Entropic Entanglement Criteria for Fermion Systems

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Abstract

Entanglement criteria for general (pure or mixed) states of systems consisting of two identical fermions are introduced. These criteria are based on appropriate inequalities involving the entropy of the global density matrix describing the total system, on the one hand, and the entropy of the one particle reduced density matrix, on the other one. A majorization-related relation between these two density matrices is obtained, leading to a family of entanglement criteria based on Rényi's entropic measure. These criteria are applied to various illustrative examples of parametrized families of mixed states. The dependence of the entanglement detection efficiency on Rényi's entropic parameter is investigated. The extension of these criteria to systems of N identical fermions is also considered.

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I. INTRODUCTION

The entanglement features exhibited by systems consisting of identical fermions have attracted the attention of several researchers in recent years [1–15]. Entanglement in fermion systems has been studied in connection with different problems, such as the entanglement between electrons in a conducting band [7], the entanglement dynamics associated with scattering processes involving two electrons [8], the role played by entanglement in the time-optimal evolution of fermionic systems [9, 10], the classification of three fermion states based on their entanglement features [11], the detection of entanglement in fermion systems through the violation of appropriate uncertainty relations [12], the entanglement features of fractional quantum Hall liquids [13] and the entanglement properties of the eigenstates of soluble two-electrons atomic models [14].

The concept of entanglement in systems of indistinguishable particles exhibits some differences from the corresponding concept as applied to systems consisting of distinguishable parts. There is general consensus among researchers that in systems of identical fermions the minimum quantum correlations between the particles that are required by the antisymmetric character of the fermionic state do not contribute to the state's amount of entanglement [1–15]. This means that the separable (that is, non-entangled) pure states of N fermions are those having Slater rank 1. These are the states whose wave function can be expressed (with respect to an appropriate single particle basis) as a single Slater determinant [3]. On the other hand, the set of mixed non-entangled states comprises those states that can be written as a statistical mixture of pure states of Slater rank 1. Here, when discussing systems of identical fermions, we are considering entanglement between particles and not entanglement between modes.

In the case of pure states of two identical fermions, necessary and sufficient separability criteria can be formulated in terms of the entropy of the single particle reduced density matrix [4, 6, 15]. Alas, no such criteria are known for general, mixed states of two fermions, except for the case of two fermions with a single particle Hilbert space of dimension 4, for which a closed analytical expression for the concurrence (akin to the celebrated Wootters' formula for two-qubits [16]) is known. In general, to determine whether a given density matrix of a two-fermion system represents a separable state or not is a notoriously difficult (and largely unexplored) problem. Consequently, there is a clear need for practical

separability criteria, or entanglement indicators, which can be extended to systems of higher dimensionality or to scenarios involving more than two fermions [15].

Entropic separability criteria have played a distinguished role in the study of the entanglement-related features of mixed states of multipartite systems constituted by distinguishable subsystems [17–23]. For this kind of composite quantum systems, non-entangled states behave classically in the sense that the entropy of a subsystem is always less or equal than the entropy of the whole system. If the entropy of a subsystem happens to be larger than the entropy of the whole system, then we know for sure that the state is entangled (that is, this constituteas a sufficient entanglement criteria). This statement can be formulated mathematically in terms of the Rényi entropic measures,

$$S_q^{(R)}[\rho] = \frac{1}{1-q} \ln(\mathrm{Tr}[\rho^q]), \tag{1}$$

leading to the following family of inequalities satisfied by separable states [17–23],

$$\begin{aligned} S_q^{(R)}[\rho_A] &\leq S_q^{(R)}[\rho_{AB}] \\ S_q^{(R)}[\rho_B] &\leq S_q^{(R)}[\rho_{AB}]. \end{aligned} \tag{2}$$

In the above equations ρ_{AB} is the joint density matrix describing a bipartite system consisting of the subsystems A and B, and $\rho_{A,B}$ are the marginal density matrices describing the subsystems. The entropic parameter in (1-2) adopts values $q \geq 1$. In the limit $q \to 1$ the Rényi entropy reduces to the von Neumann entropy. Note that the entropic criteria considered in [17–23] and in the present work, which depend on the entropies of the total and reduced density matrices, are different from those studied in [24], which involve entropic uncertainty relations associated with the measurement of particular observables.

The study of entropic entanglement criteria based upon the above considerations has been the focus of a considerable amount of research over the years [17–23]. It would be interesting to extend this approach to systems consisting of identical fermions. The aim of this paper is to investigate entanglement criteria for general (mixed) states of systems of two identical fermions based upon the comparison of the entropy of the global density matrix describing the total system and the entropy of the one particle reduced density

matrix.

The organization of the paper is as follows. A brief review of entanglement between particles in systems of identical fermions is given in Section II. Entropic entanglement criteria for systems of two identical fermions based on the von Neumann, the linear, and the Rényi entropies are derived in Section III. These entropic criteria are applied to particular families of states of two-fermion systems in Sections IV and V. The extension to systems of N fermions of the entanglement criteria based upon the Rényi entropies is considered in Section VI. Finally, some conclusions are drawn in Section VII.

