Fourier series solution to the heat conduction equation with an internal heat source linearly dependent on temperature; application to chilling of fruit and vegetables.

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Abstract

This paper proposes a separation of variables solution to the Equation for heat transfer by conduction in simply-shaped, homogeneous and isotropic bodies subjected to cooling or heating processes without a phase change and with an internal heat source that is a linear function of temperature and subject to homogeneous external conditions of the 3rd kind. The solution is given by the sum of an infinite Fourier series. Starting from this solution, the paper also proposes a simple calculation of chilling time based on an approximation to the first term of that solution (exponential zone); it further proposes a first approximation to the maximum value attained by the temperature history, and to the corresponding time.

(Key words: Transient heat transfer; Cooling; Heat of respiration, Chilling times.)

Nomenclature

A_0 = Constant of the heat source. $\mathrm{J\ m^{-3}\ s^{-1}}$

A_1 = Slope of the heat source. $\mathrm{J\ m^{-3}\ s^{-1}\ K^{-1}}$

$a = \dfrac{k}{\rho \cdot c}$ = Thermal diffusivity. $\mathrm{m^2\ s^{-1}}$

$B_0 = A_0 + A_1 T_{ex}$. $\mathrm{J\ m^{-3}\ s^{-1}}$

$Bi = \dfrac{hR}{k}$ = Biot number

c = Thermal capacity per unit of mass. $\mathrm{J\ kg^{-1}\ K^{-1}}$

$D_x, D_{Sf}, \overline{D}$ = Displacement corrections (dimensionless)

$Fo = \dfrac{a.t}{R^2}$ = Fourier number (dimensionless time)

J, J_i = Coefficients of expansion of the series

$\bar{J}, \bar{J}_i = \left(J\overline{\psi}\right)_{\delta_i}$ = Product of J_i multiplied by the average value of ψ

k = Heat conductivity. $\mathrm{W\ m^{-1}\ K^{-1}}$

R = Characteristic length. m

T = Temperature. K

T_{ex} = External temperature. K

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T_{Sf} = Temperature at surface. K

$\Delta T = T - T_{ex}$. K

t = Time. s

x = Dimensionless distance

Greek symbols

$\alpha^2 = \dfrac{A_1 R^2}{k}$ = Dimensionless slope of the heat source

$\beta = \dfrac{B_0 \cdot R^2}{k \cdot \Delta T_0}$ = Dimensionless number of the heat source

$\delta = \delta_1$ = First root of the boundary equation

$\theta = \dfrac{\Delta T}{\Delta T_0}$ = Dimensionless temperature

θ_s = Dimensionless temperature, steady-state solution

θ_{Sf} = Dimensionless temperature at surface

ρ = Density. $\mathrm{kg\,m^{-3}}$

φ = Solution to the time-related part of the equation

ψ **= Solution to the spatial part of the equation** $\psi' = \dfrac{\partial\psi}{\partial(\delta x)} = \dfrac{1}{\delta}\dfrac{\partial\psi}{\partial x}$

Introduction

Calculation of heat transfer by conduction through temperature-dependent internal heat sources as in fruits and vegetables, where there are additional problems of geometry, packaging, stowage, respiration heat, etc., is generally a highly complex problem requiring sophisticated numerical procedures only available on a computer. In this context, respiration heat is generally considered to be a function that increases exponentially with temperature (Campañone et al., 2002, Tanner et al., 2002, part 1 and part 2), but various authors use other models in practical cases. For instance, some authors take heat generation to be a constant value (Dincer 1994, 1997, Meffert et al., 1971, Stela et al., 2005); then others treat it as a constant in theory but in practice consider it to be negligible in comparison with other more powerful heat sources (Wang 2001). Other authors take a potential model (Sadashive Gowda et al., 1997, and Tanner, et al., 2002 part 1 and part 2), or even a function of time (Campañone et al., 2002). There are regression models like those of Kole & Prasad 1994 (regression to a fourth-grade polynomial) or Rao et al., 1993 (regression to a sigmoidal model). Exact analytical solutions have been derived for some particular cases, including heat generation. Jakob (1949), for example, considered the case of conduction with a linear heat source under steady state conditions, and Carslaw & Jaeger (1959) proposed a solution to the problem in transient conditions for an infinite slab. A solution was also proposed for an infinite cylinder and infinite Biot number.
This study proposes a single general solution based on analysis by separation of variables, which is valid for the three elementary geometries and in a general way for the coefficient of heat transmission.

The reason for adopting a linear rate model for heat generation is that an exact solution is still feasible using separation of variables; moreover, there is an abundance of data in the literature supporting the adoption of such a linear model as a first approximation, for example in Xu & Burfoot, 1999, who adopt a linear approach in the case of potato chilling.
Without ruling out other possible applications, the solution proposed here targets the study of cooling of plant foods whose respiratory process constitutes a heat source.

Theoretical analysis

The one-dimensional Fourier equation for heat transfer in a single dimension, with constant, homogeneous and isotropic coefficients and a linear heat source may be written thus:

$$k\frac{1}{r^\Gamma}\frac{\partial}{\partial r}\left(r^\Gamma\frac{\partial T}{\partial r}\right)+A_0+A_1.T=\rho c\frac{\partial T}{\partial t}$$

With the boundary condition of the third kind

$$k\left[\frac{\partial T}{\partial r}\right]_{r=R}=-h\left(T_{Sf}-T_{ex}\right)$$

Γ is the geometric constant taking values $\Gamma=0$ for an infinite slab, $\Gamma=1$ for an infinite cylinder and $\Gamma=2$ for a sphere.
Table I shows the values of A_0 and A_1 obtained by linear regression of the average values taken from the ASHRAE tables (1998)
Table I: Summary of regression values calculated from ASHRAE
The above equations may be written in dimensionless form, thus:

$$\frac{1}{x^\Gamma}\frac{\partial}{\partial x}\left(x^\Gamma\frac{\partial\theta}{\partial x}\right)+\alpha^2\theta+\beta=\frac{\partial\theta}{\partial Fo} \qquad (1)$$

$$\left[\frac{\partial\theta}{\partial x}\right]_{x=1}=-Bi.\theta_{Sf} \qquad (2)$$

Where:

$$\theta=\frac{\Delta T}{\Delta T_0}=\frac{T-T_{ex}}{T_0-T_{ex}}$$

$$B_0=A_0+A_1.T_{ex}$$

$x=\dfrac{r}{R}$ = Dimensionless distance

$Fo=\dfrac{a\cdot t}{R^2}$ = dimensionless time (Fourier number)

$Bi=\dfrac{hR}{k}$ = Biot number (dimensionless)

