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ALGORITHM FOR CONTROL AND ANTICONTROL OF CHAOS IN CONTINUOUS-TIME DYNAMICAL SYSTEMS

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Abstract. In this paper we present a simple algorithm that allows the control and anticontrol of chaos. Considering two identical chaotic dynamical systems wich evolve for different control parameter values, the value of the first system state variables is modified to gain the corresponding value of the second system in order to bring near the two trajectories. Thus, the behavior of the first system is adapted to that of the second one. Three examples are considered.

Keywords. Chaotic dynamical system, anticontrol, chaos control, Lorenz system, Chen system, Rabinovich-Fabrikant system.

AMS subject classification: 34E99, 37M05, 65P20

1 Introduction

In the last years the stabilization of chaos like the slight perturbations of a system parameter (the most known being the OGY method [9]), or the changes in the system variables in the form of instantaneous pulses (GM algorithm, introduced by Güemez and Matías [6], [8]) have proved to be of a real interest. While the first class of algorithms are useful when we have access to the system parameters without changing the state variables, the second class of methods are useful in the cases when the system parameters are inaccessible, namely in the cases of certain chemical experiments, biological and ecological systems, electrical circuits etc.

The control algorithms using the parameter modification imply generally a supplementary knowledge of some data related to the system as Lyapunov exponents, covariant and contravariant vectors etc. On the other hand the algorithms which change the system variables are easier to implement, but less performing since it is difficult to find a clear link between the necessary variables pulses and certain targeted trajectory.

The anticontrol of chaos (or chaotification) which makes chaotic a non-chaotic dynamical system, or enhances the chaos in chaotic systems, have also attracted increasing attention due to its great potential in many non-traditional applications (see e.g. the Chen & Lai anticontrol algorithms based on time-delay feedback [2], [12]). The anticontrol could be useful in mechanical, electronic, optical, and particularly biological and medical systems.

Both the GM control and Chen & Lai anticontrol modify slightly the systems structure because they change the system variables while the OGY algorithm performs changes in the parameter values only.

GM control, applied to time-continuous systems, performs small changes in the system variables every $\delta t \geq h$ time interval in the form

$$x(t) \leftarrow x(t)(1+\lambda), \tag{1}$$

where x is the state variable, λ is a small positive or negative real number and h is the integration step. Choosing adequately δt and λ one can obtain (but without a rigorous criteria at least to our knowledge) stable trajectories.

The GM algorithm was applied to discrete and continuous-time chaotic dynamical systems (see e.g. [6] and [8]) but also to discontinuous dynamical systems [3].

The present control-anticontrol algorithm (CA) represents a simplified variant of synchronization of chaotic orbits and is applied to continuous dynamical systems. Both phases can be switched at any evolution moment. This is an important advantage because in practical examples and in nature there are many different interactions and therefore systems do not evolve according to a unique dynamic. It is reasonable to imagine that the evolution of certain physical, biological or economical complex systems can be explained by the combination of the dynamics of (almost) identical dynamical systems.

2 The algorithm

Let us consider a continuous autonomous dynamical system modeled by the initial value problem

$$\dot{x} = f_{\,p}(x),\, x_0 = x(0)\,, \tag{2}$$

where $f_p : \mathbb{R}^n \to \mathbb{R}^n$ is a vector continuous function and $p \in \mathbb{R}$ represents the control parameter.

The CA method implies the previous knowledge of the system behavior for certain values of the control parameter in order to be able to select a "targeted" trajectory.

Let p_1 and p_2 be the parameters for which the system (2) has, respectively, chaotic and stable behavior. Corresponding to these values of p, consider now the following two identical systems starting from the same initial conditions

$$\begin{aligned} &S_1 : \dot{x} = f_{p_1}(x), \;\; x_0 = x(0)\,, \\ &S_2 : \dot{y} = f_{p_2}(y), \;\; y_0 = x_0\,, \;\; p_1 \neq p_2. \end{aligned} \tag{3}$$

Let T_1 and T_2 be the trajectories corresponding to S_1 and S_2 starting at the same moment $t = 0$.

Let's suppose that the first system S_1 evolves on a chaotic trajectory T_1, while the second one S_2 on a stable trajectory T_2.

CA algorithm allows both the control and anticontrol phases at any moment as will be seen next.

