SHARP MORAWETZ ESTIMATES

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ABSTRACT. We prove sharp Morawetz estimates – global in time with a singular weight in the spatial variables – for the linear wave, Klein–Gordon and Schrödinger equations, for which we can characterise the maximisers. We also prove refined inequalities with respect to the angular integrability.

1. INTRODUCTION

There has been a number of recent articles dedicated to sharp constants and maximisers for Strichartz estimates (see for example [2, 3, 6, 11, 14, 17]). Proving the existence of such maximisers has also received attention (see for example [5, 12, 13, 19, 24, 25]).

The purpose of this article is to characterise the maximisers for some other classical estimates. Essentially, all will follow from the same elementary proof, which employs the sharp Hardy–Littlewood–Sobolev inequality on the sphere due to Lieb, and avoids the use of the trace lemma.

We consider inequalities for linear dispersive equations which are global in time and with a singular power weight in the spatial variables. The first estimate of this kind was proven by Morawetz [22] for the nonlinear Klein–Gordon equation.

Our sharp constant $C(d, s)$, which we define once and for all, is given by

$$C(d, s) := \frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{d-1}{2} - s)}{\Gamma(\frac{d-1}{2} + s)} \frac{\Gamma(s)}{\Gamma(s + \frac{1}{2})}.$$

For the wave equation we will prove the following theorem.

Theorem 1.1. *Suppose that $\partial_{tt}u = \Delta u$, on $\mathbb{R}^{d+1}$, with $d \geq 2$, and let $0 < s < \frac{d-1}{2}$. Then*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} |u(x, t)|^2 \frac{dx dt}{|x|^{1+2s}} \leq C(d, s) (\|u(0)\|_{H^s}^2 + \|\partial_t u(0)\|_{H^{s-1}}^2).$$

Equality is attained if and only if $u(0)$ and $\partial_t u(0)$ are radially symmetric.

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Here, $\|f\|_{\dot{H}^s} = \|(-\Delta)^{s/2} f\|_2$ with $(-\Delta)^{s/2} f = (|\cdot|^s \widehat{f})^\vee$, where the Fourier transform $\widehat{\cdot}$ and its inverse $^\vee$ are defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix \cdot \xi} dx \quad \text{and} \quad f^\vee(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(\xi) e^{ix \cdot \xi} d\xi.$$

Taking $s = 1$, this yields the following real variable estimate. Vega and Visciglia [34] (see also [23]) had previously calculated the sharp constant in this case (it being evident from their identity (0.8) with an angular term).

Corollary 1.2. *Suppose that $\partial_{tt}u = \Delta u$, on $\mathbb{R}^{d+1}$, with $d \geq 4$. Then*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} |u(x, t)|^2 \frac{dx dt}{|x|^3} \leq \frac{4}{(d-1)(d-3)} (\|\nabla u(0)\|_2^2 + \|\partial_t u(0)\|_2^2).$$

Equality is attained if and only if $u(0)$ and $\partial_t u(0)$ are radially symmetric.

For the Klein–Gordon equation we obtain the following theorem.

Theorem 1.3. *Suppose that $-\partial_{tt}u + \Delta u = u$, on $\mathbb{R}^{d+1}$, with $d \geq 2$, and let $-1/2 < s < d/2 - 1$. Then*

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}^d} |(1 - \Delta)^{1/4} u(x, t)|^2 \frac{dx dt}{|x|^{2+2s}} \\ \leq C(d, s + \tfrac{1}{2}) (\|u(0)\|_{\dot{H}^s}^2 + \|u(0)\|_{\dot{H}^{s+1}}^2 + \|\partial_t u(0)\|_{\dot{H}^s}^2). \end{aligned}$$

Equality is attained if and only if $u(0)$ and $\partial_t u(0)$ are radially symmetric.

Taking $s = 0$ yields the following corollary.

Corollary 1.4. *Suppose that $-\partial_{tt}u + \Delta u = u$, on $\mathbb{R}^{d+1}$, with $d \geq 3$. Then*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} |(1 - \Delta)^{1/4} u(x, t)|^2 \frac{dx dt}{|x|^2} \leq \frac{\pi}{d-2} (\|u(0)\|_2^2 + \|\nabla u(0)\|_2^2 + \|\partial_t u(0)\|_2^2).$$

Equality is attained if and only if $u(0)$ and $\partial_t u(0)$ are radially symmetric.

For the Schrödinger equation we recover an estimate due to Kato and Yajima [18] (see also [1], [26], [30], [35]). Note that s can be negative which explains why this estimate is also known as a local smoothing estimate (see [8], [27] and [32] for the first estimates of this kind). Watanabe [37] characterised the maximisers (see also [36] and [4]). We provide an alternative proof from which one can also deduce a refinement with respect to the angular integrability. Our constant differs from the one presented by Watanabe due to a minor error in his calculation.