$a=\dfrac{k}{\rho\cdot c}$ = Thermal diffusivity

R = Semithickness in the direction of propagation
The heat source is expresse by two new dimnesionless numbers:

$$\alpha^2 = \frac{A_1 R^2}{k} \tag{3}$$

$$\beta = \frac{B_0 \cdot R^2}{k \cdot \Delta T_0} \tag{4}$$

General solution for simple geometries

Equation (1) has to be integrated with the boundary condition (2). Denominating the steady function as θ_s, the general solution may be written as follows (see appendix 1):

$$\theta(x, Fo) - \theta_s(x) = \sum_i J_i \psi(\delta_i x) \cdot e^{-\left(\delta_i^2 - \alpha^2\right)Fo} \tag{5}$$

The steady solution $\theta_s(x)$ is (see appendix 2):

$$\theta_s = \frac{T_s - T_{ex}}{\Delta T_0} = \frac{\beta}{\alpha^2}\left[\frac{Bi.\psi(\alpha x)}{\alpha \psi'_\alpha + Bi\,\psi_\alpha} - 1\right]$$

δ_i is the solution to the classical transcendental equation (as in the no heat generation case):

$$\delta.\psi'(\delta) = -Bi.\psi(\delta)$$

and coefficients J_i (see appendix 3):

$$J_i = J_{i,0}\left(1 - \frac{\beta}{\delta_i^2 - \alpha^2}\right)$$

where

$J_{i,0}$ is the coefficient in the absence of heat generation:

$$J_{i,0} = \frac{2Bi}{\psi_{\delta_i}\left[\delta_i^2 + Bi^2 - (\Gamma - 1)Bi\right]}$$

and the function ψ has the following expressions:

$\psi(\delta_i x) = \cos(\delta_i x); \psi(\alpha x) = \cos(\alpha x)$ for an infinite slab

$\psi(\delta_i x) = J_0(\delta_i x); \psi(\alpha x) = J_0(\alpha x)$ for an infinite cylinder

$\psi(\delta_i x) = \dfrac{\sin(\delta_i x)}{\delta_i x}; \psi(\alpha x) = \dfrac{\sin(\alpha x)}{\alpha x}$ for a sphere

Therefore, the temperature function becomes:

$$\theta = \frac{T - T_{ex}}{\Delta T_0} = \frac{\beta}{\alpha^2}\left(\frac{Bi.\psi(\alpha x)}{\alpha \psi'_\alpha + Bi\,\psi_\alpha} - 1\right) + \sum_i J_i \psi(\delta_i x) e^{-\left(\delta_i^2 - \alpha^2\right)Fo} \tag{6}$$

Threshold Biot number

As shown in appendix 2, α^2 must satisfy the condition:

$$\alpha^2 \le \delta^2 \le \delta_M^2 \tag{7}$$

Also, given that in most fruits and vegetables the value of α^2 is in the region of 10^{-2}, an approximate value can be found for this threshold Biot number. The first value of δ_i^2 may be approximated in elementary geometries for Biot numbers close to zero (Cuesta & Lamua 1995) with the equation (to facilitate writing, when working exclusively with the first term of the series we drop subscript 1, which denotes the order number, and the function's argument whenever this leaves no room for doubts):

$$\frac{1}{\delta^2} \approx \frac{1}{2(\Gamma+1)}\left(1+\frac{2}{Bi}\right)$$

From this equation and relationship (7) we can deduce:

$$Bi \geq \frac{2\alpha^2}{2(\Gamma+1)-\alpha^2} \tag{8}$$

And, when the value of α² is very small, this is reduced to:

$$Bi \geq \frac{\alpha^2}{\Gamma+1} \tag{9}$$

Average value

The mass average temperature of the product at constant density is as follows (see appendix 4):

$$\bar{\theta} = \bar{\theta}_s + \sum_I \bar{J}_i \cdot e^{-\left(\delta_i^2-\alpha^2\right)Fo} \tag{10}$$

With: $\bar{J}_i = \bar{J}_{i,0} \cdot \left(1 - \frac{\beta}{\delta_i^2 - \alpha^2}\right)$

and

$$\bar{J}_{i,0} = J_{i,0} \cdot \bar{\psi}_{\delta_i} = \frac{2Bi^2(\Gamma+1)}{\delta_i^2\left[\delta_i^2 + Bi^2 - (\Gamma-1)Bi\right]}$$

Therefore, the mass average temperature function becomes :

$$\bar{\theta} = \frac{\bar{T} - T_{ex}}{\Delta T_0} = -\frac{\beta}{\alpha^2}\left[\frac{(\Gamma+1)}{\alpha}\frac{Bi\,\psi'_\alpha}{\alpha\psi'_\alpha + Bi\,\psi_\alpha} + 1\right] + \sum_I \bar{J}_i \cdot e^{-\left(\delta_i^2-\alpha^2\right)Fo} \tag{11}$$

Chilling times

As in the case of chilling without an internal heat source, when the time is long enough the calculation can be done with the first term of the series. In this case equation (5) may be rewritten approximately as follows:

$$Y = \theta - \theta_s(x) \approx J\psi e^{-\left(\delta^2-\alpha^2\right)\cdot Fo} \tag{12}$$

If the subscript *c* denotes the values at the thermal core:

$$Y_c = Je^{-\left(\delta^2-\alpha^2\right)\cdot Fo}$$

We have that:

$$Y = Y_c\psi$$

Similar to the case with no internal source, chilling times for sufficiently large Fourier numbers can be calculated using a very simple linear analytical expression. Thus, from the exponential equation (12):

$$\ln Y = \ln(J\psi) - \left(\delta^2 - \alpha^2\right)\cdot Fo$$

and the Fourier number (dimensionless time) can immediately be found:

$$Fo = \frac{\ln(J\psi/Y)}{\delta^2 - \alpha^2} \tag{13}$$

In the cases of the thermal core and average value these equation will be:

$$Fo_c = \frac{\ln(J/Y)}{\delta^2 - \alpha^2} \tag{14}$$

And for the mass average value:

$$\overline{Fo} = \frac{\ln(J\overline{\psi}/Y)}{\delta^2 - \alpha^2}$$

Displacement correction

Equation (13) can be rewritten thus:

$$Fo = \frac{\ln(J\psi/Y)}{\delta^2 - \alpha^2} = \frac{\ln(J/Y)}{\delta^2 - \alpha^2} + \frac{\ln(\psi)}{\delta^2 - \alpha^2}$$

The first term is the value of the Fourier number corresponding to the core (Fo_c), and the second term represents the increase in that number due to displacement to the x coordinate.