II. ENTANGLEMENT BETWEEN PARTICLES IN FERMIONIC SYSTEMS

The concept of entanglement between the particles in system of identical fermions is associated with the quantum correlations exhibited by quantum states on top of the minimal correlations due to the indistinguishability of the particles and the anti-symmetric character of fermionic states. A pure state of Slater rank one of N identical fermions (that is, a state that can be described by one single Slater determinant) must be regarded as separable (non-entangled) [2, 3]. The correlations exhibited by such states do not provide a resource for implementing non-classical information transmission or information processing tasks. Moreover, the non-entangled character of states of Slater rank one is consistent with the possibility of assigning complete sets of properties to the parts of the composite system [4]. Consequently, a pure state of two identical fermions of the form

$$|\psi_{sl}\rangle = \frac{1}{\sqrt{2}}\{|\phi_1\rangle|\phi_2\rangle - |\phi_2\rangle|\phi_1\rangle\}, \tag{3}$$

where $|\phi_1\rangle$ and $|\phi_2\rangle$ are orthonormal single-particle states, is regarded as separable.

A pure state $|\psi\rangle$ of a system of N identical fermions has Slater rank 1, and is therefore separable, if and only if

$$\mathrm{Tr}(\rho_1^2) = \frac{1}{\mathrm{N}}, \tag{4}$$

where $\rho_1 = \mathrm{Tr}_{2,\ldots,\mathrm{N}}(\rho)$ is the single particle reduced density matrix, $\rho = |\psi\rangle\langle\psi|$, n is the dimension of the single particle state space and $N \leq n$ [15]. On the other hand, entangled pure states satisfy

$$\frac{1}{n} \leq \mathrm{Tr}(\rho_1^2) < \frac{1}{\mathrm{N}}. \tag{5}$$

Non-entangled mixed states of systems of N identical fermions are those that can be written as a mixture of Slater determinants,

$$\rho_{sl} = \sum_i \lambda_i |\psi_{sl}^{(i)}\rangle\langle\psi_{sl}^{(i)}|, \tag{6}$$

where the states $|\psi_{sl}^{(i)}\rangle$ can be expressed as single Slater determinants, and $0 \leq \lambda_i \leq 1$ with $\sum_i \lambda_i = 1$.

Systems of identical fermions with a single-particle Hilbert space of dimension $2k$ (with $k \geq 2$) can be formally regarded as systems consisting of spin-s particles, with $s = (2k-1)/2$. The members $\{|i\rangle,\ i = 1, \ldots, 2k\}$ of an orthonormal basis of the single particle Hilbert space can be identified with the states $|s, m_s\rangle$, with $m_s = s - i + 1$, $i = 1, \ldots, 2k$. We can use for these states the shorthand notation $\{|m_s\rangle,\ m_s = -s, \ldots, s\}$, because each particular example discussed here will correspond to a given value of k (and s). According to this angular momentum representation, the antisymmetric joint eigenstates $\{|j, m\rangle,\ -j \leq m \leq j,\ 0 \leq j \leq 2s\}$ of the total angular momentum operators J^2 and J_z constitute a basis for the Hilbert space associated with a system of two identical fermions. The antisymmetric states $|j, m\rangle$ are those with an even value of the quantum number j.

A closed analytical expression for the concurrence of general (pure or mixed) states of two identical fermions sharing a single particle Hilbert space of dimension 4 (corresponding to $s = 3/2$) was discovered by Eckert, Schliemann, Bruss, and Lewenstein (ESBL) in [2]. The ESBL concurrence formula is

$$\mathcal{C}_{\mathcal{F}}(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 - \lambda_5 - \lambda_6\}, \tag{7}$$

where the λ_i's are the square roots of the eigenvalues of $\rho\tilde{\rho}$ in descending order of magnitude. Here $\tilde{\rho} = \mathcal{D}\rho\mathcal{D}^{-1}$, with the operator $\mathcal{D}$ given by

$$\mathcal{D} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \mathcal{K}, \tag{8}$$

where $\mathcal{K}$ stands for the complex conjugation operator and (8) is written with respect to the total angular momentum basis, ordered as $|2,2\rangle$, $|2,1\rangle$, $|2,0\rangle$, $|2,\text{-}1\rangle$, $|2,\text{-}2\rangle$ and $i|0,0\rangle$.

In what follows we are going to consider systems comprising a given, fixed number of identical fermions. Therefore, we are going to work within the first quantization formalism.

III. ENTROPIC ENTANGLEMENT CRITERIA FOR SYSTEMS OF TWO IDENTICAL FERMIONS

A. Entanglement Criteria Based on the von Neumann and the Linear Entropies

Let ρ be a density matrix describing a quantum state of two identical fermions and ρ_r be the corresponding single particle reduced density matrix, obtained by computing the partial trace over one of the two particles.

If $\rho = |\psi_{sl}\rangle\langle\psi_{sl}|$, where $|\psi_{sl}\rangle$ represents a separable pure state of the form (3), and

$$S_{\text{vN}}[\rho] = -\text{Tr}(\rho \ln \rho) \tag{9}$$

is the von Neumann entropy of ρ, we have that $S_{\text{vN}}[\rho] = 0$ and $S_{\text{vN}}[\rho_r] = \ln 2$. That is, for separable pure states we have $S_{\text{vN}}[\rho] - S_{\text{vN}}[\rho_r] = -\ln 2$. It then follows from the concavity property of the quantum conditional entropy [25] that, for a separable mixed state ρ of the form (6), $S_{\text{vN}}[\rho] - S_{\text{vN}}[\rho_r] \geq -\ln 2$. Consequently, all separable states (pure or mixed) of a system of two identical fermions satisfy the inequality