Theorem 1.5. *Suppose that $i\partial_t u + \Delta u = 0$, on $\mathbb{R}^{d+1}$, with $d \geq 2$, and let $-1/2 < s < d/2 - 1$. Then*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} |u(x, t)|^2 \frac{dx dt}{|x|^{2+2s}} \leq C(d, s + \tfrac{1}{2}) \|u(0)\|_{\dot{H}^s}^2.$$

Equality is attained if and only if $u(0)$ is radially symmetric.

Taking $s = 0$ yields the following corollary. Simon [26] had previously calculated the sharp constant in this case (see also [33, Theorem 0.1] for an identity with angular part in the case $s = 1/2$).

Corollary 1.6. *Suppose that $i\partial_t u + \Delta u = 0$, on $\mathbb{R}^{d+1}$, with $d \geq 3$. Then*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} |u(x, t)|^2 \frac{dx dt}{|x|^2} \leq \frac{\pi}{d-2} \|u(0)\|_2^2.$$

Equality is attained if and only if $u(0)$ is radially symmetric.

We remark that in the corollaries, the restriction on the dimension is evident from the value of the sharp constants. Similarly, our constant $C(d, s)$ tends to infinity as s approaches either 0 or $(d-1)/2$.

Finally, we will provide a substitute inequality in lower dimensions, with nonsharp constant. A higher dimensional version of the following estimate is easily obtained by interpolation of Corollary 1.6 with the Strichartz estimates [29]. We remark that the first Morawetz estimate for the Schrödinger equation, due to Lin and Strauss [21], was an $L^4_{t,x}(|\cdot|^{-1})$ estimate, in $\mathbb{R}^{3+1}$, with data in the energy space.

Theorem 1.7. *Let $d = 1, 2$ and $0 \leq \alpha < d$. If $i\partial_t u + \Delta u = 0$, on $\mathbb{R}^{d+1}$, then*

$$\left(\int_{\mathbb{R}} \int_{\mathbb{R}^d} |u(x, t)|^p \frac{dx dt}{|x|^\alpha} \right)^{1/p} \leq C \|u(0)\|_2, \quad p = \frac{2(d+2-\alpha)}{d}.$$

We prove Theorems 1.1 and 1.5, in the following section, and make the necessary adjustments for Theorem 1.3 in Section 3. The proof of Theorem 1.7, in Section 5, goes via the TT^* argument for which we will need weighted dispersive estimates which we prove in Section 4. We also prove a refined version of Theorem 1.7 with respect to the angular integrability in higher dimensions, for which we prove a refined version of Pitt's inequality, also in Section 4. This makes use of the recent angular refinement of the Stein–Weiss weighted inequality for the Riesz potential due to D'Ancona and Luca' [10].

2. PROOF OF THEOREMS 1.1 AND 1.5

It is convenient to consider the equation of Schrödinger type, $i\partial_t u = D^a u$, where $a > 0$, with initial data $u(\cdot, 0) = f$. Here, $D^a u = (-\Delta)^{a/2} u = (|\cdot|^a \widehat{u})^\vee$. When the data is Schwartz, the solution can be written as

$$u(x, t) = e^{-itD^a} f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{f}(\xi) e^{-it|\xi|^a} e^{ix \cdot \xi} d\xi.$$

Via Sobolev embedding on the unit sphere, one could deduce the following estimate (without the sharp constant) from the refined inequalities of Hoshiro [16] and Sugimoto [31] (see also [7]). We remark that there are many maximisers.

Lemma 2.1. *Let $d \geq 2$ and $a > 0$, and suppose that $\frac{1-a}{2} < s < \frac{d-a}{2}$. Then*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} |D^{-s} e^{-itD^a} f(x)|^2 \frac{dx dt}{|x|^{a+2s}} \leq A(d, s, a) \int_{\mathbb{R}_+} \left(\int_{\mathbb{S}^{d-1}} |\widehat{f}(r\omega)|^p d\omega \right)^{2/p} r^{d-1} dr,$$

where $p = \frac{2(d-1)}{d+a-2+2s}$ and the sharp constant $A(d, s, a)$ is given by

$$A(d, s, a) = \frac{2^{1-a-2s} \pi^{1-\frac{a}{2}-s}}{a(2\pi)^d} \left(\frac{\Gamma(d-1)}{\Gamma(\frac{d-1}{2})} \right)^{\frac{a-1+2s}{d-1}} \frac{\Gamma(\frac{a-1}{2} + s)}{\Gamma(\frac{d+a}{2} - 1 + s)} \frac{\Gamma(\frac{d-a}{2} - s)}{\Gamma(\frac{a}{2} + s)}.$$

Equality is attained if and only if f satisfies

$$\widehat{f}(r \cdot) = \frac{h(r)}{(1 - \zeta(r) \cdot \omega)^{\frac{d+a}{2}-1+s}},$$

where $h : \mathbb{R}_+ \rightarrow \mathbb{C}$ and $\zeta : \mathbb{R}_+ \rightarrow \mathbb{B}^d$, the open unit ball in $\mathbb{R}^d$.

