A SHARP BILINEAR ESTIMATE FOR THE KLEIN–GORDON EQUATION IN $\mathbb{R}^{1+1}$

TOHRU OZAWA AND KEITH M. ROGERS

ABSTRACT. We prove a sharp bilinear estimate for the one dimensional Klein–Gordon equation. The proof involves an unlikely combination of five trigonometric identities. We also prove new estimates for the restriction of the Fourier transform to the hyperbola, where the pullback measure is not assumed to be compactly supported.

1. INTRODUCTION

Sharp constants and maximisers have been calculated for a number of space-time estimates for dispersive equations (see for example [1, 9, 13, 15, 22, 27]). Proving the existence of such maximisers has also received attention (see for example [4, 7, 8, 10, 11, 19, 24, 26]).

Regarding bilinear estimates for the Schrödinger equation, the sharp constant and maximisers were calculated in [23] for the estimate

$$\|(-\Delta)^{\frac{2-d}{4}}(e^{it\Delta}f_1 e^{it\Delta}\overline{f_2})\|_{L^2_{t,x}(\mathbb{R}^{d+1})} \leq C\|f_1\|_2\|f_2\|_2,$$

and, in [6], for the estimate

$$\|e^{it\Delta}f_1 e^{it\Delta}f_2\|_{L^2_{t,x}(\mathbb{R}^{d+1})}^2 \leq C \int_{\mathbb{R}^{2d}} |\widehat{f_1}(\xi_1)|^2 |\widehat{f_2}(\xi_2)|^2 |\xi_1 - \xi_2|^{d-2} d\xi_1 d\xi_2.$$

For the wave equation, the sharp constant and maximisers were calculated in [2] for the estimate

$$\begin{aligned} & \|e^{it\sqrt{-\Delta}}f_1 e^{it\sqrt{-\Delta}}f_2\|_{L^2_{t,x}(\mathbb{R}^{d+1})}^2 \\ & \leq C \int_{\mathbb{R}^{2d}} |\widehat{f_1}(\xi_1)|^2 |\widehat{f_2}(\xi_2)|^2 |\xi_1|^{\frac{d-1}{2}} |\xi_2|^{\frac{d-1}{2}} \left(1 - \frac{\xi_1 \cdot \xi_2}{|\xi_1||\xi_2|}\right)^{\frac{d-3}{2}} d\xi_1 d\xi_2. \end{aligned}$$

The estimates for the Schrödinger equation hold when the dimension $d = 1$, whereas no such estimate can hold for the wave equation in one dimension. This is connected to the fact that the parabola is curved, but the cone in $\mathbb{R}^{1+1}$ is not.

2010 *Mathematics Subject Classification.* Primary 35B45; Secondary 35L05, 42B05, 81Q05.

Key words and phrases. Klein–Gordon equation; sharp constants; restriction of the Fourier transform.

Supported by the JSPS grant 11133 (Japan), the ERC grant 277778 (Europe) and the MICINN grant MTM2010-16518 (Spain).

Here we consider the Klein–Gordon equation, $-\partial_{tt}u + \Delta u = u$ in space-time $\mathbb{R}^{1+1}$. The solution can be written as $u = u_+ + u_-$, where

$$u_+(t) = e^{it\sqrt{1-\Delta}}f_+ \quad \text{and} \quad u_-(t) = e^{-it\sqrt{1-\Delta}}f_-,$$

and

$$u(0) = f_+ + f_- \quad \text{and} \quad \partial_t u(0) = i\sqrt{1-\Delta}(f_+ - f_-).$$

Recall that $e^{\pm it\sqrt{1-\Delta}}$ is defined, initially on Schwartz functions, by

$$e^{\pm it\sqrt{1-\Delta}}f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\xi) e^{\pm it\sqrt{1+\xi^2}} e^{ix\xi} d\xi,$$

where $\widehat{\cdot}$ is the Fourier transform defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi} dx.$$

As in the case of the Schrödinger equation, the separation of the frequency supports is necessary in the following estimate, as otherwise the right hand side is infinite.