$$S_{\text{vN}}[\rho_r] \leq S_{\text{vN}}[\rho] + \ln 2. \tag{10}$$

Hence, if the quantity

$$D_{\rm vN} = S_{\rm vN}[\rho_r] - S_{\rm vN}[\rho] - \ln 2 \tag{11}$$

is positive the state ρ is necessarily entangled. Indeed, in the particular case of pure states this quantity has been used as a measure of entanglement in some applications (see, for instance, [13] and references therein). The inequality (10) can be extended to the more general case of systems of N identical fermions. From an argument similar to the one used to derive (10) it follows that a separable state of N fermions (that is, a state that can be written as a statistical mixture of pure states each having the form of single Slater determinant) satisfies the inequality

$$S_{\rm vN}[\rho_r] \leq S_{\rm vN}[\rho] + \ln N. \tag{12}$$

Consequently, a state of N fermions violating inequality (12) is necessarily entangled. In the case of pure states of N fermions this entanglement criteria reduces to one of the entanglement criteria previously discussed in [15]. The special case of this criterion corresponding to pure states of two fermions was first analyzed in [4]. When deriving the inequalities (10) and (12) we have used the concavity of the quantum conditional entropy. This property is usually discussed in connection with composite systems comprising distinguishable subsystems. However, within the first quantization formalism, any density matrix of two identical fermions has mathematically also the form of a density matrix describing distinguishable subsystems (in fact, it is just a density matrix that happens to be expressible as a statistical mixture of antisymmetric pure states). Consequently, any mathematical property that is satisfied by general density matrices describing distinguishable subsystems is also satisfied by the special subset of density matrices that can describe a system of identical fermions.

An entanglement criterion for states of two fermions similar to the one already discussed can be formulated in terms of the linear entropy,

$$S_L[\rho] = 1 - {\rm Tr}(\rho^2). \tag{13}$$

Given a quantum state ρ of two fermions, let's consider the quantity

$$c[\rho] \; = \; \inf \sum_i p_i c[|\phi_i\rangle], \tag{14}$$

where $c[|\phi_i\rangle] = \sqrt{2\left[1-\mathrm{Tr}[(\rho_r^{(i)})^2]\right]}$, $\rho_r^{(i)}$ is the one particle reduced density matrix corresponding to $|\phi_i\rangle$, $\rho = \sum_i p_i|\phi_i\rangle\langle\phi_i|$, and the infimum is taken over all the possible decompositions of ρ as a statistical mixture $\{p_i, |\phi_i\rangle\}$ of pure states (note that $c[\rho]$ adopts values in the range $[0, \sqrt{2}]$). The quantity defined in (14) satisfies the inequality [26]

$$c[\rho]^2 \geq 2\left[\mathrm{Tr}(\rho^2) - \mathrm{Tr}\left(\rho_r^2\right)\right]. \tag{15}$$

If ρ corresponds to a separable state of the two fermions, we have that $\rho = \sum_i p_i|\psi_{sep}^{(i)}\rangle\langle\psi_{sep}^{(i)}|$ with $c[|\psi_{sep}^{(i)}\rangle] = 1$ for all i. Therefore, for a separable state we have $c[\rho] \leq 1$ and, from (15), $1 \geq (c[\rho])^2 \geq 2\left[\mathrm{Tr}(\rho^2) - \mathrm{Tr}\left(\rho_r^2\right)\right]$. Consequently, separable states (pure or mixed) of a system of two identical fermions comply with the inequality,

$$S_L[\rho_r] \leq S_L[\rho] + \frac{1}{2}. \tag{16}$$

In other words, states for which the quantity

$$D_L = S_L[\rho_r] - S_L[\rho] - \frac{1}{2} \tag{17}$$

is positive are necessarily entangled. In the particular case of pure states of two identical fermions, the positivity of (17) becomes both a necessary and sufficient entanglement criterion ([15] and references therein). Moreover, a quantity basically equal to (17) has been proposed as an entanglement measure for pure states of two fermions and indeed constitutes one of the most useful entanglement measures for these states [8].

B. Entropic Entanglement Criteria Based on the Rényi Entropies

On the basis of the Rényi family of entropies we are going to derive now a generalization of the separability criterion associated with inequality (10). We are going to prove that a (possibly mixed) quantum state ρ of a system of two identical fermions satisfying the inequality

$$S_q^{(R)}[\rho] + \ln 2 < S_q^{(R)}[\rho_r], \tag{18}$$

for some $q \geq 1$, is necessarily entangled. Here $S_q^{(R)}$ stands for the Rényi entropy,

$$S_q^{(R)}[\rho] = \frac{1}{1-q}\ln(\mathrm{Tr}[\rho^q]). \tag{19}$$

The inequality (18) leads to an entropic entanglement criterion that detects entanglement whenever the quantity

$$R_q = S_q^{(R)}[\rho_r] - S_q^{(R)}[\rho] - \ln 2 \tag{20}$$

is strictly positive. In the limit $q \to 1$ the Rényi measure reduces to the von Neumann entropy and we recover the entanglement criterion given by inequality (10). When $q \to \infty$ the Rényi entropy becomes

$$S_\infty^{(R)}[\rho] = -\ln\left(\lambda_{\mathrm{max.}}\right), \tag{21}$$

where $\lambda_{\mathrm{max.}}$ is the largest eigenvalue of ρ. In this limit case, the entropic criterion says that any state satisfying

$$2\,\lambda_{\mathrm{max.}}^{(\rho_r)} < \lambda_{\mathrm{max.}}^{(\rho)} \tag{22}$$

is entangled, where $\lambda_{\mathrm{max.}}^{(\rho)}$ and $\lambda_{\mathrm{max.}}^{(\rho_r)}$ are, respectively, the largest eigenvalues of ρ and ρ_r.