Proof. As in [32], by polar coordinates and the change of variables $r^a \rightarrow r$,

$$\begin{aligned} D^{-s} e^{-itD^a} f(x) &= \frac{1}{(2\pi)^d} \int |\xi|^{-s} \widehat{f}(\xi) e^{-it|\xi|^a} e^{i\xi \cdot x} d\xi \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}_+} \int_{\mathbb{S}^{d-1}} r^{d-1-s} \widehat{f}(r\omega) e^{-itr^a} e^{ir\omega \cdot x} d\omega dr \\ &= \frac{1}{a(2\pi)^d} \int_{\mathbb{R}_+} \int_{\mathbb{S}^{d-1}} r^{\frac{d-a-s}{a}} \widehat{f}(r^{1/a}\omega) e^{-itr} e^{ir^{1/a}\omega \cdot x} d\omega dr, \end{aligned}$$

so that, by Plancherel's theorem followed by the reverse change of variables,

$$\begin{aligned} \int_{\mathbb{R}} |D^{-s} e^{-itD^a} f(x)|^2 dt &= \frac{1}{a^2(2\pi)^{2d-1}} \int_{\mathbb{R}_+} \left| \int_{\mathbb{S}^{d-1}} r^{\frac{d-a-s}{a}} \widehat{f}(r^{1/a}\omega) e^{ir^{1/a}\omega \cdot x} d\omega \right|^2 dr \\ (1) \quad &= \frac{1}{a(2\pi)^{2d-1}} \int_{\mathbb{R}_+} \left| \int_{\mathbb{S}^{d-1}} r^{d-\frac{a+1}{2}-s} \widehat{f}(r\omega) e^{ir\omega \cdot x} d\omega \right|^2 dr. \end{aligned}$$

Now, by squaring out the integral, we see that the left hand side of the desired inequality is equal to

$$\frac{1}{a(2\pi)^{2d-1}} \int_{\mathbb{R}^d} \int_{\mathbb{R}_+} \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} r^{2d-a-1-2s} \widehat{f}(r\omega) \overline{\widehat{f}(r\sigma)} e^{ir(\omega-\sigma) \cdot x} d\omega d\sigma dr \frac{dx}{|x|^{a+2s}}.$$

By Fubini's theorem, and using the fact that the Fourier transform of $|\cdot|^{-\alpha-d}$ is a constant multiple of $|\cdot|^{-\alpha}$, this is equal to

$$\frac{1}{a(2\pi)^{2d-1}} \frac{2^\alpha \pi^{\frac{d}{2}} \Gamma(\frac{\alpha}{2})}{\Gamma(\frac{d-\alpha}{2})} \int_{\mathbb{R}_+} \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \frac{\widehat{f}(r\omega) \overline{\widehat{f}(r\sigma)}}{|\omega - \sigma|^\alpha} d\omega d\sigma r^{d-1} dr,$$

where $\alpha = d - a - 2s$. We see that the less obvious restriction on the regularity, that $\frac{1-a}{2} < s$, is in fact necessary.

We now apply the sharp Hardy–Littlewood–Sobolev inequality on the unit sphere due to Lieb [20];

$$\int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \frac{g(\omega) h(\sigma)}{|\omega - \sigma|^\alpha} d\omega d\sigma \leq \pi^{\frac{\alpha}{2}} \left(\frac{\Gamma(d-1)}{\Gamma(\frac{d-1}{2})} \right)^{1-\frac{\alpha}{d-1}} \frac{\Gamma(\frac{d-1-\alpha}{2})}{\Gamma(d-1-\frac{\alpha}{2})} \|g\|_{L^p} \|h\|_{L^p},$$

where $\frac{2}{p} + \frac{\alpha}{d-1} = 2$, which is attained if and only if

$$g(\omega) = \frac{c_1}{(1 - \zeta \cdot \omega)^{d-1-\frac{\alpha}{2}}}, \quad \text{and} \quad h(\omega) = \frac{c_2}{(1 - \zeta \cdot \omega)^{d-1-\frac{\alpha}{2}}},$$

where $|\zeta| < 1$ and $c_1, c_2 \in \mathbb{C}$ (see also [15, Theorem 2.2] for an explicit statement of this result). To apply the inequality we require that $0 < d - a - 2s < d - 1$ which is the restriction on the regularity in the statement of the theorem. We get the desired inequality with