$$Fo = Fo_c + \frac{\ln(\psi)}{\delta^2 - \alpha^2} = Fo_c - \frac{\ln(1/\psi)}{\delta^2 - \alpha^2}$$

$$Fo_{x,Y} = Fo_{0,Y} - D_x \tag{15}$$

with

$$D_x = \frac{\ln[1/\psi(\delta x)]}{\delta^2 - \alpha^2}$$

Clearly this term is dependent not on time but solely on the coordinate. Hence, other than for low Fourier numbers, if we wish to calculate the time required at a coordinate x to reach a given temperature θ, it will suffice to know the time needed to calculate it at the core and adjust this with the appropriate term for displacement. In the particular cases of the surface and the average value we get:

A) Surface:

$$D_{Sf} = \frac{\ln(1/\psi_\delta)}{\delta^2 - \alpha^2} \tag{16}$$

B) Average value:

$$\overline{D} = \frac{\ln(1/\overline{\psi})}{(\delta^2 - \alpha^2)} \tag{17}$$

Summary of the procedure

Hence, in the exponential zone (where the first term is enough), if we wish to know the time Fo needed to attain the absolute dimensionless difference θ, it will suffice to apply the following procedure:

i) Take the difference value: $Y = \theta - \theta_s$

ii) Calculate the time for the core (equation 14)

$$Fo_c = \frac{\ln(J/Y)}{\delta^2 - \alpha^2}$$

iii) Carry out the appropriate displacement if applicable (equation 16 or 17):

$$D_x = \frac{\ln(1/\psi)}{\delta^2 - \alpha^2}$$

and finally import to (15)

$$Fo_{x,Y} = Fo_{0,Y} - D_x$$

Illustrative example

Let us take the case of an individual potato which we wish to cool from 25ºC to 5ºC. According to Xu & Burfoot (1999), we may assume that this potato is like a sphere 0.065 m in diameter (that is, $\Gamma + 1 = 3$) with respiration heat rising linearly with temperature.

$$q_r = A_0 + A_1 T$$

where the parameters A_0 and A_1, as previously stated in SI units, are

$$A_0 = 0.01739\ \mathrm{W/kg} \ \text{ and } \ A_1 = 0.001942\ \mathrm{W/kg\,K}$$

Since the parameters A_0 and A_1 in this work are measured in:

$[A_0] = \mathrm{W/m^3}$ and $[A_1] = \mathrm{W/m^3\ K}$,

the first two will have to be multiplied by the density to get the last two. Table 3 in ASHRAE (1998) gives us the following components for potato:

$x_{Water} = 0.79$; $x_{\mathrm{Pr}ot} = 0.0207$; $x_{Fat} = 0.001$; $x_{Carboh} = 0.1798$; $x_{Ash} = 0.0089$

From these, taking the equations in tables 1 and 2 as reference, we get the following values for the thermophysical parameters:

ρ = 1123.5 kg m$^{-3,}$, k = 0.485 W m^{-1} K^{-1}, a = 0.1253 10^{-6} m^2 s^{-1}, c_p = 3636.6 J kg^{-1} K^{-1}

so that the respective values of A_0 and A_1 are finally:

$$A_0 = 0.01739 \times 1123.5 = 19.54\ \mathrm{W/m^3}$$

$$A_1 = 0.001942 \times 1123.5 = 2.18\ \mathrm{W/m^3\ K}$$

With this and the conductivity value deduced previously, it is possible to calculate the dimensionless parameters α^2 and β (equations 3 and 4):

$\alpha^2 = \dfrac{A_1 R^2}{k} = \dfrac{2.1816 \times \left(\dfrac{0.065}{2}\right)^2}{0.485} \dfrac{\mathrm{W \cdot m^2 \cdot m \cdot K}}{\mathrm{m^3 \cdot K \cdot W}} = 0.00475$ whose value depends solely on the specific fruit and not on the process. The external temperature is 5ºC and the total temperature difference is $\Delta T_0 = 25 - 5 = 20^{\mathrm{o}}\,\mathrm{C}$. So, the value of β will be:

$$\beta = \frac{B_0 \cdot R^2}{k \cdot \Delta T_0} = \frac{(A_0 + A_1 T_{ex}) \cdot R^2}{k \cdot \Delta T_0} = \frac{30.45 \times 0.0325^2}{0.4854 \times 20} = 0.00331$$

which depends both on the potato itself and on the cooling/heating process, since it is affected by the external temperature and the total temperature difference.

Since this is only an illustrative example, we shall further assume that cooling takes place with an air velocity such that the value of the Biot number is $Bi = 0.2$; the corresponding value of $\delta = 0.7593$, and hence $\delta^2 = 0.5765$, which is greater than the value calculated previously for α^2. Thus the condition is satisfied.

The parameter J_0 corresponding to the Biot number $Bi = 0.2$ is $J_0 = 1.0592$.

Hence:

$$J = J_0\left(1 - \frac{\beta}{\delta^2 - \alpha^2}\right) = 1.0592\left(1 - \frac{0.00331}{0.5765 - 0.00475}\right) = 1.0531$$

The steady temperature at the core is:

$$\theta_s(0)=\frac{\beta}{\alpha^2}\left[\frac{Bi}{\alpha\psi'_\alpha+Bi\,\psi_\alpha}-1\right]=0.0061$$

The displacement of Fo on the surface will be:

$$D_{Sf}=-\frac{\ln\left(\frac{\sin\delta}{\delta}\right)}{\delta^2-\alpha^2}=0.1714$$

and the displacement for the average value:

$$\overline{D_Y}=D_{Sf}-\frac{\ln\left[\frac{(\Gamma+1)Bi}{\delta^2}\right]}{(\delta^2-\alpha^2)}=0.1714-\frac{\ln\left[\frac{(2+1)\cdot 0.2}{0.5765}\right]}{(0.5765-0.00475)}=0.1017$$

If we take $Y=1/2$ as the reference for the core, the half-cooling time will be:

$$Fo_{1/2}=\frac{\ln(2J)}{(\delta^2-\alpha^2)}=\frac{\ln(2\times1.0530)}{(0.5765-0.00475)}=1.3026$$