C. Proof of the Entropic Criteria Based on the Rényi Entropies

The following proof is based on the powerful techniques related to the majorization concept [27, 28] that were introduced to the field of quantum entanglement by Nielsen and Kempe in [27]. These authors proved that non-entangled states of quantum systems having distinguishable subsystems are such that the total density matrix is always majorized by the marginal density matrix associated with one of the subsystems. In the case of non-entangled states of a system of identical fermions the total density matrix ρ is not necessarily majorized by the one particle reduced density matrix ρ_r. However, as we are going to prove, *there is still a definite majorization-related relation between ρ and ρ_r that yields a family of inequalities between the Rényi entropies of these two matrices, which leads in turn to a family of entropic entanglement criteria.*

In our proof of the entropic criterion associated with the inequality (18) we are going to use the following fundamental property of quantum statistical mixtures. If $\rho = \sum_i p_i |a_i\rangle\langle a_i| =$

$\sum_j q_j|b_j\rangle\langle b_j|$ are two statistical mixtures representing the same density matrix ρ, then there exists a unitary matrix $\{U_{ij}\}$ such that [27, 29]

$$\sqrt{p_i}|a_i\rangle = \sum_j U_{ij}\sqrt{q_j}|b_j\rangle. \tag{23}$$

Let us now consider a separable state of two identical fermions,

$$\rho = \sum_j \frac{p_j}{2}(|\psi_1^{(j)}\rangle|\psi_2^{(j)}\rangle - |\psi_2^{(j)}\rangle|\psi_1^{(j)}\rangle)(\langle\psi_1^{(j)}|\langle\psi_2^{(j)}| - \langle\psi_2^{(j)}|\langle\psi_1^{(j)}|) \tag{24}$$

where $0 \leq p_j \leq 1$, $\sum_j p_j = 1$ and $|\psi_1^{(j)}\rangle$, $|\psi_2^{(j)}\rangle$ are normalized single-particle states with $\langle\psi_1^{(j)}|\psi_2^{(j)}\rangle = 0$.

Let us consider now a spectral representation

$$\rho = \sum_k \lambda_k|e_k\rangle\langle e_k| \tag{25}$$

of ρ. That is, the $|e_k\rangle$ constitute an orthonormal basis of eigenvectors of ρ and the λ_k are the corresponding eigenvalues. Then, (24) and (25) are two different representations of ρ as a mixture of pure states. Therefore, there is a unitary matrix U with matrix elements $\{U_{kj}\}$ such that

$$\sqrt{\lambda_k}|e_k\rangle = \sum_j U_{kj}\sqrt{\frac{p_j}{2}}(|\psi_1^{(j)}\rangle|\psi_2^{(j)}\rangle - |\psi_2^{(j)}\rangle|\psi_1^{(j)}\rangle). \tag{26}$$

The single particle reduced density matrix corresponding to the two fermions density matrix (24) is

$$\rho_r = \sum_j \frac{p_j}{2}(|\psi_1^{(j)}\rangle\langle\psi_1^{(j)}| + |\psi_2^{(j)}\rangle\langle\psi_2^{(j)}|), \tag{27}$$

admitting a spectral representation

$$\rho_r = \sum_l \alpha_l|f_l\rangle\langle f_l|. \tag{28}$$

We now define,

$$\begin{aligned} q_{2j} = \; q_{2j-1} \; &= \frac{1}{2}p_j \qquad (j = 1, 2, 3, \ldots) \\ |\phi_{2j-1}\rangle \quad &= \quad |\psi_1^{(j)}\rangle \end{aligned} \tag{29}$$

$$|\phi_{2j}\rangle \quad = \quad |\psi_2^{(j)}\rangle \qquad (j = 1, 2, 3, \ldots). \tag{30}$$

Now, since (27) and (28) correspond to two statistical mixtures yielding the same density matrix, there must exist a unitary matrix W with matrix elements $\{W_{jl}\}$ such that,

$$\sqrt{q_i}|\phi_i\rangle = \sum_l W_{il}\sqrt{\alpha_l}|f_l\rangle \qquad (i = 1, 2, 3, \ldots). \tag{31}$$

Now, eq.(26) can be rewritten as

$$\sqrt{\lambda_k}|e_k\rangle = \sum_j U_{kj}\left(\sqrt{q_{2j-1}}|\phi_{2j-1}\rangle|\phi_{2j}\rangle - \sqrt{q_{2j}}|\phi_{2j}\rangle|\phi_{2j-1}\rangle\right). \tag{32}$$

Combining (31) and (32) gives

$$\sqrt{\lambda_k}|e_k\rangle = \sum_l \left[\sum_j U_{kj}\Big(W_{2j-1,l}|\phi_{2j}\rangle - W_{2j,l}|\phi_{2j-1}\rangle\Big)\right]\sqrt{\alpha_l}|f_l\rangle. \tag{33}$$

Therefore, since $\langle e_k|e_{k'}\rangle = \delta_{kk'}$ and $\langle f_l|f_{l'}\rangle = \delta_{ll'}$, we have that

$$\lambda_k = \sum_l M_{kl}\alpha_l, \tag{34}$$

where

$$M_{kl} = \left(\sum_{j'} U^*_{kj'}\left\{W^*_{2j'-1,l}\langle\phi_{2j'}| - W^*_{2j',l}\langle\phi_{2j'-1}|\right\}\right)\left(\sum_{j''} U_{kj''}\left\{W_{2j''-1,l}|\phi_{2j''}\rangle - W_{2j'',l}|\phi_{2j''-1}\rangle\right\}\right). \tag{35}$$