Theorem 1.1. *Suppose that $\text{supp}(\widehat{f}_1) \cap \text{supp}(\widehat{f}_2) = \emptyset$. Then*

$$\begin{aligned} & \|e^{it\sqrt{1-\Delta}}f_1 e^{it\sqrt{1-\Delta}}f_2\|_{L_{t,x}^2(\mathbb{R}^{1+1})}^2 \\ & \leq \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} |\widehat{f}_1(\xi_1)|^2 |\widehat{f}_2(\xi_2)|^2 (1+\xi_1^2)^{3/4} (1+\xi_2^2)^{3/4} \frac{d\xi_1 d\xi_2}{|\xi_2 - \xi_1|}. \end{aligned}$$

The constant is sharp, but attained only if $f_1 = 0$ or $f_2 = 0$.

The first estimates of this type were proven by Klainerman and Machedon [16, 17, 18] for the wave equation (see also [3], [14], [20], [21], [31], [32], [33] for $L_{x,t}^p$ estimates).

2. PROOF OF THEOREM 1.1

Amusingly, the proof relies on five trigonometric identities which we collect for the convenience of the reader:

(Pythagorean)	$\sin^2 \theta = 1 - \cos^2 \theta$
(Difference)	$\sin(\phi - \theta) = \sin \phi \cos \theta - \cos \phi \sin \theta$
(Double-angle)	$\sin 2\theta = 2 \sin \theta \cos \theta$
(Product-to-sum)	$2 \sin \theta \sin \phi = \cos(\theta - \phi) - \cos(\theta + \phi)$
(Power-reduction)	$2 \sin^2 \theta = 1 - \cos 2\theta$

We have that $e^{it\sqrt{1-\Delta}}f_1(x) e^{it\sqrt{1-\Delta}}f_2(x)$ is equal to

$$\begin{aligned} & \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2) e^{ix(\xi_1+\xi_2)+it(\sqrt{1+\xi_1^2}+\sqrt{1+\xi_2^2})} d\xi_1 d\xi_2 \\ & = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} F(\xi_1, \xi_2) e^{ix(\xi_1+\xi_2)+it(\sqrt{1+\xi_1^2}+\sqrt{1+\xi_2^2})} d\xi_1 d\xi_2, \end{aligned}$$

where $F(\xi_1, \xi_2) = \frac{1}{2}(\widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2) + \widehat{f_1}(\xi_2)\widehat{f_2}(\xi_1))$. As the integrand is now symmetric, we can write

$$\begin{aligned} e^{it\sqrt{1-\Delta}}f_1(x) e^{it\sqrt{1-\Delta}}f_2(x) \\ = \frac{2}{(2\pi)^2} \int_{\xi_2 \geq \xi_1} F(\xi_1, \xi_2) e^{ix(\xi_1+\xi_2)+it(\sqrt{1+\xi_1^2}+\sqrt{1+\xi_2^2})} d\xi_1 d\xi_2, \end{aligned}$$

By the one-to-one change of variables

$$\eta_1 = \xi_1 + \xi_2 \quad \text{and} \quad \eta_2 = \sqrt{1+\xi_1^2} + \sqrt{1+\xi_2^2}$$

we obtain

$$(1) \quad e^{it\sqrt{1-\Delta}}f_1(x) e^{it\sqrt{1-\Delta}}f_2(x) = \frac{2}{(2\pi)^2} \int_{\xi_2 \geq \xi_1} F(\xi_1, \xi_2) e^{ix\eta_1+it\eta_2} \frac{d\eta_1 d\eta_2}{|J(\xi_1, \xi_2)|},$$

where the Jacobian J is given by

$$J(\xi_1, \xi_2) = \frac{\xi_2}{\sqrt{1+\xi_2^2}} - \frac{\xi_1}{\sqrt{1+\xi_1^2}}.$$