Let us imagine that we wish to attain the absolute dimensionless temperature $\theta=0.3$.
This means that

$$Y_{0.3}=\theta-\theta_s=0.3-0.0061=0.2939$$

and hence the dimensionless time required to attain the dimensionless temperature $\theta=0.3$ at the core is:

$$Fo_Y=Fo_{1/2}-\frac{\ln(2Y)}{\delta^2-\alpha^2}=1.3026-\frac{\ln(2\times0.2939)}{0.5765-0.00475}=2.2320$$

which coincides with the value derived using the entire series.
As we have seen, the term of displacement to the surface is:

$$D_{Sf,Y}=0.1714$$

and hence the dimensionless time needed to attain $\theta=0.3$ at the surface is:

$$Fo_{Sf,Y}=Fo_{0,Y}-D_{Sf}=2.2320-0.1714=2.0606$$

The value produced by the complete series is:

$$Fo=2.0575$$

Therefore, the error is 0.15 %

As we have seen, the average value of its displacement is $\overline{D_Y}=0.1017$. Therefore, the time needed to attain the average value $\overline{\theta}=0.3$ will be:

$$\overline{Fo_Y}=Fo_{0.Y}-\overline{D}_Y=2.2320-0.1017=2.1303$$

The value of the complete series is $\overline{Fo}=2.1284$, and therefore the error is 0.09 %
The values for this example in the case when there is no respiration are:
Core: $Fo=2.1879$ (difference -1.98 %)
Surface: $Fo=2.0079$ (difference -2.47%)
Average value: $\overline{Fo}=2.0770$ (difference -2.41 %)
That is, in this particular case because the values of A_0 and A_1 are low, the mass considered is small (an individual potato) and the Biot number is much larger than the threshold value, the influence of the respiration heat is relatively small.

Figures 1 and 2 illustrate the influence of the parameters α^2 and β by plotting the temperature history for a fixed Biot number (Bi = 5) and for different values of α^2 and β. In figure 1 the value of β ($\beta = 1$) is fixed and that of α^2 varies between $\alpha^2 = 1$ and $\alpha^2 = 5$ (always lower than the first value of δ^2 corresponding to the Biot number), while in figure 2 α is fixed ($\alpha^2 = 1$) and β varies between $\beta = 1$ and $\beta = 5$.

By way of illustration, figure 3 depicts the temperature history for the three elementary geometries for the values $Bi = 5$, $\alpha^2 = 0.5$ and $\beta = 0.5$

Figure 1: Temperature history for $\beta = 1 = Cte.$ and different values of α^2

Figure 2: Temperature history for $\alpha^2 = 1 = Cte.$ and different values of β

Figure 3: Temperature history for the three elementary geometries

Approximation to maximum value at the core

As we can see in both figures, depending on the particular conditions in each case the temperature may rise at the outset up to a given maximum value θ_M in a time Fo_M, falling thereafter down to its steady value. A first approximation to this value can be achieved by considering that its time derivative must also vanish. Hence, at this point the following condition is required:

$$\left.\frac{dY}{dFo}\right|_{Fo_M} = -\sum_i \left(\delta_i^2 - \alpha^2\right) J_i \psi(\delta_i x) \cdot e^{-\left(\delta_i^2 - \alpha^2\right) Fo_M} = 0$$

To calculate a first approximation to this value, the first two terms of the series can be taken as significant:

$$\left.\frac{dY}{dFo}\right|_{Fo_M} \approx -\left(\delta_1^2 - \alpha^2\right) J_1 \psi(\delta_1 x) \cdot e^{-\left(\delta_1^2 - \alpha^2\right) Fo_M} - \left(\delta_2^2 - \alpha^2\right) J_2 \psi(\delta_2 x) \cdot e^{-\left(\delta_2^2 - \alpha^2\right) Fo_M} = 0$$

Specifically, at the core $\psi = 1$, which leaves:

$$\left.\frac{dY}{dFo}\right|_{Fo_M} \approx -\left(\delta_1^2 - \alpha^2\right) J_1 \cdot e^{-\left(\delta_1^2 - \alpha^2\right) Fo_M} - \left(\delta_2^2 - \alpha^2\right) J_2 \cdot e^{-\left(\delta_2^2 - \alpha^2\right) Fo_M} = 0$$

and from this we can find the value of Fo_M:

$$Fo_M \approx \frac{\ln\left[-\dfrac{\left(\delta_2^2 - \alpha^2\right) J_2}{\left(\delta_1^2 - \alpha^2\right) J_1}\right]}{\left(\delta_2^2 - \delta_1^2\right)} \qquad (18)$$

If we substitute this value in the sum of the first two terms of the series, we get an approximation to the maximum value attained.

$$\theta_M \approx \theta_{s,0} + J_1 \cdot e^{-\left(\delta_1^2 - \alpha^2\right) Fo_M} + J_2 \cdot e^{-\left(\delta_2^2 - \alpha^2\right) Fo_M} \qquad (19)$$

Table II shows the maximum values of the Fourier number (Fo_M) calculated with equation (18) for the cases depicted in figure 1, and table III shows the corresponding maximum values attained by (θ_M) calculated with equation (19). Both tables also show the values derived using the complete series and the differences expressed in %.

Table II: Time required to attain maximum temperature

Table III: Value of maximum temperature

Conclusions

1. This paper describes the analytical Fourier series solution to the equation for heat transfer by conduction in simple geometries with an internal heat source linearly dependent on temperature.
2. The threshold condition for chilling is established.
3. A simple method based on the first term is proposed for calculation of chilling times at the core, mass average and surface.
4. Two simple equations are proposed for approximate calculation of the maximum temperature value attained at the core, and also the corresponding time.

Appendix 1. General solution

Equation (A1-1) is to be integrated:

$$\frac{1}{x^\Gamma}\frac{\partial}{\partial x}\left(x^\Gamma\frac{\partial\theta}{\partial x}\right)+\alpha^2\theta+\beta=\frac{\partial\theta}{\partial Fo} \qquad \text{(A1-1)}$$

with the boundary condition (A1-2):

$$\left[\frac{\partial\theta}{\partial x}\right]_{x=1}=-Bi.\theta_{Sf} \qquad \text{(A1-2)}$$

By definition the steady-state solution does not depend on time, and therefore the second term of equation (A1-1) is cancelled when applied to that case. If we mark the steady-state function with subscript “***s***”, we get:

$$\frac{1}{x^\Gamma}\frac{\partial}{\partial x}\left(x^\Gamma\frac{\partial\theta_s}{\partial x}\right)+\alpha^2\theta_s+\beta=\left[\frac{\partial\theta_s}{\partial Fo}\right]_{Fo=\infty}=0 \qquad \text{(A1-3)}$$

$$\left[\frac{\partial\theta_s}{\partial x}\right]_{x=1}=-Bi.(\theta_s)_{Sf} \qquad \text{(A1-4)}$$

where:

$$\theta_s=\frac{T_s-T_{ex}}{T_0-T_{ex}}$$

If we subtract equations (A1-3) and (A1-4) from equations (A1-1) and (A1-2) respectively, this leaves:

$$\frac{1}{x^\Gamma}\frac{\partial}{\partial x}\left(x^\Gamma\frac{\partial(\theta-\theta_s)}{\partial x}\right)+\alpha^2(\theta-\theta_s)=\frac{\partial(\theta-\theta_s)}{\partial Fo} \qquad \text{(A1-5)}$$

$$\left[\frac{\partial(\theta-\theta_s)}{\partial x}\right]_{x=1}=-Bi.(\theta-\theta_s)_{Sf} \qquad \text{(A1-6)}$$