We now investigate the properties of the matrix M with matrix elements $\{M_{kl}\}$. First of all, we have

$$M_{kl} \geq 0, \tag{36}$$

since the matrix elements of M are of the form $M_{kl} = \langle\Sigma|\Sigma\rangle$, with

$$|\Sigma\rangle = \sum_j U_{kj}\left(W_{2j-1,l}|\phi_{2j}\rangle - W_{2j,l}|\phi_{2j-1}\rangle\right). \tag{37}$$

We now consider the sum of the elements within a given row or column of M. The sum of a row yields,

$$\sum_k M_{kl} = \sum_{j'j''} \delta_{j'j''} \left(W^*_{2j'-1,l} \langle \phi_{2j'}| - W^*_{2j',l} \langle \phi_{2j'-1}| \right) \left(W_{2j''-1,l} |\phi_{2j''}\rangle - W_{2j'',l} |\phi_{2j''-1}\rangle \right)$$
$$= \sum_j \left(W^*_{2j-1,l} W_{2j-1,l} + W^*_{2j,l} W_{2j,l} \right) = \sum_i \left(W^\dagger \right)_{li} W_{il} = 1, \tag{38}$$

while the sum of a column is,

$$\sum_l M_{kl} = \sum_{j'j''} U^*_{kj'} U_{kj''} \left(\langle \phi_{2j'} | \phi_{2j''} \rangle \left[\sum_l W^*_{2j'-1,l} W_{2j''-1,l} \right] + \langle \phi_{2j'-1} | \phi_{2j''-1} \rangle \left[\sum_l W^*_{2j',l} W_{2j'',l} \right] \right.$$
$$\left. - \langle \phi_{2j'} | \phi_{2j''-1} \rangle \left[\sum_l W^*_{2j'-1,l} W_{2j'',l} \right] - \langle \phi_{2j'-1} | \phi_{2j''} \rangle \left[\sum_l W^*_{2j',l} W_{2j''-1,l} \right] \right)$$
$$= \sum_{j'j''} U^*_{kj'} U_{kj''} \left(\langle \phi_{2j'} | \phi_{2j''} \rangle \delta_{j'j''} + \langle \phi_{2j'-1} | \phi_{2j''-1} \rangle \delta_{j'j''} \right)$$
$$= 2 \sum_j \left(U^\dagger \right)_{jk} U_{kj} = 2. \tag{39}$$

When deriving the above two equations we made use of the unitarity of the matrices $\{U_{kj}\}$ and $\{W_{il}\}$. Summing up, we have,

$$\sum_k M_{kl} = 1$$
$$\sum_l M_{kl} = 2. \tag{40}$$

We now define a new set of variables $\{\lambda'_i\}$ and a new matrix M' with elements M'_{ij}, respectively given by,

$$\lambda'_{2k-1} = \lambda'_{2k} = \frac{1}{2} \lambda_k \qquad (k = 1, 2, 3, \ldots) \tag{41}$$
$$M'_{2k-1,l} = M'_{2k,l} = \frac{1}{2} M_{kl} \qquad (k = 1, 2, 3, \ldots), \tag{42}$$

and so we have

$$\lambda'_n = \sum_l M'_{nl} \alpha_l. \tag{43}$$

By construction, then, we have

$$\{\lambda_k\} = \{\lambda_1, \lambda_2, \lambda_3, \ldots\}$$

$$\{\lambda'_n\} \;=\; \left\{\frac{\lambda_1}{2}, \frac{\lambda_1}{2}, \frac{\lambda_2}{2}, \frac{\lambda_2}{2}, \frac{\lambda_3}{2}, \frac{\lambda_3}{2}, \ldots\right\}. \tag{44}$$

Let us now compare the matrices $\{M_{kl}\}$ and $\{M'_{nl}\}$. The matrix $\{M'_{nl}\}$ has twice as many rows as $\{M_{kl}\}$, but the rows of $\{M'_{nl}\}$ can be grouped in pairs of consecutive rows such that within each pair the rows are equal to 1/2 a row of $\{M_{kl}\}$. It follows that

$$\begin{aligned} \sum_k M_{kl} = 1 \quad &\Longrightarrow \quad \sum_n M'_{nl} = 1 \\ \sum_l M_{kl} = 2 \quad &\Longrightarrow \quad \sum_l M'_{nl} = 1. \end{aligned} \tag{45}$$

Thus,

$$\sum_n M'_{nl} = \sum_l M'_{nl} = 1 \tag{46}$$

and, therefore, $\{M'_{nl}\}$ is a doubly stochastic matrix. Interpreting the λ'_n's and the α_l's as probabilities, it follows from (43) and (46) that the probability distribution $\{\lambda'_n\}$ is more "mixed" than the probability distribution $\{\alpha_l\}$ [25] (or, alternatively that $\{\alpha_l\}$ majorizes $\{\lambda'_n\}$ [27]). This, in turn, implies that for any Rényi entropy $S_q^{(R)}$ with $q \geq 1$, we have

$$S_q^{(R)}[\lambda'_n] \geq S_q^{(R)}[\alpha_l]. \tag{47}$$

Thus,

$$\begin{aligned} S_q^{(R)}[\lambda'_n] \;&=\; \frac{1}{1-q} \ln\left(2 \sum_k \left(\frac{\lambda_k}{2}\right)^q\right) \\ &=\; \ln 2 + S_q^{(R)}[\lambda_k]. \end{aligned} \tag{48}$$

Therefore, all separable states of the two-fermion system comply with the inequality $S_q^{(R)}[\lambda_k] + \ln 2 \geq S_q^{(R)}[\alpha_l]$ and since $\{\lambda_k\}$ and $\{\alpha_l\}$ are the eigenvalues of ρ and ρ_r respectively,

$$S_q^{(R)}[\rho] + \ln 2 \geq S_q^{(R)}[\rho_r]. \tag{49}$$

The above inequality leads to an entanglement criterion that detects entanglement when the indicator R_q defined in equation (19) is strictly positive.