Thus, by Plancherel's theorem and reversing the change of variables,

$$\begin{aligned} \|e^{it\sqrt{1-\Delta}}f_1 e^{it\sqrt{1-\Delta}}f_2\|_{L^2_{t,x}(\mathbb{R}^{1+1})}^2 &= \frac{4}{(2\pi)^2} \int_{\xi_2 \geq \xi_1} |F(\xi_1, \xi_2)|^2 \frac{d\eta_1 d\eta_2}{|J(\xi_1, \xi_2)|^2} \\ (2) \quad &= \frac{4}{(2\pi)^2} \int_{\xi_2 \geq \xi_1} |F(\xi_1, \xi_2)|^2 \frac{d\xi_1 d\xi_2}{|J(\xi_1, \xi_2)|}. \end{aligned}$$

We will require the sharp estimate

$$(3) \quad \frac{\xi_2 - \xi_1}{(1+\xi_2^2)^{3/4}(1+\xi_1^2)^{3/4}} \leq \frac{\xi_2}{\sqrt{1+\xi_2^2}} - \frac{\xi_1}{\sqrt{1+\xi_1^2}}, \quad \xi_2 \geq \xi_1.$$

An unsharp version of this estimate, for large ξ_1, ξ_2 , was obtained by Segal [25]. In order to see this, we note that by (Product-to-sum),

$$2 \sin \theta \sin \phi \leq 1 - \cos(\theta + \phi),$$

so that by (Power-reduction),

$$\sqrt{\sin \theta \sin \phi} \leq \sin\left(\frac{\phi + \theta}{2}\right).$$

Now if $0 \leq \theta \leq \phi \leq \pi$, then both sides are nonnegative. Using the fact that $\cos(\frac{\phi-\theta}{2}) \leq 1$ and multiplying both sides by $2 \sin(\frac{\phi-\theta}{2})$, we obtain

$$2 \sin\left(\frac{\phi-\theta}{2}\right) \cos\left(\frac{\phi-\theta}{2}\right) \sqrt{\sin \theta \sin \phi} \leq 2 \sin\left(\frac{\phi-\theta}{2}\right) \sin\left(\frac{\phi+\theta}{2}\right).$$

Using (Double-angle) to change the left hand side and (Product-to-sum) to change the right, this yields the trigonometric inequality

$$(4) \quad \sin(\phi - \theta) \sqrt{\sin \theta \sin \phi} \leq \cos \theta - \cos \phi, \quad 0 \leq \theta \leq \phi \leq \pi.$$

Now using (Difference), this is equivalent to

$$(\sin \theta)^{1/2} (\sin \phi)^{3/2} \cos \theta - (\sin \theta)^{3/2} (\sin \phi)^{1/2} \cos \phi \leq \cos \theta - \cos \phi,$$

and finally by (Pythagorean), this is equivalent to

$$(1 - \cos^2 \theta)^{1/4} (1 - \cos^2 \phi)^{3/4} \cos \theta - (1 - \cos^2 \theta)^{3/4} (1 - \cos^2 \phi)^{1/4} \cos \phi \leq \cos \theta - \cos \phi.$$

Finally, by making the substitution

$$(5) \quad \cos \phi = \frac{\xi_1}{\sqrt{1 + \xi_1^2}} \quad \text{and} \quad \cos \theta = \frac{\xi_2}{\sqrt{1 + \xi_2^2}},$$

and noting that

$$1 - \cos^2 \phi = \frac{1}{1 + \xi_1^2} \quad \text{and} \quad 1 - \cos^2 \theta = \frac{1}{1 + \xi_2^2},$$

we obtain (3).

Inserting (3) into (2), we see that

$$\begin{aligned} \|e^{it\sqrt{1-\Delta}} f_1 e^{it\sqrt{1-\Delta}} f_2\|_2^2 &\leq \frac{4}{(2\pi)^2} \int_{\xi_2 \geq \xi_1} |F(\xi_1, \xi_2)|^2 (1 + \xi_1^2)^{3/4} (1 + \xi_2^2)^{3/4} \frac{d\xi_1 d\xi_2}{|\xi_2 - \xi_1|} \\ &= \frac{2}{(2\pi)^2} \int_{\mathbb{R}^2} |F(\xi_1, \xi_2)|^2 (1 + \xi_1^2)^{3/4} (1 + \xi_2^2)^{3/4} \frac{d\xi_1 d\xi_2}{|\xi_2 - \xi_1|}, \end{aligned}$$

