Therefore although the solution of *Yang et al.* [2006] converges progressively to the observation as the coupling strength increases, it eventually starts to deviate from the reality for too strong couplings. In our case, on the other hand, reality and observation are identical and the solution converges monotonously to the reality (in terms of the amplitude of the synchronization error) as the coupling strength is increased. Interestingly, despite the different

nature of the optima in the two cases, they both appear at, or nearby, the value of the coupling that corresponds to the critical coupling strength in the noise free case.

[36] At this point an important question is whether the spatial correlations of the synchronized state that appear with a given coupling are able to survive when the model is left to evolve freely, i.e., when using the model to predict the future states of the reality. To check this, we proceed as follows. We couple observation and model as described in equations (1) and (2) until a partial synchronized state is attained (assimilation). Here we assimilated the system during a time interval of length $\tau_{\text{assim}} = 419.4304$. Then, the coupling is switched off and the model and observation are run freely in parallel (prediction). The error between model and reality is then computed as time evolves. We monitored the evolution of the error during an time interval of length $\tau_{\text{free}} = 26.2144$. In order to compare the results for different coupling strengths it is convenient to scale the error amplitudes to some small value before the systems are uncoupled. In Figures 5a–5d we plot the evolution of amplitude and width of the synchronization error \mathbf{w} , whose amplitude is normalized to $\ln \varepsilon_0 = -8$ before κ is set to zero. All figures correspond to a system size $L = 256$ and various choices of the coupling strength and noise strength $D = 10^{-8}$ (Figures 5a and 5b), and $D = 10^{-6}$ (Figures 5c and 5d), respectively. Each of the curves was averaged over 1000 realizations. The amplitude grows as time elapses and, therefore, the temporal evolution can be read of from left to right.

[37] To make the temporal scales more visible we plot the temporal evolution of the squared width versus time in Figure 6. The curves correspond to the same cases as in Figure 5. Moreover, the insets of Figure 6 show the temporal evolution of the amplitude. Note that the amplitude grows exponentially with a growth rate corresponding to the first Lyapunov exponent until nonlinear effects set in causing a saturation of the amplitude. The saturation occurs roughly at the same time as the break down of the width.

[38] At first a decrease of the width is observed. This decrease indicates a slight destruction of spatial correlations due to an incompatibility of the rescaled synchronization error structure with the dynamics of the uncoupled system [see *Primo et al.*, 2007]. As can be seen by comparing the different cases presented here, this effects gets stronger for

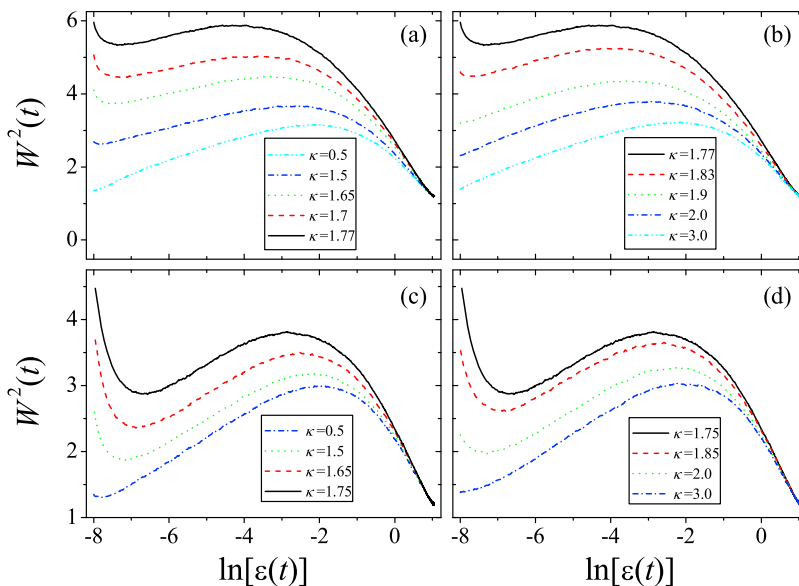


Figure 5. For (a, b) $D = 10^{-8}$ and (c, d) $D = 10^{-6}$, we plot the dynamics of the squared width versus amplitude of the rescaled synchronization error after switching κ to zero. The curves correspond to different coupling strengths, (a, c) below and (b, d) above coupling strength κ_m , respectively. Temporal evolution can be read of from left to right. Note the decline of the width at small times.

larger values of D . The initial decrease is followed by a period of increase of the correlations as the finite perturbation behaves similar to an infinitesimal one. Eventually, as errors grow too large, all correlations are destroyed.