The key of the algorithm is to verify at fixed interval of time δt if the two trajectories remain close enough one to another e.g. the Euclidean distance between T_1 and T_2 at any time moment t verifies the following condition

$$|\, x - y \,| < \varepsilon\,, \;\; t \in [0,\, t_{\max}], \tag{4}$$

where ε is a small real positive number. This closeness is possible due to the ergodicity property of continuous dynamical systems and because the systems are identical.
When (4) is not verified, i.e.

$$|\, x - y \,| \geq \varepsilon\,, \tag{5}$$

we modify the values of the variables. The parameter δt is of major importance: it should be enough small in order that T_1 and T_2 be close enough one to another and, the same, be not too small for computing time reasons. Thus, if we want to obtain the anticontrol of chaos we perturb the value of the state variable y of the stable system S_2 in order to reaches the value of the state variable x corresponding to the chaotic evolution of system S_1

$$y \leftarrow y + sgn(x - y)\varepsilon, \tag{6}$$

and the stable trajectory T_2 becomes unstable following the chaotic behavior of T_1.
If we want to control the chaos of the first trajectory then, when (5) is fulfilled, we use the reverse transformation

$$x \leftarrow x + sgn(y - x)\varepsilon, \tag{7}$$

and the chaotic trajectory T_1 becomes stable.

Remark 1 *i) By practical reasons we are motivated to consider that the switches (6) and (7) are performed inertially. However, in this paper we consider for the sake of simplicity that these transformations act instantly.*

ii) We do not intend to prove the chaotic character of anticontrol since the anticontrol acts such that T_2 follows the known chaotic trajectory T_1.

Relations (6) and (7) are practically applicable for small enough ε as follows from the continuation of solutions theorem (see e.g. [10]).

In other words to obtain the control/anticontrol, the variables of one of the systems are perturbed in order to evolve nearby the targeted trajectory.

The anticontrol/control phases (6), (7) can start at any moment of time in the interval $[0,\, t_{\max}]$. In both cases one of the trajectories $T_{1,2}$ becomes only "ε-identical" to the other i.e. the trajectories remain in an ε-tube.

Proposition 1 *The GM substitution* (1) *and CA substitutions* (6), (7) *are equivalent.*

Proof. *Following (4) the perturbation of any of the two variables (6) or (7) is*

$$\alpha = \pm \mid x - y \mid , \tag{8}$$

with $\mid \alpha \mid < \varepsilon$ *a sufficiently small real number. Thus (6) could be written as follows*

$$y \leftarrow y + \alpha. \tag{9}$$

When (5) is checked, at small enough δt, *one can find a real small number* λ *such that* $\alpha = \lambda y$ *and (9) becomes*

$$y \leftarrow y + \alpha = y(1 + \lambda), \tag{10}$$

which represents the transformations (1). The same for (7).

The moments when the CA algorithm becomes active, $t \mod \delta\, t = 0$, can be chosen in the same way in both CA algorithm phases. In our examples we chosen $\delta\, t = (1 \div 100)\, h$, h being the time step integration, but also bigger values were found in other considered applications.

One of the advantages of CA algorithm over the GM algorithm is that CA algorithm allows to chose prior which of the possible stable (unstable) trajectories are target.

Remark 2 *i) A faster and easier variant for computer implementation can be used instead of* (5): *it is easy to see that* (5) *is also verified if the following relations are true*

$$\mid x_i - y_i \mid \geq \varepsilon',\ i = 1...n, \quad \varepsilon' > 0.$$

Indeed

$$\mid x - y \mid = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2},$$

and if

$$|x_i - y_i| \geq \varepsilon / \sqrt{n},\ i = 1...n,$$

then choosing $\varepsilon' = \varepsilon/\sqrt{n}$ *one obtains (5).*

ii) The condition that the both systems start from the same initial conditions is not a necessary one: because of ergodicity, even if the systems start from different initial conditions, after some finite time, T_1 *and* T_2 *will be* $\varepsilon-$*close enough to start the CA algorithm.*

iii) While the Chen & Lai anticontrol algorithm makes chaotic an originally non-chaotic dynamical system, CA algorithm allows, obviously, the anticontrol of only chaotic systems. Since in the practical examples the dynamical systems are in the great majority chaotic, this restriction does not affect the algorithm utility.

3 Applications

In this section the CA algorithm is applied to three representative dynamical systems. For this purpose a special program, using standard Runge-Kutta method, was written to plot time series, phase portraits and the difference between the two trajectories.

The switch from the anticontrol to the control phase (starting at the moment $t = t_1$) presents transients which are typical "inertia" phenomena for dynamical systems (see e.g. Fig.3). For $t \in [t_2, t_{\max}]$ the two systems evolve uncontrolled.