$$\begin{aligned} A(d, s, a) &= \pi^{\frac{a}{2}} \left(\frac{\Gamma(d-1)}{\Gamma(\frac{d-1}{2})} \right)^{1-\frac{a}{d-1}} \frac{\Gamma(\frac{d-1-\alpha}{2})}{\Gamma(d-1-\frac{\alpha}{2})} \frac{1}{a(2\pi)^{2d-1}} \frac{2^\alpha \pi^{\frac{d}{2}} \Gamma(\frac{\alpha}{2})}{\Gamma(\frac{d-\alpha}{2})} \\ &= \frac{2^{1-a-2s} \pi^{1-\frac{a}{2}-s}}{a(2\pi)^d} \left(\frac{\Gamma(d-1)}{\Gamma(\frac{d-1}{2})} \right)^{\frac{a-1+2s}{d-1}} \frac{\Gamma(\frac{a-1}{2}+s)}{\Gamma(\frac{d+a}{2}-1+s)} \frac{\Gamma(\frac{d-a}{2}-s)}{\Gamma(\frac{a}{2}+s)}, \end{aligned}$$

and the maximisers are as described. $\square$

Proof of Theorem 1.5. First we note that by taking $\zeta(r) \equiv 0$, equality is attained for radial functions in Lemma 2.1. Apply Hölder's inequality in the angular integral;

$$(2) \quad \|\widehat{f}(r \cdot)\|_{L^p(\mathbb{S}^{d-1})} \leq |\mathbb{S}^{d-1}|^{\frac{1}{p}-\frac{1}{2}} \|\widehat{f}(r \cdot)\|_{L^2(\mathbb{S}^{d-1})},$$

followed by Plancherel's theorem,

$$(3) \quad \int_{\mathbb{R}} \int_{\mathbb{R}^d} |D^{-s} e^{-itD^a} f(x)|^2 \frac{dx dt}{|x|^{a+2s}} \leq B(d, s, a) \|f\|_2^2,$$

where $B(d, s, a) = (2\pi)^d A(d, s, a) |\mathbb{S}^{d-1}|^{\frac{2}{p}-1}$ and $p = \frac{2(d-1)}{d+a-2+2s}$. Note that

$$\begin{aligned} B(d, s, a) &= \frac{2^{1-a-2s} \pi^{1-\frac{a}{2}-s}}{a} \left(\frac{2\pi^{d/2} \Gamma(d-1)}{\Gamma(\frac{d}{2}) \Gamma(\frac{d-1}{2})} \right)^{\frac{a-1+2s}{d-1}} \frac{\Gamma(\frac{a-1}{2}+s)}{\Gamma(\frac{d+a}{2}-1+s)} \frac{\Gamma(\frac{d-a}{2}-s)}{\Gamma(\frac{a}{2}+s)} \\ &= \frac{\sqrt{\pi}}{a} \frac{\Gamma(\frac{a-1}{2}+s)}{\Gamma(\frac{d+a}{2}-1+s)} \frac{\Gamma(\frac{d-a}{2}-s)}{\Gamma(\frac{a}{2}+s)}, \end{aligned}$$

which, taking $a = 2$, is equal to $C(d, s + 1/2)$. Equality is attained in (2) and thus (3) if and only if $\widehat{f}(r \cdot)$ is constant almost everywhere. That is to say, equality is attained if and only if f is radial. $\square$

Proof of Theorem 1.1. The solution to the wave equation can be written as $u = u_+ + u_-$, where

$$u_+(t) = e^{itD} f_+ \quad \text{and} \quad u_-(t) = e^{-itD} f_-,$$

and

$$u(0) = f_+ + f_- \quad \text{and} \quad \partial_t u(0) = iD(f_+ - f_-).$$

Now the temporal Fourier transforms of u_+ and u_- ,

$$\begin{aligned} \widehat{u}_\pm^t(\tau) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \widehat{f}_\pm(\xi) e^{\pm it|\xi|} e^{ix \cdot \xi} d\xi e^{-i\tau t} dt \\ &= \frac{1}{(2\pi)^{d-1}} \int_{\mathbb{R}^d} \delta(\pm|\xi| - \tau) \widehat{f}_\pm(\xi) e^{\pm it|\xi|} e^{ix \cdot \xi} d\xi, \end{aligned}$$

changes are supported on disjoint (apart from the origin) halflines, so that by Fubini's theorem and Plancherel,

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} |u(x, t)|^2 \frac{dx dt}{|x|^{1+2s}} = \int_{\mathbb{R}} \int_{\mathbb{R}^d} |u_+(x, t)|^2 \frac{dx dt}{|x|^{1+2s}} + \int_{\mathbb{R}} \int_{\mathbb{R}^d} |u_-(x, t)|^2 \frac{dx dt}{|x|^{1+2s}}.$$