(A1-5) admits a solution in separation of variables in the same way as without a heat source. The complete solution will therefore be a serial expansion in the form:

$$\theta(x,Fo)=\theta_s(x)+\sum_{i=1}^{i=\infty}J_i.\psi(\delta_i x).e^{-(\delta_i^2-\alpha^2).Fo} \qquad \text{(A1-7)}$$

where δ_i are the infinite solutions of the boundary equation:

$$\delta.\psi'(\delta)=-Bi.\psi(\delta)$$

and where (see appendix 2) if $\alpha^2>0$:

$$\theta_s = \frac{\beta}{\alpha^2}\left[\frac{Bi\,\psi(\alpha x)}{\alpha\psi'_\alpha + Bi\,\psi_\alpha} - 1\right] \qquad \text{(A1-8)}$$

The J_i constants may be written as follows (see Appendix 3):

$$J_i = J_{i,0}\left(1 - \frac{\beta}{\delta_i^2 - \alpha^2}\right) \qquad \text{(A1-9)}$$

where the values $J_{i,0}$ are the constants of the serial expansion for the case with no heat source:

$$J_{i,0} = \frac{2Bi}{\psi_{\delta_i}\left[\delta_i^2 + Bi^2 - (\Gamma - 1)Bi\right]} \qquad \text{(A1-10)}$$

Appendix 2. Steady-state

Equation (A1-7) will be the desired solution if the function θ_e corresponding to $Fo = \infty$ is known, for which it is necessary to solve the equation:

$$\frac{1}{x^\Gamma}\frac{\partial}{\partial x}\left(x^\Gamma \frac{\partial \theta_s}{\partial x}\right) + \alpha^2\theta_s + \beta = 0 \qquad \text{(A2-1)}$$

with the transcedental equation:

$$\left[\frac{\partial \theta_s}{\partial x}\right]_{x=1} = -Bi.(\theta_s)_{Sf} \qquad \text{(A2-2)}$$

where:

$$\theta_s = \frac{T_s - T_{ex}}{T_0 - T_{ex}} \qquad \text{(A2-3)}$$

The homogeneous differential equation corresponding to (A2-1) will be:

$$\frac{1}{x^\Gamma}\frac{\partial}{\partial x}\left(x^\Gamma \frac{\partial \theta_h}{\partial x}\right) + \alpha^2\theta_h = 0 \qquad \text{(A2-4)}$$

As the constant source case ($A_0 > 0;\ A_1 = 0$) was considered by Beek & Meffert, in this paper we consider only the case where $\alpha^2 > 0$. In this case the solution is the same as in the case with no heat source, except that here the coefficient α^2 is imposed by the product's internal heat source and hence is unique. In this case a particular solution of (A2-1) is:

$$\theta_p = -\frac{\beta}{\alpha^2}$$

The steady solution is then equal to:

$$\theta_s = M.\psi(\alpha x) - \frac{\beta}{\alpha^2} \qquad \text{(A2-5)}$$

So that (A2-1) will be satisfied except for the constant M.

Hereafter, as long as there is no risk of confusion, we will use the notation:

$$\psi_\alpha \equiv \psi(\alpha)$$

$$\psi'_\alpha \equiv \psi'(\alpha) = \left[\partial\psi(\alpha x)/\partial(x)\right]$$

To derive the constant M , (A2-5) is substituted in the boundary condition (A2-2). After rearranging terms this leads to:

$$M = \frac{Bi.\beta}{\alpha^2\left[\alpha\psi'_\alpha + Bi\,\psi_\alpha\right]} \qquad \text{(A2-6)}$$

If eq. (A2-6) is substituted in (A2-5) we have:

$$\theta_s = \frac{\beta}{\alpha^2}\left[\frac{Bi\,\psi(\alpha x)}{\alpha\psi'_\alpha + Bi\,\psi_\alpha} - 1\right] \qquad \text{(A2-7)}$$

which is the same expression as found by Jakob (1949) for the same particular case in each of the three elementary geometries.

Following the reasoning of Jakob (1949), at this point we can state that the conditions that equation (6) must meet to remain positive are:

$$Bi.\psi(\alpha x) \geq \alpha\psi'_\alpha + Bi\,\psi_\alpha$$

and

$$\alpha\psi'_\alpha + Bi\,\psi_\alpha \geq 0$$

The first relationship can also be written as:

$$Bi.\left[\psi(\alpha x) - \psi(\alpha)\right] \geq \alpha\psi'(\alpha) \qquad \text{(A2-8)}$$

This is always true, since the left hand side of equation (A2-8) is always positive or null, whereas the right hand side is always null or negative. In fact $\psi(\alpha x) \geq \psi(\alpha)$ given that because $0 \leq x \leq 1$, $\alpha x \leq \alpha$ (e.g. in the case of a flat slab $\cos\alpha x \geq \cos\alpha$), and hence the left hand side of equation (A2-8) is always positive or null. And within this range $\psi'(\alpha)$ (in the infinite slab case $-\sin\alpha$) and so the right hand side, is always ≤ 0)

The second relationship indicates that there is a lower threshold Biot number for each value of α, so that the Biot number must satisfy the condition:

$$Bi \geq \frac{-\alpha\psi'(\alpha)}{\psi(\alpha)}$$

At the same time, the transcendent equation for the general case is

$$\delta.\psi'(\delta) + Bi.\psi(\delta) = 0$$

so that:

$$Bi = \frac{-\delta\psi'(\delta)}{\psi(\delta)}$$

In other words:

$$\frac{-\delta\psi'_\delta}{\psi_\delta} \geq \frac{-\alpha\psi'_\alpha}{\psi_\alpha}$$

Since the general function

$$y = \frac{-x\psi'(x)}{\psi(x)}$$

is continuous and monotonously positive within the range $0 \leq x \leq \delta_M$ for all three elementary geometries, this condition is only fulfilled if:

$$\alpha \leq \delta$$

which for convenience will be expressed in the form:

$$\alpha^2 \leq \delta^2 \qquad \text{(A2-9)}$$

and obviously δ will always be less than its maximum value δ_M (for an infinite Biot number):

$\delta^2 \le \delta_M^2$

So, the conditions are finally:

$\alpha^2 \le \delta^2 \le \delta_M^2$

Thus, as the Biot number decreases, α approaches δ and the denominator $\alpha\psi'_\alpha + Bi\,\psi_\alpha$ will approach $\delta\psi'_\delta + Bi\,\psi_\delta$, which is zero, and therefore in the limit case $\alpha\psi'_\alpha + Bi\,\psi_\alpha = 0$, θ_s – and hence also equation (6) for θ – will tends towards infinity. Following both Jakob's (1949) and Carslaw's (1959) reasoning, all this means that in such a hypothetical case, extraction of heat via the surface would not be sufficient to eliminate the heat generated in the interior and the temperature rise would be unbounded.