3.1 Lorenz system

The mathematical model of the known system is

$$\begin{aligned} \dot{x}_1 &= -\,a\,x_1 + a\,x_2, \\ \dot{x}_2 &= -\,x_1\,x_3 + b\,x_1 - x_2, \\ \dot{x}_3 &= x_1\,x_2 - c\,x_3, \end{aligned} \qquad (11)$$

with $a = 10$, $c = 8/3$ and b the control parameter. For the chaotic behavior we chose $b_1 = 28$ while for the stable trajectory $b_2 = 8$. The control phase (for $t \in [0, t_1]$) and the anticontrol phase ($t \in [t_1,\, t_2]$) were obtained for $\delta = 20$ and $\varepsilon = 0.01$ (Fig 1).

3.2 Chen system

The well-known Chen's system modelling an electronic circuit [1]

$$\begin{aligned} \dot{x}_1 &= a\,x_1 + x_2\,x_3, \\ \dot{x}_2 &= -\,x_1\,x_3 + b\,x_1 - x_2, \\ \dot{x}_3 &= x_1\,x_2 - c\,x_3, \end{aligned} \qquad (12)$$

evolves on a stable trajectory for the control parameter $c = -4$, while for $c = -2.5$ it behaves chaotically. The other parameters are $a = 0.5$, $b = -10$. The CA values used in this case are $\delta = 20$ and $\varepsilon = 0.1$. (Fig.2).

3.3 Rabinovich-Fabrikant system

The Rabinovich-Fabrikant (R-F) system models the stochasticity arising from the modulation instability in a non-equilibrium dissipative medium [11]

$$\begin{aligned} \dot{x_1} &= x_2(x_3 - 1 + x_1^2) + ax_1, \\ \dot{x_2} &= x_1(3x_3 + 1 - x_1^2) + ax_2, \\ \dot{x_3} &= -2x_3(b + x_1x_2)\,. \end{aligned} \tag{13}$$

The complex dynamical behavior of this system is presented in [4] and [7] and was numerically studied using a special numerical method [5]. For $a = 0.1$ and the control parameter $b = 0.98$ the system behaves chaotically while for $b = 0.14$ the system evolves on a stable limit cycle. The CA values are $\delta = 50$ and $\varepsilon = 0.01$ (Fig.3).

3.4 Circuitry implementation

The CA algorithm can be realized physically (see Fig.4). Suppose that S_1 and are S_2 two circuits. If we want to obtain the anticontrol of chaos we must perturb the value of the state variable y according to (6). The *sgn* function is implemented by an operational amplifier that compares the x and y values and controls a switch. This switch selects the reference values $+\varepsilon$ or -ε. Finally an adder circuit obtains $y + sgn(x - y)\varepsilon$.

4 Conclusions

In this paper we present a simple method to achieve both control and anti-control of chaos in chaotic dynamical systems.

The algorithm performs variables perturbation to one of two identical dynamical systems evolving with different values of the control parameter. Knowing the system behavior as function of the control parameter we can obtain, starting from a stable or unstable trajectory, any desired kind of motions (stable or chaotic).

CA algorithm allows to switch at any moment the two phases (anticontrol-control) which could be of useful in many practical situations, especially in the cases of the dynamical systems with accessible variables.

No special link between ε and δt was found yet.

The CA algorithm can in fact be regarded as a synchronization problem with one system as the master system and the other system as the slave system.