IV. TWO-FERMION SYSTEMS WITH A SINGLE PARTICLE HILBERT SPACE OF DIMENSION FOUR

Now we are going to apply our above derived entropic entanglement criteria to some parameterized families of two-fermion states. To this end, consider systems consisting of two fermions with a single particle Hilbert space of dimension 4. In this case there is an exact, analytical expression for the state's concurrence. It is then possible to compare the range of parameters for which entanglement is detected by the criteria with the exact range of parameters for which the states under consideration are entangled. As mentioned in Section II, in this case the two-fermions states can be mapped into the states of two $s = \frac{2}{3}$ spins. The antisymmetric eigenstates $|j, m\rangle$ of the total angular momentum operators J^2 and J_z constitute then a basis of the system's Hilbert space. These states are $|0, 0\rangle$, $|2, -2\rangle$, $|2, -1\rangle$, $|2, 0\rangle$, $|2, 1\rangle$, and $|2, 2\rangle$.

A. Werner-Like States

First we are going to consider a family of states consisting of a mixture of the maximally entangled state $|0, 0\rangle$ and a totally mixed state. These states are of form,

$$\rho_W = p|0,0\rangle\langle 0,0| + \frac{1-p}{6}\, I \tag{50}$$

where $0 \leq p \leq 1$, and

$$I \;=\; |0,0\rangle\langle 0,0| + \sum_{m=-2}^{2} |2,m\rangle\langle 2,m| \tag{51}$$

is the identity operator acting on the six-dimensional Hilbert space corresponding to the two-fermion system. Evaluation of the concurrence shows that these states are entangled when $p > 0.4$. For these states we have,

$$\begin{aligned} D_{\mathrm{vN}}[\rho_W] \;&=\; -\ln 2 + \ln 4 - \frac{5}{6}(-1+p)\ln\left(\frac{1-p}{6}\right) + \frac{1}{6}(1+5p)\ln\left(\frac{1}{6}(1+5p)\right) \\ D_L[\rho_W] \;&=\; -\frac{7}{12} + \frac{5p^2}{6}. \end{aligned} \tag{52}$$

The minimum values p_m of the parameter p such that for $p > p_m$ the entanglement indicators D_{vN}, D_L, R_2, and $R_{q\to\infty}$ are positive (and thus entanglement is detected by the corresponding

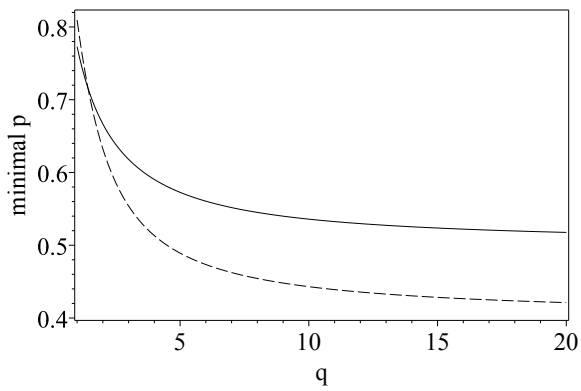

FIG. 1: Minimum p-value p_{min} for which entanglement is detected in the case of the state ρ_W defined in eq.(50) (dashed line) and of the state ρ_G given by eq.(55) (solid line).

criteria) are given in the following table (that is, in each case, entanglement is detected when p is larger than the listed value).

	$D_{\rm vN} > 0$	$D_L > 0$	$R_{q\to\infty} > 0$	$R_{q=2} > 0$
p_{min}	≈ 0.809	$\sqrt{0.7} \approx 0.837$	0.4	≈ 0.632

The entanglement detection efficiency of the entropic criterion based upon Rényi entropy increases with q. Indeed, in the limit $q \to \infty$ the Rényi entropic criterion detects all the entangled states within the family of states (50). The behaviour of the minimum value of p for which entanglement is detected as a function of the entropic parameter q is depicted in Figure 1.

B. θ-State

As second illustration we consider the following pure state,

$$|\psi\rangle = \frac{\sin\theta}{\sqrt{2}}\left[\left|-\frac{3}{2}\frac{3}{2}\right\rangle - \left|\frac{3}{2}-\frac{3}{2}\right\rangle\right] + \frac{\cos\theta}{\sqrt{2}}\left[\left|-\frac{1}{2}\frac{1}{2}\right\rangle - \left|\frac{1}{2}-\frac{1}{2}\right\rangle\right], \tag{53}$$

for which

$$D_{\rm vN}[|\psi\rangle\langle\psi|] \;=\; -\ln 2 - \cos^2\theta \ln\left[\frac{\cos^2\theta}{2}\right] - \ln\left[\frac{\sin^2\theta}{2}\right]\sin^2\theta$$

$$D_L[|\psi\rangle\langle\psi|] \;=\; \cos^2\theta\sin^2\theta. \tag{54}$$

Thus, $D_{\rm vN}$, $D_L = 0$ for $\theta = 0, \pi/2, \pi$. So $S_q^{(R)}[\rho] + \ln 2 < S_q^{(R)}[\rho_r]$ for all $\theta \in (0,\pi), \theta \neq \pi/2$. Therefore, all entangled states are detected.