Now by symmetry, (3) holds for u_+ as well as u_- , so that

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}^d} |u(x, t)|^2 \frac{dx dt}{|x|^{1+2s}} &\leq B(d, s, 1) (\|f_+\|_{\dot{H}^s}^2 + \|f_-\|_{\dot{H}^s}^2) \\ &= \frac{B(d, s, 1)}{2} (\|u(0)\|_{\dot{H}^s}^2 + \|\partial_t u(0)\|_{\dot{H}^{s-1}}^2), \end{aligned}$$

where the final identity is by the parallelogram law. Again, equality in the inequality is attained if and only if f_+ and f_- are radially symmetric. This occurs if and only if $u(0)$ and $\partial_t u(0)$ are radially symmetric, which completes the proof. $\square$

3. PROOF OF THEOREM 1.3

The proof is very similar to that of the previous section, so we will be brief. First we consider the one sided operator $e^{-it\sqrt{1-\Delta}}$ defined by

$$e^{-it\sqrt{1-\Delta}}f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{f}(\xi) e^{-it\sqrt{1+|\xi|^2}} e^{ix \cdot \xi} d\xi.$$

Lemma 3.1. *Let $d \geq 2$, and suppose that $0 < s < \frac{d-1}{2}$. Then*

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left| \frac{(\sqrt{-\Delta})^{1/2-s}}{(\sqrt{1-\Delta})^{1/2}} e^{-it\sqrt{1-\Delta}}f(x) \right|^2 \frac{dx dt}{|x|^{1+2s}} \\ \leq A(d, s, 1) \int_{\mathbb{R}_+} \left(\int_{\mathbb{S}^{d-1}} |\widehat{f}(r\omega)|^p d\omega \right)^{2/p} r^{d-1} dr, \end{aligned}$$

where $p = \frac{2(d-1)}{d-1+2s}$. Equality is attained if and only if f satisfies

$$\widehat{f}(r \cdot) = \frac{h(r)}{(1 - \zeta(r) \cdot \omega)^{\frac{d-1}{2}+s}},$$

where $h : \mathbb{R}_+ \rightarrow \mathbb{C}$ and $\zeta : \mathbb{R}_+ \rightarrow \{\xi \in \mathbb{R}^d : |\xi| < 1\}$.

Proof. By the change of variables $\sqrt{1+r^2} \rightarrow r$,

$$\begin{aligned} &\frac{(\sqrt{-\Delta})^{1/2-s}}{(\sqrt{1-\Delta})^{1/2}} e^{-it\sqrt{1-\Delta}}f(x) \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}_+} \int_{\mathbb{S}^{d-1}} (1+r^2)^{-1/4} r^{d-1/2-s} \widehat{f}(r\omega) e^{-it\sqrt{1+r^2}} e^{ir\omega \cdot x} d\omega dr \\ &= \frac{1}{(2\pi)^d} \int_1^\infty \int_{\mathbb{S}^{d-1}} r^{1/2} (r^2-1)^{d/2-3/4-s/2} \widehat{f}(\sqrt{r^2-1}\omega) e^{-itr} e^{i\sqrt{r^2-1}\omega \cdot x} d\omega dr, \end{aligned}$$

so that, by Plancherel's theorem followed by the reverse change of variables,

$$\begin{aligned} &\int_{\mathbb{R}} \left| \frac{(\sqrt{-\Delta})^{1/2-s}}{(\sqrt{1-\Delta})^{1/2}} e^{-it\sqrt{1-\Delta}}f(x) \right|^2 dt \\ &= \frac{1}{(2\pi)^{2d-1}} \int_1^\infty \left| \int_{\mathbb{S}^{d-1}} r^{1/2} (r^2-1)^{d/2-3/4-s/2} \widehat{f}(\sqrt{r^2-1}\omega) e^{i\sqrt{r^2-1}\omega \cdot x} d\omega \right|^2 dr \\ &= \frac{1}{(2\pi)^{2d-1}} \int_{\mathbb{R}_+} \left| \int_{\mathbb{S}^{d-1}} r^{d-1-s} \widehat{f}(r\omega) e^{ir\omega \cdot x} d\omega \right|^2 dr. \end{aligned}$$

Thus, we see that

$$(4) \quad \int_{\mathbb{R}} \left| \frac{(\sqrt{-\Delta})^{1/2-s}}{(\sqrt{1-\Delta})^{1/2}} e^{-it\sqrt{1-\Delta}} f(x) \right|^2 dt = \int_{\mathbb{R}} |D^{-s} e^{-itD} f(x)|^2 dt,$$

and so the result follows by Lemma 2.1. $\square$

By the shift in regularity $s \rightarrow s - 1/2$, it will suffice to prove that for $0 < s < \frac{d-1}{2}$,