where we have again used the fact that the integrand is symmetric. Given that f_1 and f_2 have disjoint frequency supports,

$$|F(\xi_1, \xi_2)|^2 = \frac{1}{4} \left(|\widehat{f_1}(\xi_1)|^2 |\widehat{f_2}(\xi_2)|^2 + |\widehat{f_1}(\xi_2)|^2 |\widehat{f_2}(\xi_1)|^2 \right),$$

so that $\|e^{it\sqrt{1-\Delta}} f_1 e^{it\sqrt{1-\Delta}} f_2\|_2^2$ is less than or equal to

$$\frac{1}{2(2\pi)^2} \int_{\mathbb{R}^2} \left(|\widehat{f_1}(\xi_1)|^2 |\widehat{f_2}(\xi_2)|^2 + |\widehat{f_1}(\xi_2)|^2 |\widehat{f_2}(\xi_1)|^2 \right) (1 + \xi_1^2)^{3/4} (1 + \xi_2^2)^{3/4} \frac{d\xi_1 d\xi_2}{|\xi_2 - \xi_1|},$$

which yields the desired inequality.

To see that this is sharp, we first show that (3) is sharp. Note that

$$(6) \quad \frac{\xi_2}{\sqrt{1 + \xi_2^2}} - \frac{\xi_1}{\sqrt{1 + \xi_1^2}} = \int_{\xi_1}^{\xi_2} \frac{d\omega}{(1 + \omega^2)^{3/2}} \leq \frac{\xi_2 - \xi_1}{(1 + \xi_1^2)^{3/2}}.$$

Thus, if we could improve on (3), and it were true that

$$\frac{\xi_2 - \xi_1}{(1 + \xi_2^2)^{3/4} (1 + \xi_1^2)^{3/4}} \leq c \left(\frac{\xi_2}{\sqrt{1 + \xi_2^2}} - \frac{\xi_1}{\sqrt{1 + \xi_1^2}} \right), \quad \xi_2 \geq \xi_1,$$

with $c < 1$, then by combining with (6), we would have

$$\frac{1}{(1 + \xi_2^2)^{3/4}} \leq c \frac{1}{(1 + \xi_1^2)^{3/4}}, \quad \xi_2 \geq \xi_1,$$

which is manifestly false. Thus (3), and hence our bilinear estimate, is sharp.

On the other hand, to see that there are no nontrivial maximisers, it will suffice to see that equality in (3) is only attained in trivial circumstances. Now equality in (3) is attained if and only if the equality in our trigonometric inequality (4) is

attained. This happens if and only if $\cos(\frac{\phi-\theta}{2}) = 1$, which happens if and only if $\phi = \theta$. By (5), this happens if and only if

$$\frac{\xi_1}{\sqrt{1+\xi_1^2}} = \frac{\xi_2}{\sqrt{1+\xi_2^2}},$$

which happens if and only if $\xi_1 = \xi_2$. Thus, there are no nontrivial maximisers, and we are done.

3. RESTRICTION OF THE FOURIER TRANSFORM

We require the following estimate which we will again see is equivalent to a trigonometric inequality. In the previous section we proved this estimate with $\alpha = 1$ and $C_\alpha = 1$.

Lemma 3.1. *There is a constant C_α such that*

$$\frac{|\xi_2 - \xi_1|^\alpha}{(1 + \xi_1^2)^{1/2+\alpha/4}(1 + \xi_2^2)^{1/2+\alpha/4}} \leq C_\alpha \left| \frac{\xi_2}{(1 + \xi_2^2)^{1/2}} - \frac{\xi_1}{(1 + \xi_1^2)^{1/2}} \right|$$

if and only if $1 \leq \alpha \leq 2$.