C. Gisin-Like States

As a final example let us consider the parameterized family of mixed states given by,

$$\rho_G = p|0,0\rangle\langle 0,0| + \frac{1-p}{2}\left(|2,-2\rangle\langle 2,-2| + |2,2\rangle\langle 2,2|\right), \tag{55}$$

with $0 \leq p \leq 1$. In this case we have,

$$\begin{aligned} D_{\rm vN}[\rho_G] &= -(-1+p)\ln(1-p) + p\ln(2p) \\ D_L[\rho_G] &= \frac{1}{4}\left(-1-4p+6p^2\right). \end{aligned} \tag{56}$$

The critical p values at which the entropic criteria based on the indicators $D_{\rm vN}$, D_L, $R_{q\to\infty}$, and $R_{q=2}$ begin to detect entanglement are listed in the Table bellow.

	$D_1 > 0$	$D_2 > 0$	$R_{q\to\infty} > 0$	$R_{q=2} > 0$
p_{min}	≈ 0.773	$\frac{2+\sqrt{10}}{6} \approx 0.860$	0.5	≈ 0.667

From the evaluation of the concurrence it follows that the Gisin-like states are entangled for $p > 0.5$. Thus, once again, the Rényi based entropic criterion based on the indicator $R_{q\to\infty}$ detects all the entangled states in the family (55).

V. TWO-FERMION SYSTEMS WITH A SINGLE PARTICLE HILBERT SPACE OF DIMENSION SIX

Two identical fermions with a 4-dimensional single particle Hilbert space (the simplest fermionic system admitting the phenomenon of entanglement) constitutes the only fermion system for which an exact analytical formula for the concurrence has been obtained. It is thus of interest to apply the entropic entanglement criteria to systems of higher dimensionality, for which such an expression for the concurrence is not known. Here we are going to consider a system consisting of two identical fermions with a single particle Hilbert space

of dimension 6. The Hilbert space of this system is 15-dimensional. Using the angular momentum representation the two-fermion system can be mapped onto a system of two spins with $s = \frac{5}{2}$. It is useful to introduce the following notation,

$$|m_1 m_2| = \frac{1}{\sqrt{2}} \left[|m_1\rangle |m_2\rangle - |m_2\rangle |m_1\rangle \right]. \tag{57}$$

We are going to study three particular families of mixed states of the form

$$\rho_i = p|\varphi_i\rangle\langle\varphi_i| + \frac{1-p}{15} I, \tag{58}$$

where $0 \leq p \leq 1$ and

$$I \;=\; |0,0\rangle\langle 0,0| + \sum_{m=-2}^{2} |2,m\rangle\langle 2,m| + \sum_{m=-4}^{4} |4,m\rangle\langle 4,m| \tag{59}$$

is the identity operator acting on the 15-dimensional Hilbert space describing the two-fermion system, and $|\varphi_i\rangle$ is an entangled two-fermion pure state. We consider three particular instances of $|\varphi_i\rangle$. In each case we provide the expressions for the indicators $D_{\rm vN}$ and D_L, and the minimum values p_m of the parameter p such that for $p > p_m$ entanglement is detected by the criteria based on the positivity of the quantities $D_{\rm vN}$, D_L, $R_{q\to\infty}$ and $R_{q=2}$.

The first illustration corresponds to

$$|\varphi_1\rangle = \frac{1}{\sqrt{3}} \left[\left|\frac{5}{2}\frac{3}{2}\right| + \left|\frac{1}{2}\text{-}\frac{1}{2}\right| - \left|\text{-}\frac{3}{2}\text{-}\frac{5}{2}\right| \right], \tag{60}$$

for which

$$\begin{aligned} D_{vN}[\rho_1] \;&=\; \ln 3 - \frac{14}{15}(-1+p)\ln\left(\frac{1-p}{15}\right) + \frac{1}{15}(1+14p)\ln\left(\frac{1}{15}(1+14p)\right) \\ D_L[\rho_1] \;&=\; \frac{1}{15}\left(-9+14p^2\right), \end{aligned} \tag{61}$$

resulting in

	$D_{vN} > 0$	$D_L > 0$	$R_{q\to\infty} > 0$	$R_{q=2} > 0$
p_{min}	≈ 0.767	$\frac{3}{\sqrt{14}} \approx 0.802$	$\frac{2}{7}$	≈ 0.535

The second example is given by

$$|\varphi_2\rangle = -\frac{2}{3}\left|\frac{5}{2}\frac{3}{2}\right| - \frac{2}{3}\left|\frac{1}{2}\text{-}\frac{1}{2}\right| + \frac{1}{3}\left|\text{-}\frac{3}{2}\text{-}\frac{5}{2}\right|, \tag{62}$$

with,

$$
\begin{aligned}
D_{vN}[\rho_2] &= \frac{1}{45}\left(-45\ln 2 - 42(-1+p)\ln\left(\frac{1-p}{15}\right) + 5(-3+2p)\ln\left(\frac{1}{6}-\frac{p}{9}\right) - 10(3+p)\ln\left(\frac{3+p}{18}\right)\right. \\
&\quad \left. +3(1+14p)\ln\left(\frac{1}{15}(1+14p)\right)\right) \\
D_L[\rho_2] &= -\frac{3}{5} + \frac{121p^2}{135},
\end{aligned} \tag{63}
$$

and

	$D_{vN} > 0$	$D_L > 0$	$R_{q\to\infty} > 0$	$R_{q=2} > 0$
p_{min}	≈ 0.788	$\frac{9}{11}$	≈ 0.324	≈ 0.557