[39] Whenever the noise level is moderate (i.e., for a good enough model) the magnitude of the correlations is still dominated by its magnitude during the assimilation procedure (see Figures 5a and 5b). On the other hand, if the noise is stronger, the correlations that have been gained previously in the assimilation process are greatly lost and the effect of choosing the optimal coupling is diminished (see Figures 5c

and 5d), however, the effect of choosing an optimal coupling can still be seen. Nonetheless, if the noise is too strong (i.e., our model is too bad), one can expect that all the correlations produced during the assimilation procedure are lost and it would not make sense any more to talk about an optimal coupling strength.

6. Conclusions

[40] We have investigated the use of synchronization for the assimilation of observations in a computer model. We

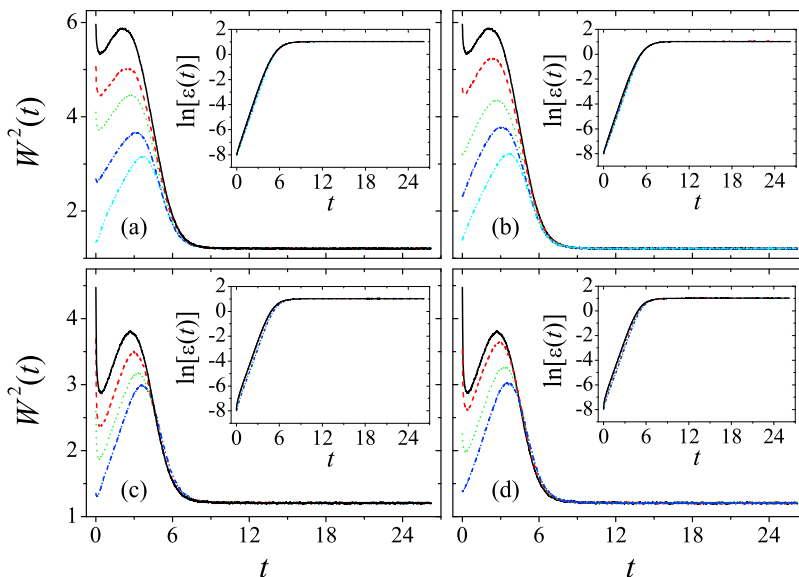


Figure 6. For (a, b) $D = 10^{-8}$ and (c, d) $D = 10^{-6}$, we plot the temporal evolution of the squared width of the rescaled synchronization error after switching κ to zero. The curves correspond to the same coupling strengths as in Figure 5. In the insets we show the temporal evolution of the corresponding amplitudes.

studied a system that is simple enough to allow us to learn about the basic questions, but is still able to capture the essential ingredients of more complex weather systems, including high dimensional chaos. In our study observations are generated by a noisy Lorenz 96 equation so that the effect of noise on data assimilation performance can be pondered. The computer model is then synchronized to the (noisy) reality in the simplest fashion through a linear term proportional to a coupling parameter κ . Our analysis has revealed that not only the magnitude of the synchronization error is important but also its spatial structure. Indeed, our study shows that the spatial structure is the key element in order to quantify the quality of assimilation by means of synchronization in complex spatially extended systems.

[41] Our main result concerns the existence of an optimal coupling κ_{op} such that the partial synchronization of the model to the observations is largely enhanced. This finding is very important and may have implications for other assimilation techniques apart from the particular application in assimilation by means of synchronization. Our results strongly suggest that the optimal solution of the assimilation process may not necessarily be the one that produces the smallest error. Rather, the extent and magnitude of spatial correlations are the quantities that determine the quality of the assimilation in spatially extended systems.