Proof. By the substitution (5) and (Pythagorean), this is equivalent to

$$\begin{aligned} & \left((\sin \theta)^{1/\alpha-1/2} (\sin \phi)^{1/\alpha+1/2} \cos \theta - (\sin \theta)^{1/\alpha+1/2} (\sin \phi)^{1/\alpha-1/2} \cos \phi \right)^\alpha \\ & \leq C_\alpha (\cos \theta - \cos \phi), \end{aligned}$$

where $0 \leq \theta \leq \phi \leq \pi$. By (Difference), this is equivalent to

$$((\sin \theta \sin \phi)^{1/\alpha-1/2} \sin(\phi - \theta))^\alpha \leq C_\alpha (\cos \theta - \cos \phi).$$

By (Double-angle) and (Product-to-sum), this is the same as

$$(7) \quad (\sin \theta \sin \phi)^{1-\frac{\alpha}{2}} \left(2 \sin \left(\frac{\phi - \theta}{2} \right) \cos \left(\frac{\phi - \theta}{2} \right) \right)^\alpha \leq C_\alpha 2 \sin \left(\frac{\phi - \theta}{2} \right) \sin \left(\frac{\phi + \theta}{2} \right).$$

When $1 \leq \alpha \leq 2$, combining (Product-to-sum) and (Power-reduction) as before, it suffices to show that

$$\left(\sin \left(\frac{\phi + \theta}{2} \right) \right)^{1-\alpha} \left(2 \sin \left(\frac{\phi - \theta}{2} \right) \right)^{\alpha-1} \left(\cos \left(\frac{\phi - \theta}{2} \right) \right)^\alpha \leq C_\alpha.$$

As $\sin \left(\frac{\phi - \theta}{2} \right) \leq \sin \left(\frac{\phi + \theta}{2} \right)$ and $\cos \vartheta \leq 1$, we have proven the inequality.

To see that the inequality of the statement is not true when $\alpha \notin [1, 2]$, we note that it is equivalent to (7) which, by an application of (Double-angle), is equivalent to

$$(\sin \theta \sin \phi)^{1-\alpha/2} (\sin(\phi - \theta))^{\alpha-1} \cos \left(\frac{\phi - \theta}{2} \right) \leq C_\alpha \sin \left(\frac{\phi + \theta}{2} \right).$$

Using the fact that $\sin \vartheta \approx \vartheta$ when $0 \leq \vartheta \leq \pi/2$ and $\cos \vartheta \approx 1$ when $0 \leq \vartheta \leq \pi/4$, for $0 \leq \phi, \theta \leq \pi/2$, this is in turn equivalent to

$$(\theta \phi)^{1-\alpha/2} (\phi - \theta)^{\alpha-1} \leq C_\alpha (\phi + \theta).$$

Taking θ close to zero, we see that such an inequality is not possible when $\alpha > 2$, and when ϕ and θ are close we see that such an inequality is not possible when $\alpha < 1$. $\square$

The first estimate of the following kind was proven by Stein, who meaningfully restricted the Fourier transform to the sphere. Fefferman, Stein and Zygmund [12, 34] restricted to the circle in the sharp range. The same estimates holds for a compact piece of the parabola. For certain Lebesgue exponents, the estimate for the parabola is scale invariant, and so the whole parabola can be considered.

The following estimate is well-known when the measure is assumed to be compactly supported (see [29, pp. 412]). The following global version was proven by Segal [25] in the case $q = 2$ and $p = 6/5$, and extended by Strichartz [30] to the range $q = 2$ and $1 < p \leq 6/5$. In fact the result of Segal followed directly (by duality) from the earlier work of Sjölin [28] who treated the whole line $3q = p'$ with $1 \leq p < 4/3$ (the authors thank J. Wright for bringing this to their attention). We remark that the power of the weight in the definition of the measure is not predicted by scaling (considering $\sqrt{1 - \Delta} \approx \sqrt{-\Delta}$). However, we will prove that it is sharp in the sense that a weight with less decay would not suffice. It is the precise price one has to pay for the fact that the hyperbola has almost no curvature near to infinity.