As a third instance we tackle,

$$
|\varphi_3\rangle = \frac{1}{\sqrt{2}}\left[\left|\frac{5}{2}\frac{3}{2}\right| + \left|\frac{1}{2}\text{-}\frac{1}{2}\right|\right], \tag{64}
$$

leading to,

$$
\begin{aligned}
D_{vN}[\rho_3] &= \frac{1}{15}\left(-p\ln 7776 + p\ln 248832 - 9(-1+p)\ln(1-p) - 5(2+p)\ln(2+p)\right. \\
&\quad \left. + \ln\left(\frac{1024(1+14p)}{30517578125}\right) + 14p\ln(1+14p)\right) \\
D_L[\rho_3] &= -\frac{3}{5} + \frac{17p^2}{20},
\end{aligned} \tag{65}
$$

and

	$D_{vN} > 0$	$D_L > 0$	$R_{q\to\infty} > 0$	$R_{q=2} > 0$
p_{min}	≈ 0.825	$2\sqrt{\frac{3}{17}} \approx 0.840$	≈ 0.348	≈ 0.590

VI. SYSTEMS OF N IDENTICAL FERMIONS

Let us consider the general case of N fermions with single particle Hilbert space of general (even) dimension $n > N$. The dimension of the Hilbert space associated with the N-fermion system is then $d = \frac{n!}{(n-N)!N!}$. The Rényi based entropic criterion for two fermions that we derived in Section III can be extended to the case of N fermions. According to the extended criterion a state ρ of N identical fermions satisfying the inequality

$$
S_q^{(R)}[\rho_r] > S_q^{(R)}[\rho] + \ln N, \tag{66}
$$

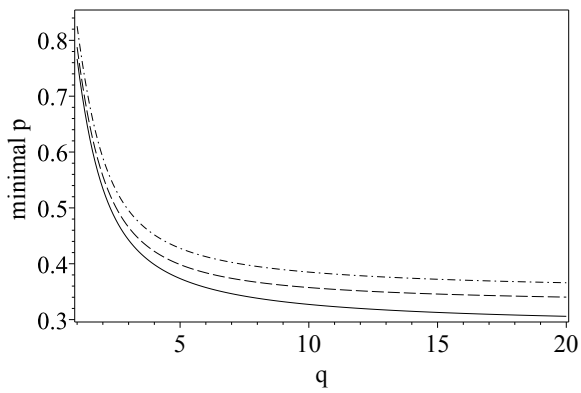

FIG. 2: Minimum value of p, as a function of the entropic parameter q, for entanglement detection in the states (58) with $|\varphi_1\rangle$ (solid line), $|\varphi_2\rangle$ (dashed line) and $|\varphi_3\rangle$ (dashdotted line).

for some $q \geq 1$ is necessarily entangled. This criterion can be derived following a procedure similar to the one detailed in Section III for the case of two fermions.

As an illustration of the entanglement criterion based on the inequality (66) let us consider a family of states of a system of N fermions having the form

$$p\,|\Phi\rangle\langle\Phi| + \frac{(1-p)}{d}\,I_d, \tag{67}$$

where $0 \leq p \leq 1$, I_d is the identity operator acting on the N-fermions Hilbert space, and the single particle Hilbert space has dimension $n = kN$, with $k \geq 2$ integer. We also assume that the (pure) N-fermion state $|\Phi\rangle$ is of the form

$$|\Phi\rangle = \frac{1}{\sqrt{k}}\Big(|1,2,\ldots,N| + |N+1,N+2,\ldots,2N| + \ldots + |(k-1)N+1,(k-1)N+2,\ldots,kN|\Big), \tag{68}$$

where $|i_1, i_2, \ldots, i_N|$ denotes the Slater determinant (as in equation (57)) constructed with N different members $\{|i_1\rangle, \ldots, |i_N\rangle\}$ of an orthonormal basis $\{|1\rangle, \ldots, |n\rangle\}$ of the single particle Hilbert space. The single particle, reduced density matrix associated with the (pure) state $|\Phi\rangle$ corresponds to the totally mixed (single particle) state, $\frac{1}{n}I_n$, where I_n is the identity operator corresponding to the single particle Hilbert space. On the basis of the Rényi entropic criterion corresponding to $q \to \infty$ we identify as entangled the states of the form

(67) satisfying the inequality,

$$\ln n + \ln\left(p + \frac{(1-p)}{d}\right) - \ln N > 0 \tag{69}$$

and hence entanglement is detected for

$$p \; > \; \frac{N\,(n-1)! - (n-N)!\,N!}{n! - (n-N)!\,N!}. \tag{70}$$

With N fixed, we find that the efficiency of the entanglement criterion grows as the dimension of the single particle states, n, increases (that is, p_{min} decreases with n).

VII. SUMMARY

In the present work new entropic entanglement criteria for systems of two identical fermions have been advanced. These criteria have the form of appropriate inequalities involving the entropy of the density matrix associated with the total system, on the one hand, and the entropy of the single particle reduced density matrix, on the other one. We obtained entanglement criteria based upon the von Neumann, the linear, and the Rényi entropies. The criterion associated with the von Neumann entropy constitutes a special instance, corresponding to the particular value $q = 1$ of the Rényi entropic parameter, of the more general criteria associated with the Rényi family of entropies. Extensions of these criteria to systems constituted by N identical fermions where also considered.

We applied our entanglement criteria to various illustrative examples of parametrized families of mixed states, and studied the dependence of the entanglement detection efficiency on the entropic parameter q. The entanglement criterion improves as q increases and is the most efficient in the limit $q \to \infty$.

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