Appendix 3. Core: Deducing the expansion constants

At the starting point ($Fo = 0$) the solution may be written as:

$$\theta(x,0) - \theta_s(x) \equiv \theta_0 - \theta_s = \sum_i J_i \psi(\delta_i x) \qquad \text{(A3-1)}$$

where (see appendix 2, eq. (A2-5)):

$$\theta_s = M.\psi(\alpha x) - \frac{\beta}{\alpha^2} \qquad \text{(A3-2)}$$

and:

$$M = \frac{Bi.\beta}{\alpha^2\left[\alpha\psi'_\alpha + Bi\,\psi_\alpha\right]} \qquad \text{(A3-3)}$$

(A3-3) may be written:

$$M\left(\alpha\psi'_\alpha + Bi\,\psi_\alpha\right) = \frac{Bi.\beta}{\alpha^2} \qquad \text{(A3-4)}$$

To calculate the coefficients J_i, we multiply both sides of equation (A3-1) by $x^\Gamma \psi\left(\delta_j x\right)$ and integrate between $x = 0$ and $x = 1$. This gives:

$$\int_0^1 (\theta_0 - \theta_s) x^\Gamma \psi\left(\delta_j x\right) dx = \sum_i J_i \int_0^1 x^\Gamma \psi(\delta_i x) \psi\left(\delta_j x\right) dx$$

For two solutions δ_i and δ_j corresponding to the same Biot number, the following two integral orthogonality properties are deduced:

$$\delta_i \ne \delta_j;\quad \int_0^1 x^\Gamma \cdot \psi(\delta_i \cdot x) \cdot \psi\left(\delta_j \cdot x\right) \cdot dx = 0$$

$$\delta_i = \delta_j;\quad \int_0^1 x^\Gamma \cdot \psi^2(\delta_i \cdot x) \cdot dx = \frac{\psi^2(\delta_i)}{2\delta_i^2}\left[\delta_i^2 + Bi^2 + (\Gamma - 1) \cdot Bi\right]$$

So that:

$$J_i = \frac{\int_0^1 (\theta_0 - \theta_s) x^\Gamma \psi(\delta_i x) dx}{\int_0^1 x^\Gamma [\psi(\delta_i x)]^2 dx} = \frac{\int_0^1 (\theta_0 - \theta_s) x^\Gamma \psi(\delta_i x) dx}{\frac{\psi^2(\delta_i)}{2\delta_i^2}\left[\delta_i^2 + Bi^2 + (\Gamma - 1)\cdot Bi\right]} \quad \text{(A3-5)}$$

As

$\theta_0(x) = 1 = cte.$

we have, by substituting (A3-2):

$$J_i = \frac{\left(1 + \frac{\beta}{\alpha^2}\right)\cdot \int_0^1 x^\Gamma \psi(\delta_i x) dx}{\frac{\psi^2(\delta_i)}{2\delta_i^2}\left[\delta_i^2 + Bi^2 + (\Gamma - 1)\cdot Bi\right]} - \frac{M\int_0^1 x^\Gamma \psi(\delta_i x)\psi(\alpha x) dx}{\frac{\psi^2(\delta_i)}{2\delta_i^2}\left[\delta_i^2 + Bi^2 + (\Gamma - 1)\cdot Bi\right]}$$

The first term in the right hand side contains the J_i coefficient for the case without internal heat sources (Cuesta et al. 1990):

$$J_{i,0} = \frac{\int_0^1 x^\Gamma \psi(\delta_i x) dx}{\frac{\psi^2(\delta_i)}{2\delta_i^2}\left[\delta_i^2 + Bi^2 + (\Gamma - 1)\cdot Bi\right]} = \frac{2Bi}{\psi_{\delta_i}\left[\delta_i^2 + Bi^2 - (\Gamma - 1)Bi\right]}$$

(A3-6)

so that J_i may be written:

$$J_i = \left(1 + \frac{\beta}{\alpha^2}\right) J_{i,0} - \frac{M\int_0^1 x^\Gamma \psi(\delta_i x)\cdot \psi(\alpha x) dx}{\frac{\psi^2(\delta_i)}{2\delta_i^2}\left[\delta_i^2 + Bi^2 + (\Gamma - 1)\cdot Bi\right]} \quad \text{(A3-7)}$$

If we solve the integral of the numerator of the second term (considering the boundary condition $\delta_i \psi'_{\delta_i} = -Bi\,\psi_{\delta_i}$) and equation (A3-4) and rearrange terms, this gives:

$$J_i = \left(1 + \frac{\beta}{\alpha^2}\right) J_{i,0} + \frac{2\delta_i^2}{\alpha^2 - \delta_i^2} \frac{Bi\,\frac{\beta}{\alpha^2}}{\psi_{\delta_i}\left[\delta_i^2 + Bi^2 + (\Gamma - 1)\cdot Bi\right]}$$

Taking (A3-6) into account, we have that:

$$J_i = J_{i,0}\left(1 - \frac{\beta}{\delta_i^2 - \alpha^2}\right) \quad \text{(A3-8)}$$

Appendix 4. Average temperature: deducing the expansion constants

In simple geometries, the mass average temperature of the product is calculated according to the expression:

$$\bar{\theta} = (\Gamma + 1)\int_0^1 x^\Gamma \theta(x, Fo) dx \quad \text{(A4-1)}$$

θ may be written as:

$$\theta = \theta_s + \sum J_i \psi(\delta_i x) e^{-(\delta_i^2 - \alpha^2)\cdot Fo}$$