Theorem 3.2. *Let $3q \leq p' \leq 4q$ and $\frac{2}{p} + \frac{2}{q} < 3$. Then*

$$\left(\int_{\mathbb{R}} |\widehat{F}(\xi, \sqrt{1 + \xi^2})|^q \frac{d\xi}{\sqrt{1 + \xi^2}} \right)^{1/q} \leq C \|F\|_{L^p(\mathbb{R}^2)}.$$

Proof. By duality, the desired estimate is equivalent to

$$\|e^{it\sqrt{1-\Delta}} f\|_{L_{x,t}^q} \leq C \|(1 + |\cdot|^2)^{\frac{1}{2p'}} \widehat{f}\|_p, \quad 1 < \frac{2}{p} + \frac{2}{q},$$

where $3p' \leq q \leq 4p'$. We follow the well-known argument which can be traced back to the work of Carleson–Sjölin [5] for Bochner–Riesz means. By (1), we have

$$e^{it\sqrt{1-\Delta}} f(x) e^{it\sqrt{1-\Delta}} f(x) = \frac{2}{(2\pi)^2} \int_{\xi_2 \geq \xi_1} \widehat{f}(\xi_1) \widehat{f}(\xi_2) e^{ix\eta_1 + it\eta_2} \frac{d\eta_1 d\eta_2}{|J(\xi_1, \xi_2)|},$$

where the Jacobian J is given by

$$J(\xi_1, \xi_2) = \frac{\xi_2}{\sqrt{1 + \xi_2^2}} - \frac{\xi_1}{\sqrt{1 + \xi_1^2}}.$$

This time we use the Hausdorff–Young inequality, instead of using Plancherel’s theorem, so that

$$\begin{aligned} \|e^{it\sqrt{1-\Delta}} f\|_{L_{x,t}^q}^2 &= \|e^{it\sqrt{1-\Delta}} f e^{it\sqrt{1-\Delta}} f\|_{L_{x,t}^{q/2}} \\ &\leq C \left(\int_{\xi_2 \geq \xi_1} |\widehat{f}(\xi_1) \widehat{f}(\xi_2)|^{(q/2)'} \frac{d\eta_1 d\eta_2}{|J(\xi_1, \xi_2)|^{(q/2)'}} \right)^{\frac{1}{(q/2)'}} \\ &\leq C \left(\int_{\mathbb{R}^2} |\widehat{f}(\xi_1) \widehat{f}(\xi_2)|^{(q/2)'} \frac{d\xi_1 d\xi_2}{|J(\xi_1, \xi_2)|^{(q/2)'-1}} \right)^{\frac{1}{(q/2)'}}. \end{aligned}$$

Using Lemma 3.1, this is bounded by

$$C_\alpha \left(\int_{\mathbb{R}^2} |\widehat{f}(\xi_1) \widehat{f}(\xi_2)|^{(q/2)'} \left(\frac{(1 + \xi_1^2)^{1/2+\alpha/4} (1 + \xi_2^2)^{1/2+\alpha/4}}{|\xi_2 - \xi_1|^\alpha} \right)^{((q/2)'-1)} d\xi_1 d\xi_2 \right)^{\frac{1}{(q/2)'}} ,$$

for all $1 \leq \alpha \leq 2$. The proof is completed by Hölder's inequality followed by an application of the Hardy-Littlewood-Sobolev inequality (see for example [30]). For this we require that $q > 2(1 + \alpha)$ and $\frac{1}{p'} = \frac{2+\alpha}{q}$, which yields the result. $\square$

We remark that a larger range of p and q can be obtained by interpolating between this estimate and that due to Strichartz mentioned previously.

The fact that $3q \leq p'$ is a necessary condition is well-known (see for example [30] or [32] or the references therein). When $3q = p'$ our condition $\frac{2}{p} + \frac{2}{q} < 3$ becomes $q > 4/3$ which is also well-known to be sharp, however we have no reason to believe that the condition is sharp when $3q < p'$. Similarly we have no reason to believe that the condition $p' \leq 4q$ is necessary, however we cannot extend the range via these arguments as Lemma 3.1 is false when $\alpha > 2$.