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}^d} |(1-\Delta)^{1/4} u(x,t)|^2 \frac{dxdt}{|x|^{1+2s}} \\ \leq C(d,s) (\|u(0)\|_{\dot{H}^{s-1/2}}^2 + \|u(0)\|_{\dot{H}^{s+1/2}}^2 + \|\partial_t u(0)\|_{\dot{H}^{s-1/2}}^2). \end{aligned}$$

The solution to the Klein-Gordon equation can be written as $u = u_+ + u_-$, where

$$u_+(t) = e^{it\sqrt{1-\Delta}} f_+ \quad \text{and} \quad u_-(t) = e^{-it\sqrt{1-\Delta}} f_-,$$

and

$$u(0) = f_+ + f_- \quad \text{and} \quad \partial_t u(0) = i\sqrt{1-\Delta} (f_+ - f_-).$$

As before the supports of the temporal Fourier transforms of u_+ and u_- are disjoint, so that by Fubini's theorem and Plancherel,

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left| \left(\frac{\sqrt{-\Delta}}{\sqrt{1-\Delta}} \right)^{1/2} u(x,t) \right|^2 \frac{dxdt}{|x|^{1+2s}} &= \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left| \left(\frac{\sqrt{-\Delta}}{\sqrt{1-\Delta}} \right)^{1/2} u_+(x,t) \right|^2 \frac{dxdt}{|x|^{1+2s}} \\ &+ \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left| \left(\frac{\sqrt{-\Delta}}{\sqrt{1-\Delta}} \right)^{1/2} u_-(x,t) \right|^2 \frac{dxdt}{|x|^{1+2s}}. \end{aligned}$$

By Hölder's inequality, Lemma 3 yields

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} \left| \left(\frac{\sqrt{-\Delta}}{\sqrt{1-\Delta}} \right)^{1/2} e^{\pm it\sqrt{1-\Delta}} f(x) \right|^2 \frac{dxdt}{|x|^{1+2s}} \leq B(d,s,1) \|f\|_{\dot{H}^s}^2,$$

with equality if and only if f is radially symmetric. Together, we see that

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left| \left(\frac{\sqrt{-\Delta}}{\sqrt{1-\Delta}} \right)^{1/2} u(x,t) \right|^2 \frac{dxdt}{|x|^{1+2s}} &\leq A(d,s) (\|f_+\|_{\dot{H}^s}^2 + \|f_-\|_{\dot{H}^s}^2) \\ &= \frac{A(d,s)}{2} (\|u(0)\|_{\dot{H}^s}^2 + \|(\sqrt{1-\Delta})^{-1} \partial_t u(0)\|_{\dot{H}^s}^2), \end{aligned}$$

where the final identity is by the parallelogram law. By considering

$$u(0) \rightarrow \frac{\sqrt{1-\Delta}}{(\sqrt{-\Delta})^{1/2}} u(0), \quad \partial_t u(0) \rightarrow \frac{\sqrt{1-\Delta}}{(\sqrt{-\Delta})^{1/2}} \partial_t u(0)$$

and using the fact that $\|(1-\Delta)^{1/2} u(0)\|_{\dot{H}^{s-1/2}}^2 = \|u(0)\|_{\dot{H}^{s-1/2}}^2 + \|u(0)\|_{\dot{H}^{s+1/2}}^2$, this yields the inequality. We used Lemma 3.1 and so equality is attained if and only if f_+ and f_- are radially symmetric. This occurs if and only if $u(0)$ and $\partial_t u(0)$ are radially symmetric, which completes the proof of Theorem 1.3.

4. PITT'S INEQUALITY AND WEIGHTED DISPERSIVE ESTIMATES

Pitt's inequality states that for $2 \leq p < \infty$,

$$(5) \quad \left\| |\cdot|^{-\gamma} \widehat{f} \right\|_{L_x^p(\mathbb{R}^d)} \leq C \left\| |\cdot|^\gamma f \right\|_{L_x^{p'}(\mathbb{R}^d)}$$

when $0 \leq \gamma < d/p$. We improve this using the recent refinement of the Stein–Weiss weighted inequality for Riesz potentials due to D'Ancona and Luca' [10].