[42] For this optimal coupling the spatial extent of connected synchronized regions becomes maximal and the predictive power of the model is thus the best we can achieve with the synchronization setting in the presence of a noisy reality. The effect of using couplings larger than the optimal one is twofold. On the one hand, the error magnitude is lower, as the spatial average error is concerned. At first sight, this could be naively interpreted as if our model becomes closer to the observation. However, our exhaustive analysis clearly shows that this conclusion is incorrect. The reason being that for couplings above the optimal value there is a second effect – a progressive destruction of spatial correlation at all scales. This phenomenon reminds of an overfitting of the model to the reality and indicates an incorrect spatial structure of the error for couplings above κ_{op} .

[43] Our results allow us to suggest that the correlation length l_c (conversely W_a^2) may be used as a measure of the degree of synchronization or, equivalently, it can be used as a measure of the quality of data assimilation by means of synchronization schemes. Note that the error surface width W_a^2 measures deviations after a logarithmic transformation. Thus W_a^2 is a kind of entropic estimator that is quite robust to describe statistical properties. Furthermore, we have related the appearance of a maximal correlation length to the critical phase transition existing in the limit of a perfect model (noiseless observations).

[44] Much work remains to be done as major problems, which may appear in actual weather models, have not been addressed in this paper. It is possible that the effects of parameter mismatches, observation noise, or incomplete information about the system (i.e., coupling restricted to discrete times or a fraction of the spatial points) could hamper the effects we have observed. Furthermore, in realistic applications, such as weather prediction, the models are generally inhomogeneous. It is known that quenched inhomogeneities lead to a strong localization of errors

[Szendro et al., 2008] and similar effects might be expected in systems where long-lived spatially correlated structures appear due to the intrinsic dynamics of the system. The effect of strongly localized errors on the results presented here is, however, difficult to anticipate at this stage. No doubt, all these questions need further investigation.

[45] Although beyond the scope of this article, it is very likely that similar phenomena also appear in commonly used variational approaches. Therefore it becomes apparent that it would be of great interest to investigate the enhancement of predictive power for variational assimilation schemes when one tries to maximize the spatial correlation, instead of solely minimizing the size of the error.

[46] Finally, we should point out that studies comparing the performance of synchronization methods with the most common variational approaches will be needed. An obvious advantage of synchronization with respect to variational methods is that the former is far less expensive in computational terms. However, it will be the performance with respect to predictive power what should eventually be the most important factor in such an evaluation.