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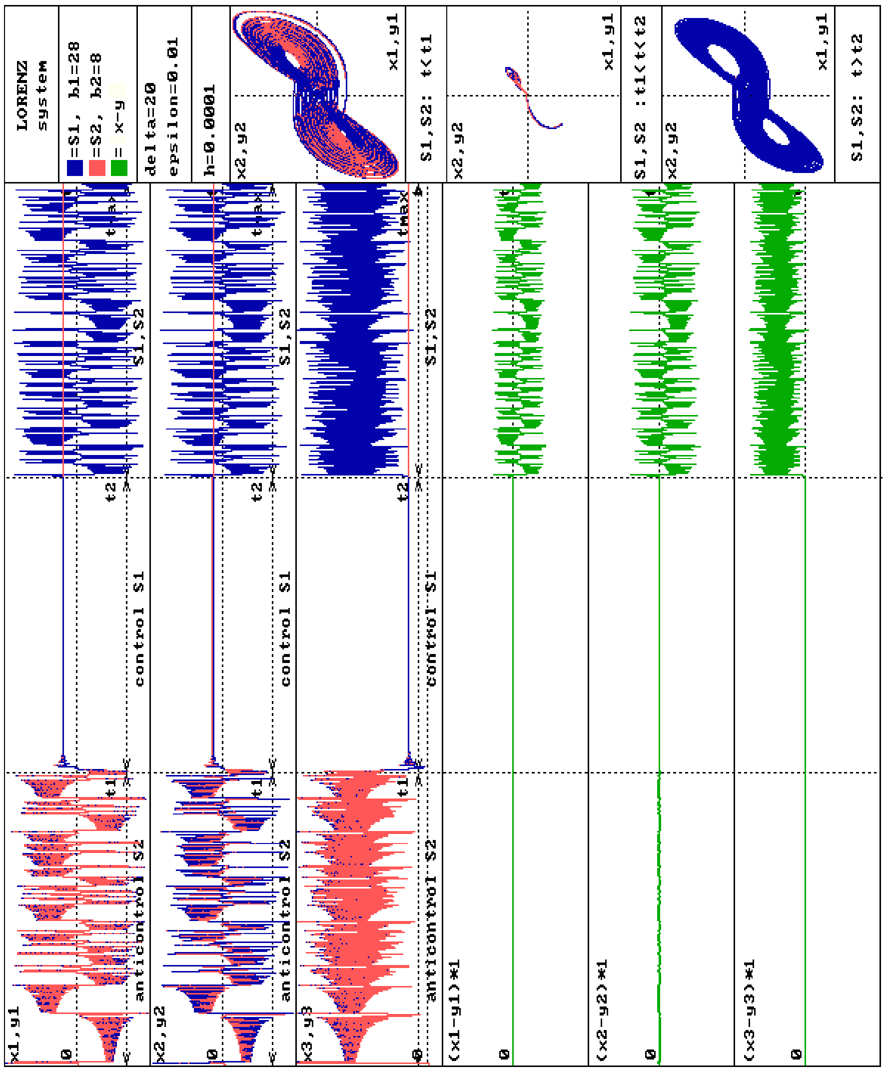

Figure 1: The CA algorithm applied to Lorenz system (11) for $\delta = 20$ and $\varepsilon = 0.01$.