To see that the power of the weight is sharp, we consider the dual inequality

$$(8) \quad \|e^{it\sqrt{1-\Delta}}f\|_{L^q_{x,t}} \leq C\|(1+|\cdot|^2)^{s/2}\widehat{f}\|_p.$$

No such inequality can hold when $\sqrt{1-\Delta}$ is replaced by $\sqrt{-\Delta}$, because then the waves do not disperse, making the left hand side infinite. For the Klein-Gordon equation, these waves keep together for only a finite amount of time, and it is such a wave which proves that $s \geq 1/p'$ is necessary.

Indeed, consider the datum $\widehat{f}(\xi) = \psi(R^{-1}\xi)$, where ψ is smooth and supported in the unit annulus. Then, by scaling,

$$e^{it\sqrt{1-\Delta}}f(x) = \frac{R}{2\pi} \int_{\mathbb{R}} \psi(\xi) e^{i(Rx\xi + Rt\sqrt{\xi^2 + R^{-2}})} d\xi.$$

Now as $\sqrt{\xi^2 + R^{-2}} = \xi + \frac{1}{2\xi}R^{-2} + O(R^{-4})$, we see that if $|x+t| \leq \frac{1}{8}R^{-1}$ and $|t| \leq \frac{1}{8}R$, then

$$\begin{aligned} |Rx\xi + Rt\sqrt{\xi^2 + R^{-2}}| &= |R(x+t)\xi - Rt\xi + Rt(\xi + \frac{1}{2\xi}R^{-2} + O(R^{-4}))| \\ &\leq |R(x+t)\xi + \frac{t}{2\xi}R^{-1} + O(R^{-2})| \leq 1, \end{aligned}$$

so that the phase is not large enough for the exponential function to oscillate. This yields

$$\|e^{it\sqrt{1-\Delta}}f\|_{L^q_{x,t}} \geq cRR^{1/q}R^{-1/q} = cR.$$

On the other hand, it is easy to see that

$$\|(1+|\cdot|^2)^{s/2}\widehat{f}\|_p \leq CR^{1/p+s},$$

so that a necessary condition for (8) to hold is that $s \geq 1/p'$.

This is a natural continuation of the results obtained in [2] and [23]. The authors would like to thank Neal Bez and Yoshio Tsutsumi for the numerous conversations which have influenced this work.