And therefore:

$$\overline{\theta} = \overline{\theta}_s + \sum J_i \overline{\psi(\delta_i)} e^{-\left(\delta_i^2 - \alpha^2\right)Fo}$$

which again is composed of two terms. The first corresponds to the average steady temperature:

$$\overline{\theta}_s = (\Gamma+1)\int_0^1 x^\Gamma \cdot \theta_s(x)dx \qquad \text{(A4-2)}$$

and the second formally corresponds to the average value for the case with no respiration, except that the coefficients J_i is now given by equation (A3-12). Hence, the average value of $\psi(\delta_i)$, $\overline{\psi}_{\delta_i}$, will be:

$$\overline{\psi} = -\frac{(\Gamma+1)\psi'_\delta}{\delta} = \frac{(\Gamma+1)Bi\,\psi_\delta}{\delta^2} \qquad \text{(A4-3)}$$

so that, for $\overline{\theta}_0 = 1$, the products $\overline{J}_i \equiv J_i \overline{\psi}_{\delta_i}$ become:

$$\overline{J}_i = \overline{J}_{i,0} \cdot \left(1 - \frac{\beta}{\delta_i^2 - \alpha^2}\right) \qquad \text{(A4-4)}$$

$$\overline{J}_{i,0} = J_{i,0} \cdot \overline{\psi}_{\delta_i} = \frac{2Bi^2(\Gamma+1)}{\delta_i^2\left[\delta_i^2 + Bi^2 - (\Gamma-1)Bi\right]} \qquad \text{(A4-5)}$$

As to the average steady value:

$$\overline{\theta}_s = (\Gamma+1)\int_0^1 x^\Gamma \cdot [\theta_s(x)]dx = (\Gamma+1)\int_0^1 x^\Gamma \cdot \left[M.\psi(\alpha x) - \frac{\beta}{\alpha^2}\right]dx$$

$$\overline{\theta}_s = M(\Gamma+1)\int_0^1 x^\Gamma \cdot \psi(\alpha x)dx - (\Gamma+1)\frac{\beta}{\alpha^2}\int_0^1 x^\Gamma dx = I_1 - I_2$$

which can be shown to have the following solutions:

$$I_1 = -\frac{M(\Gamma+1)}{\alpha^2}\alpha\psi'_\alpha$$

$$I_2 = (\Gamma+1)\frac{\beta}{\alpha^2}\int_0^1 x^\Gamma dx = \frac{\beta}{\alpha^2}$$

And hence:

$$\overline{\theta}_s = I_1 - I_2 = -\frac{M(\Gamma+1)}{\alpha}\psi'_\alpha - \frac{\beta}{\alpha^2} \qquad \text{(A4-6)}$$

and if we introduce (A2-6):

$$\overline{\theta}_s = -\frac{\beta}{\alpha^2}\left[\frac{(\Gamma+1)}{\alpha}\frac{Bi\,\psi'_\alpha}{\alpha\psi'_\alpha + Bi\,\psi_\alpha} + 1\right] \qquad \text{(A4-7)}$$

Thus, the average temperature is:

$$\overline{\theta} = \overline{\theta}_s + \sum \overline{J}_i \cdot e^{-\left(\delta_i^2 - \alpha^2\right)Fo}$$

That is:

$$\overline{\theta} = \frac{\overline{T} - T_{ex}}{\Delta T_0} = -\frac{\beta}{\alpha^2}\left[\frac{(\Gamma+1)}{\alpha}\frac{Bi\,\psi'_\alpha}{\alpha\psi'_\alpha + Bi\,\psi_\alpha} + 1\right] + \sum \overline{J}_i \cdot e^{-\left(\delta_i^2 - \alpha^2\right)\cdot Fo} \qquad \text{(A4-8)}$$

with:

$$\overline{J}_i = \overline{J}_{i,0} \cdot \left(1 - \frac{\beta}{\delta_i^2 - \alpha^2} \right)$$

and

$$\overline{J}_{i,0} = J_{i,0} \cdot \overline{\psi}_{\delta_i} = \frac{2Bi^2(\Gamma + 1)}{\delta_i^2 \left[\delta_i^2 + Bi^2 - (\Gamma - 1)Bi \right]}$$

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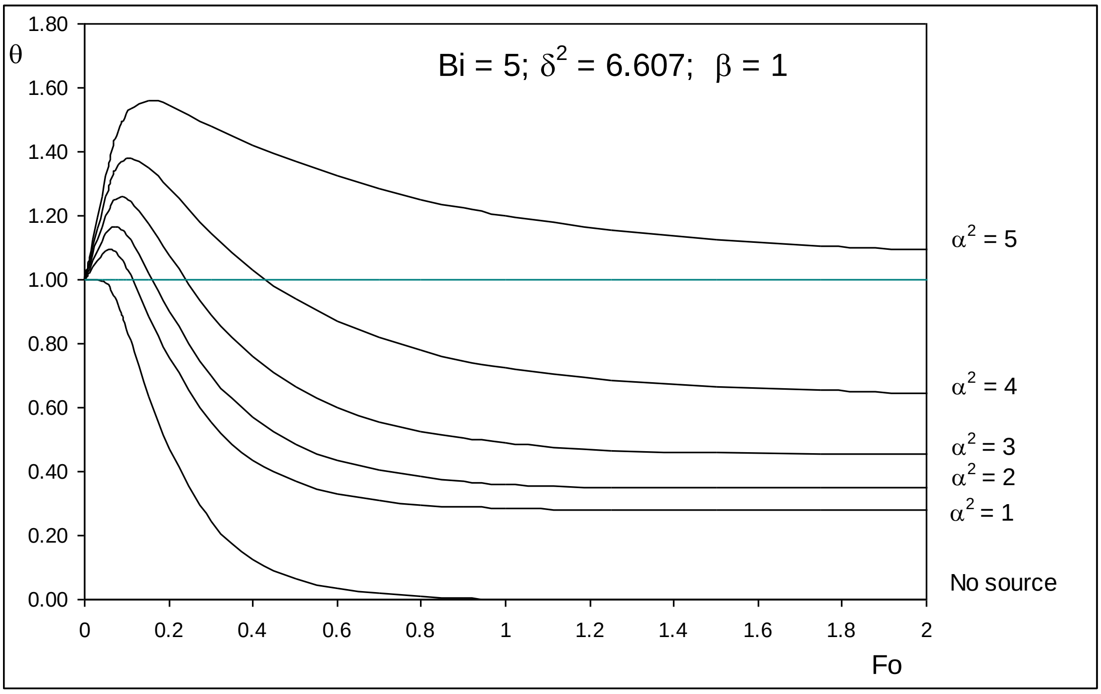