Lemma 4.1. *Let $d \geq 2$ and $2 \leq p \leq \tilde{p} < \infty$, and suppose that*

$$(d-1)\left(\frac{2}{p} - \frac{2}{\tilde{p}}\right) \leq \gamma < \frac{d}{p}.$$

Then

$$\left\| |\cdot|^{-\gamma} \widehat{f} \right\|_{L_r^p L_\omega^{\tilde{p}}} \leq C \left\| |\cdot|^\gamma f \right\|_{L_r^{p'} L_\omega^{q'}}.$$

Proof. By interpolation, it will suffice to prove

$$(6) \quad \left\| |\cdot|^{-\gamma} \widehat{f} \right\|_{L_r^p L_\omega^p} \leq C \left\| |\cdot|^\gamma f \right\|_{L_r^{p'} L_\omega^{q'}}$$

and

$$\left\| |\cdot|^{-\gamma} \widehat{f} \right\|_{L_r^p L_\omega^q} \leq C \left\| |\cdot|^\gamma f \right\|_{L_r^{p'} L_\omega^{q'}},$$

where $\frac{1}{\tilde{p}} = \frac{1}{2q} + \frac{1}{2p}$. Note that by duality, the latter follows from former, so it will suffice to prove (6).

To this end, we note that

$$\left\| |\cdot|^{-\gamma} \widehat{f} \right\|_{L_r^p L_\omega^p} = C \left(\int |\widehat{I_\gamma f}|^p d\xi \right)^{1/p},$$

where I_γ is the Riesz potential defined by $I_\gamma f(x) = \int \frac{f(y)}{|y-x|^{d-\gamma}} dy$, so that, by the Hausdorff–Young inequality,

$$(7) \quad \left\| |\cdot|^{-\gamma} \widehat{f} \right\|_{L_r^p L_\omega^p} \leq C \left(\int |I_\gamma f|^{p'} d\xi \right)^{1/p'}.$$

Now D'Ancona and Luca' [10] refined the Stein–Weiss [28] weighted inequality for the Riesz potential;

$$(8) \quad \left(\int |I_\gamma f|^{p'} d\xi \right)^{1/p'} \leq C \left\| |\cdot|^\gamma f \right\|_{L_r^{p'} L_\omega^{q'}}$$

when $(d-1)(\frac{1}{q'} - \frac{1}{p'}) \leq \gamma < \frac{d}{p}$, so that combining (7) with (8), we obtain (6). $\square$

Proposition 4.2. *Let $2 \leq p < \infty$ and $0 \leq \gamma < \frac{d}{p}$. Then*

$$\|e^{it\Delta} f\|_{L_x^p(|\cdot|^{-p\gamma})} \leq \frac{C}{|t|^{d(\frac{1}{2}-\frac{1}{p})+\gamma}} \|f\|_{L_x^{p'}(|\cdot|^{p'\gamma})}.$$

Moreover, when $d \geq 2$ and $p \leq \tilde{p} < \infty$, if $(d-1)(\frac{2}{p} - \frac{2}{\tilde{p}}) \leq \gamma < \frac{d}{p}$, then

$$\|e^{it\Delta} f\|_{L_r^p L_\omega^{\tilde{p}}(|\cdot|^{-p\gamma})} \leq \frac{C}{|t|^{d(\frac{1}{2}-\frac{1}{p})+\gamma}} \|f\|_{L_r^{p'} L_\omega^{\tilde{p}'}(|\cdot|^{p'\gamma})}.$$

Proof. It is well-known that the Schrödinger operator can be written more directly in terms of the Fourier transform (see for example [9]). Indeed, using the kernel representation

$$e^{it\Delta}f(x) = \frac{1}{(4\pi it)^{d/2}} \int e^{\frac{i|x-y|^2}{4t}} f(y) dy$$

and writing

$$\mathcal{F}[f](\xi) = \widehat{f}(\xi) = \int f(x) e^{-ix \cdot \xi} dx,$$

by squaring and a change of variables, we obtain

$$\begin{aligned} e^{it\Delta}f(x) &= \frac{1}{(4\pi it)^{d/2}} \int e^{\frac{i|x-y|^2}{4t}} f(y) dy \\ &= \frac{(2|t|)^d}{(4\pi it)^{d/2}} e^{\frac{i|x|^2}{4t}} \int e^{-ix \cdot y} e^{it|y|^2} f(2ty) dy \\ &= \frac{(2|t|)^d}{(4\pi it)^{d/2}} e^{\frac{i|x|^2}{4t}} \mathcal{F}[e^{it|\cdot|^2} f(2t \cdot)](x). \end{aligned}$$

Thus, by (5) in the case $p = \tilde{p}$, or by Lemma 4.1 otherwise, we obtain

$$\begin{aligned} \|e^{it\Delta}f\|_{L_r^p L_{\omega}^{\tilde{p}}(|\cdot|^{-p\gamma})} &\leq C|t|^{d/2} \|f(2t \cdot)\|_{L_r^{p'} L_{\omega}^{\tilde{p}'}(|\cdot|^{p'\gamma})} \\ &\leq \frac{C}{|t|^{d(\frac{1}{2} - \frac{1}{p}) + \gamma}} \|f\|_{L_r^{p'} L_{\omega}^{\tilde{p}'}(|\cdot|^{p'\gamma})}, \end{aligned}$$

and we are done. $\square$

5. PROOF OF THEOREM 1.7

In Lemma 2.1, we saw that it is possible to obtain refined inequalities with respect to the angular integrability. The proof of the following refinement, also serves as a proof of Theorem 1.7 in the one dimensional case.