REFERENCES

- [1] J. Bennett, N. Bez, A. Carbery and D. Hundertmark, *Heat-flow monotonicity of Strichartz norms*, Anal. PDE **2** (2009), 147–158.
- [2] N. Bez and K.M. Rogers, *A sharp Strichartz estimate for the wave equation with data in the energy space*, J. Eur. Math. Soc., to appear.
- [3] J. Bourgain, *Estimates for cone multipliers*, Geometric Aspects of Functional Analysis, Oper. Theory Adv. Appl. **77**, Birkhäuser (1995), 41–60.
- [4] A. Bulut, *Maximizers for the Strichartz inequalities for the wave equation*, Differential and Integral Equations **23** (2010), 1035–1072.
- [5] L. Carleson and P. Sjölin, *Oscillatory integrals and a multiplier problem for the disc*, Studia Math. **44** (1972), 287–299.
- [6] E. Carneiro, *A sharp inequality for the Strichartz norm*, Int. Math. Res. Not. **16** (2009), 3127–3145.
- [7] M. Christ and S. Shao, *Existence of extremals for a Fourier restriction inequality*, Anal. PDE, **5** (2012), 261–312.
- [8] ———, *On the extremizers of an adjoint Fourier restriction inequality*, Adv. Math. **230** (2012), 957–977.
- [9] T. Duyckaerts, F. Merle and S. Roudenko, *Maximizers for the Strichartz norm for small solutions of mass-critical NLS*, Ann. Sc. Norm. Super. Pisa Cl. Sci. **X** (2011), 427–476.
- [10] L. Fanelli, L. Vega and N. Visciglia, *On the existence of maximizers for a family of restriction theorems*, Bull. Lond. Math. Soc. **43** (2011), no. 4, 811–817.
- [11] ———, *Existence of maximizers for Sobolev–Strichartz inequalities*, Adv. Math. **229** (2012), no. 3, 1912–1923.
- [12] C. Fefferman, *Inequalities for strongly singular convolution operators*, Acta Math. **124** (1970), 9–36.
- [13] D. Foschi, *Maximizers for the Strichartz inequality*, J. Eur. Math. Soc. **9** (2007), 739–774.
- [14] D. Foschi and S. Klainerman, *Bilinear space-time estimates for homogeneous wave equations*, Ann. Sci. École Norm. Sup. (4) **33** (2000), 211–274.
- [15] D. Hundertmark and V. Zharnitsky, *On sharp Strichartz inequalities in low dimensions*, Int. Math. Res. Not. (2006), Art. ID 34080, 18 pp.
- [16] S. Klainerman and M. Machedon, *Space-time estimates for null forms and the local existence theorem*, Comm. Pure Appl. Math. **46** (1993), 1221–1268.
- [17] ———, *Remark on Strichartz-type inequalities*. With appendices by J. Bourgain and D. Tataru. Int. Math. Res. Not. **5** (1996), 201–220.
- [18] ———, *On the regularity properties of a model problem related to wave maps*, Duke Math. J. **87** (1997), 553–589.
- [19] M. Kunze, *On the existence of a maximizer for the Strichartz inequality*, Comm. Math. Phys. **243** (2003), 137–162.
- [20] S. Lee, K.M. Rogers, and A. Vargas, *Sharp null form estimates for the wave equation in $\mathbb{R}^{3+1}$* , Int. Math. Res. Not. (2008), Art. ID rnn 096, 18 pp.
- [21] S. Lee and A. Vargas, *Sharp null form estimates for the wave equation*, Amer. J. Math. **130** (2008), 1279–1326.
- [22] T. Ozawa and K.M. Rogers, *Sharp Morawetz estimate*, J. Anal. Math., to appear.
- [23] T. Ozawa and Y. Tsutsumi, *Space-time estimates for null gauge forms and nonlinear Schrödinger equations*, Differential Integral Equations **11** (1998), no. 2, 201–222.
- [24] J. Ramos, *A refinement of the Strichartz inequality for the wave equation with applications*, Adv. Math. **230** (2012), no. 2, 649–698.
- [25] I. Segal, *Space-time decay for solutions of wave equations*, Adv. Math. **22** (1976), no. 3, 305–311.
- [26] S. Shao, *Maximizers for the Strichartz and the Sobolev–Strichartz inequalities for the Schrödinger equation*, Electron. J. Differential Equations (2009), No. 3, 13 pp.
- [27] B. Simon, *Best constants in some operator smoothness estimates*, J. Funct. Anal. **107** (1992), no. 1, 66–71.
- [28] P. Sjölin, *Fourier multipliers and estimates of the Fourier transform of measures carried by smooth curves in $\mathbb{R}^2$* , Studia Math. **107** (1974), 169–182.
- [29] E.M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, 43, Princeton Univ. Press, Princeton, NJ, 1993.

- [30] R.S. Strichartz, *Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations*, Duke Math. J. **44** (1977), 705–714.
- [31] T. Tao, *Endpoint bilinear restriction theorems for the cone, and some sharp null form estimates*, Math. Z. **238** (2001), 215–268.
- [32] ———, *A sharp bilinear restrictions estimate for paraboloids*, Geom. Funct. Anal. **13** (2003), 1359–1384.
- [33] T. Wolff, *A sharp bilinear cone restriction estimate*, Ann. of Math. (2) **153** (2001), 661–698.
- [34] A. Zygmund, *On Fourier coefficients and transforms of functions of two variables*, Studia Math. **50** (1974), 189–201.

TOHRU OZAWA, DEPARTMENT OF APPLIED PHYSICS, WASEDA UNIVERSITY, TOKYO 169-8555, JAPAN.

E-mail address: `txozawa@waseda.jp`

KEITH ROGERS, INSTITUTO DE CIENCIAS MATEMATICAS CSIC-UAM-UC3M-UCM, MADRID 28049, SPAIN

E-mail address: `keith.rogers@icmat.es`