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References

- Ahlers, V., and A. Pikovsky (2002), Critical properties of the synchronization transition in space-time chaos, *Phys. Rev. Lett.*, *88*, 254,101.
- Bagnoli, F., and R. Rechtman (2006), Synchronisation universality classes and stability of smooth coupled map lattices, *Phys. Rev. E*, *73*, 026202.
- Barabási, A.-L., and H. E. Stanley (1995), *Fractal Concepts in Surface Growth*, 386 pp., Cambridge Univ. Press, Cambridge, U. K.
- Boccaletti, S., J. Kurths, G. Osipov, D. L. Valladares, and C. S. Zhou (2002), The synchronisation of chaotic systems, *Phys. Rep.*, *366*, 1–101.
- Cohen, A. B., B. Ravoori, T. E. Murphy, and R. Roy (2008), Using synchronisation for prediction of high-dimensional chaotic dynamics, *Phys. Rev. Lett.*, *101*, 154,102.
- Duane, G. S., and J. J. Tribbia (2007), Dynamical synchronisation of truth and model as an approach to data assimilation, parameter estimation, and model learning, in *Nonlinear Dynamics in the Geosciences*, edited by A. Tsonis, pp. 291–310, Springer, New York.
- Duane, G. S., J. J. Tribbia, and J. B. Weiss (2006), Synchronicity in predictive modelling: A new view of data assimilation, *Nonlinear Processes Geophys.*, *13*, 601–612.
- Fernández, J., C. Primo, A. Cofiño, J. Gutiérrez, and M. Rodríguez (2008), MVL spatiotemporal analysis for model intercomparison in eps: Application to the demeter multi-model ensemble, *Clim. Dyn.*, *33*, 2-3, doi:10.1007/s00382-008-0456-9.
- Gutiérrez, J. M., C. Primo, M. A. Rodríguez, and J. Fernández (2008), Spatiotemporal characterization of ensemble prediction systems: The mean-variance of logarithms (MVL) diagram, *Nonlinear Processes Geophys.*, *15*, 109–114.
- Kalnay, E. (2002), *Atmospheric Modeling, Data Assimilation and Predictability*, 1st ed., 364 pp., Cambridge Univ. Press, Cambridge, U. K.
- Kardar, M., G. Parisi, and Y.-C. Zhang (1986), Dynamic scaling of growing interfaces, *Phys. Rev. Lett.*, *56*, 889.
- López, J. M., C. Primo, M. A. Rodríguez, and I. G. Szendro (2004), Scaling properties of growing noninfinitesimal perturbations in space-time chaos, *Phys. Rev. E*, *70*, 056224.
- Lorenz, E. (1996), Predictability: A problem partially solved, in *Proceedings of a Seminar held at ECMWF on Predictability*, vol. 1, edited by European Centre for Medium-Range Weather Forecasts, pp. 1–18, Reading, U. K.
- Parlitz, U., L. Junge, and L. Kocarev (1996), Synchronization-based parameter estimation from time series, *Phys. Rev. E*, *54*, 6253.
- Pazó, D., I. G. Szendro, J. M. López, and M. A. Rodríguez (2008), Structure of characteristic Lyapunov vectors in spatiotemporal chaos, *Phys. Rev. E*, *78*, 016209.
- Pazó, D., J. M. López, and M. A. Rodríguez (2009), Exponential localization of singular vectors in spatiotemporal chaos, *Phys. Rev. E*, *79*, 036202.

- Pikovsky, A., and A. Politi (1998), Dynamic localization of Lyapunov vectors in spacetime chaos, *Nonlinearity*, *11*(4), 1049–1062.
- Pikovsky, A., M. Rosenblum, and J. Kurths (2003), *Synchronisation: A Universal Concept in Nonlinear Sciences*, 1st ed., 432 pp., Cambridge Univ. Press, Cambridge, U. K.
- Pikovsky, A. S., and J. Kurths (1994), Roughening interfaces in the dynamics of perturbations of spatiotemporal chaos, *Phys. Rev. E*, *49*, 898.
- Primo, C., M. A. Rodríguez, J. M. López, and I. Szendro (2005), Predictability, bred vectors, and generation of ensembles in space-time chaotic systems, *Phys. Rev. E*, *72*, 015201.
- Primo, C., I. G. Szendro, M. A. Rodríguez, and J. M. López (2006), Dynamic scaling of bred vectors in spatially extended chaotic systems, *Europhys. Lett.*, *76*(5), 767–773.
- Primo, C., I. G. Szendro, M. A. Rodríguez, and J. M. Gutierrez (2007), Error growth patterns in systems with spatial chaos: From coupled map lattices to global weather models, *Phys. Rev. Lett.*, *98*, 108,501.
- Szendro, I. G., D. Pazó, M. A. Rodríguez, and J. M. López (2007), Spatiotemporal structure of Lyapunov vectors in chaotic coupled-map lattices, *Phys. Rev. E*, *76*, 025202.
- Szendro, I. G., J. M. López, and M. A. Rodríguez (2008), Dynamics of perturbations in disordered chaotic systems, *Phys. Rev. E*, *78*, 036202.
- Szendro, I. G., M. A. Rodríguez, and J. M. López (2009), Spatial correlations of synchronization errors in extended chaotic systems, *Europhys. Lett.*, *86*(2), 20,008.
- Yang, S.-C., et al. (2006), Data assimilation as synchronisation of truth and model: Experiments with the three-variable Lorenz system, *J. Atmos. Sci.*, *63*, 2340–2354.
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