Figure 1: Temperature history for $\beta = 1 = Cte.$ and different values of α^2

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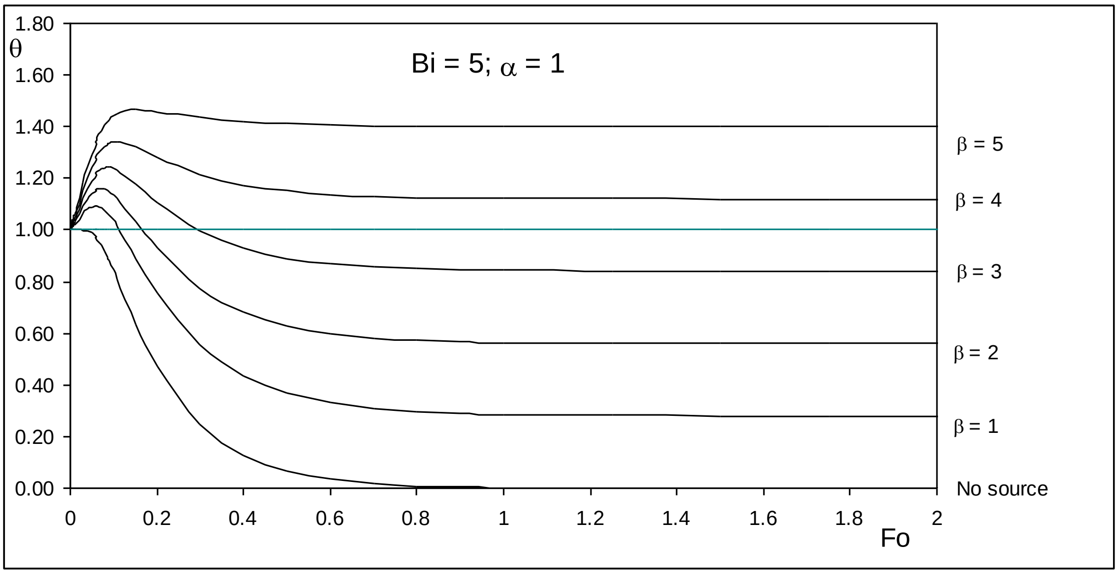

Figure 2: Temperature history for $\alpha^2 = 1$ and different values of β

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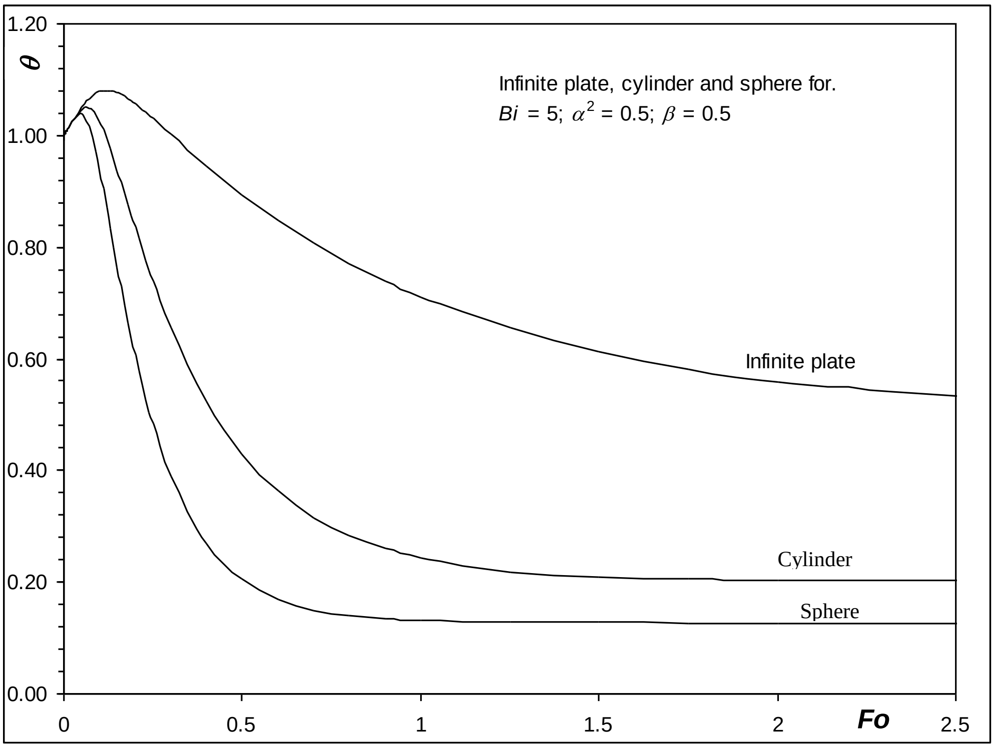

Figure 3: Temperature history for the three elementary geometries for the values $Bi = 5$, $\alpha^2 = 0.5$ and $\beta = 0.5$

Table 1.
Summary of regression values calculated from ASHRAE

Variety	**A_0 (W/kg)**	**A_1 W/(kg.K)**
Apples «Y transparent»	0.0097	0.0073
Apples Average	0.0061	0.0037
Apples Early Cultivars	0.0101	0.0040
Apples Late Cultivars	0.0052	0.0025
Apricots	0.0060	0.0038
Artichokes Globe	0.0723	0.0124
Asparagus	0.1040	0.0478
Beans Lima Unshelled	0.0232	0.0211
Beans Lima Shelled	0.0303	0.0334
Beans Snap	-0.0153	0.0189
Beets Red Roots	0.0160	0.0026
Black Berries	0.0251	0.0213
Blue Berries	0.0028	0.0097
Broccoli	0.0291	0.0506
Brussels Sprouts	0.0436	0.0179
Cabbage Penn State	0.0080	0.0048
Cabbage Red Early	0.0168	0.0070
Cabbage Savoy	0.0165	0.0182
Cabbage White Spring	0.0235	0.0085
Cabbage White Winter	0.0104	0.0045
Cauliflower	0.0284	0.0089
Cauliflower	0.0298	0.0117
Celery	0.0120	0.0060
Gooseberries	0.0214	0.0039
Peas Green	0.0486	0.0400
Potatoes[a]	0.0174	0.0019
Raspberries	-0.0028	0.0222
Strawberries	-0.0033	0.0213

[a] From Xu & Burfoot, 1999

Table 2.
Time Fo_M taken to reach maximum temperature θ_M

α^2	Fo_M (estimated, eq. 28)	Fo_M (Complete series)	Error (%)
1	0,068	0,059	14,68
2	0,077	0,072	6,97
3	0,090	0,087	3,14
4	0,110	0,109	1,10
5	0,152	0,152	-0,02

Table 3.
Maximum temperature value θ_M

α^2	θ_M (estimated, eq. 29)	θ_M (Complete series)	Error (%)
1	1,081	1,094	-1,19
2	1,159	1,166	-0,59
3	1,256	1,259	-0,23
4	1,381	1,382	-0,06
5	1,558	1,558	0,00