Theorem 5.1. *Let $d \geq 2$ and $2 \leq p \leq \tilde{p} < \infty$. Suppose that*

$$(d-1)\left(\frac{2}{p} - \frac{2}{\tilde{p}}\right) \leq \frac{\alpha}{p} < \frac{d}{p} - \frac{d}{2} + 1, \quad \text{and} \quad \frac{2}{q} = d\left(\frac{1}{2} - \frac{1}{p}\right) + \frac{\alpha}{p}.$$

Then

$$\left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}_+} \left(\int_{\mathbb{S}^{d-1}} |e^{it\Delta}f(r\omega)|^{\tilde{p}} d\omega \right)^{p/\tilde{p}} r^{d-1-\alpha} dr \right)^{q/p} dt \right)^{1/q} \leq C \|f\|_2.$$

Proof. Setting $\gamma = \alpha/p$, we use the TT^* argument (see for example [29]). By duality we are required to prove

$$\left\| \int e^{-it\Delta} F(\cdot, t) dt \right\|_{L_x^2(\mathbb{R}^d)} \leq C \|F\|_{L_t^{q'} L_r^{p'} L_{\omega}^{\tilde{p}'}(|\cdot|^{p'\gamma})},$$

which by squaring is equivalent to

$$\int \int \langle e^{-i\tau\Delta} F(\cdot, \tau), e^{-it\Delta} G(\cdot, t) \rangle_x dt d\tau \leq C \|F\|_{L_t^{q'} L_r^{p'} L_{\omega}^{\tilde{p}'}(|\cdot|^{p'\gamma})} \|G\|_{L_t^{q'} L_r^{p'} L_{\omega}^{\tilde{p}'}(|\cdot|^{p'\gamma})}.$$

The adjoint of $e^{-it\Delta}$ is $e^{it\Delta}$ and so this can be rewritten as

$$\int \left\langle \int e^{i(t-\tau)\Delta} F(\cdot, \tau) d\tau, G(\cdot, t) \right\rangle_x dt \leq C \|F\|_{L_t^{q'} L_r^{p'} L_\omega^{\bar{p}'}(|\cdot|^{p'\gamma})} \|G\|_{L_t^q L_r^{p'} L_\omega^{\bar{p}}(|\cdot|^{p'\gamma})},$$

which would follow, by Hölder's inequality, from

$$\left\| \int e^{i(t-\tau)\Delta} F(\cdot, \tau) d\tau \right\|_{L_t^q L_r^p L_\omega^{\bar{p}}(|\cdot|^{-p\gamma})} \leq C \|F\|_{L_t^{q'} L_r^{p'} L_\omega^{\bar{p}'}(|\cdot|^{p'\gamma})}.$$

This is a consequence of Minkowski's integral inequality, Proposition 4.2, and the one dimensional Hardy–Littlewood–Sobolev inequality;

$$\begin{aligned} \left\| \int e^{i(t-\tau)\Delta} F(\cdot, \tau) d\tau \right\|_{L_t^q L_r^p L_\omega^{\bar{p}}(|\cdot|^{-p\gamma})} &\leq \left\| \int \left\| e^{i(t-\tau)\Delta} F(\cdot, \tau) \right\|_{L_r^p L_\omega^{\bar{p}}(|\cdot|^{-p\gamma})} d\tau \right\|_{L_t^q} \\ &\leq \left\| \int \frac{\|F(\cdot, \tau)\|_{L_r^{p'} L_\omega^{\bar{p}'}(|\cdot|^{p'\gamma})}}{|t-\tau|^{d(\frac{1}{2}-\frac{1}{p})+\gamma}} d\tau \right\|_{L_t^q} \\ &\leq C \|F\|_{L_t^{q'} L_r^{p'} L_\omega^{\bar{p}'}(|\cdot|^{p'\gamma})}. \end{aligned}$$

For the application of Proposition 4.2 we require that $(d-1)(\frac{2}{p}-\frac{2}{\bar{p}}) \leq \gamma < \frac{d}{p}$, and for the application of the Hardy–Littlewood–Sobolev inequality we require that $\gamma < 1 - d(\frac{1}{2}-\frac{1}{p})$, which equate to the announced restrictions on the exponents, and so we are done. $\square$

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