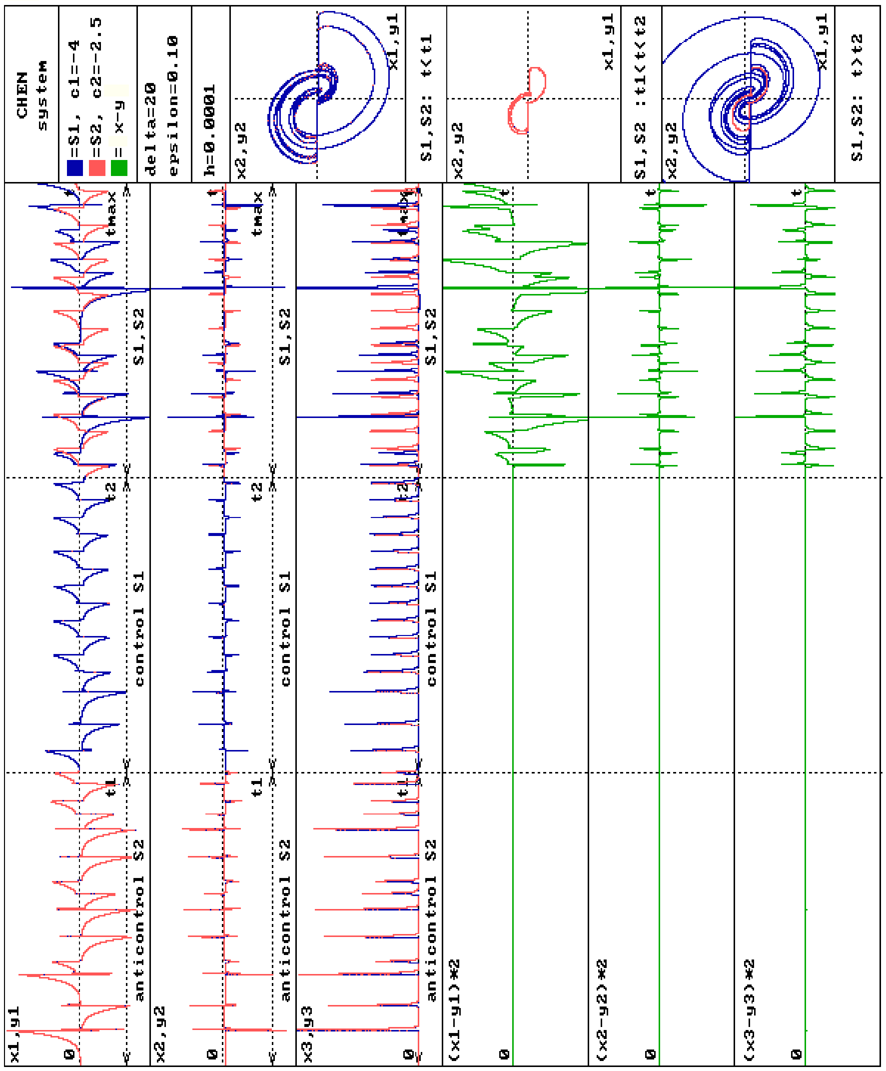

Figure 2: The CA algorithm applied to Chen system (12) for $\delta = 20$ and $\varepsilon = 0.1$.

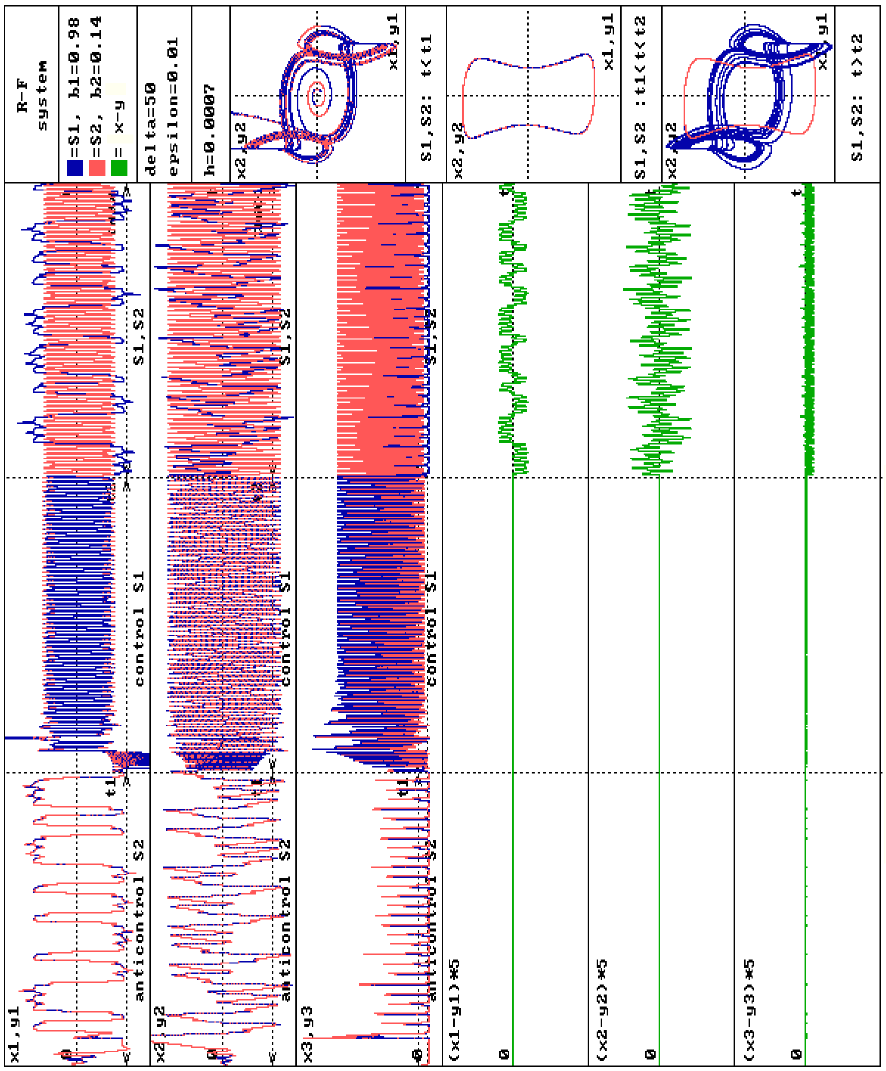

Figure 3: The CA algorithm applied to Rabinovich-Fabrikant system (13) for $\delta = 50$ and $\varepsilon = 0.01$.

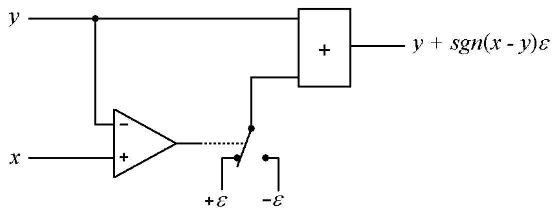

Figure 4: Circuitry implementation of CA algorithm.

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