Narrow banded wave propagation from very deep waters to the shore

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Abstract

A fully nonlinear Boussinesq-type model with several free coefficients is considered as a departure point. The model is monolayer and low order so as to simplify numerical solvability. The coefficients of the model are here considered functions of the local water depth. In doing so, we allow to improve the dispersive and shoaling properties for narrow banded wave trains in very deep waters. In particular, for monochromatic waves the dispersion and shoaling errors are bounded by $\simeq 2.8\%$ up to $kh = 100$, being k the wave number and h the water depth. The proposed model is fully nonlinear in weakly dispersive conditions, so that nonlinear wave decomposition in shallower waters is

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well reproduced. The model equations are numerically solved using a fourth order scheme and tested against analytical solutions and experimental data.
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1. Introduction

In deep waters water wave propagation does not depend on water depth. For instance, the wave celerity c for a wave with period T is $c = gT/2\pi$, with g the gravity acceleration. Because each wave period, T , travels with a different velocity, deep waters are called *dispersive*. Furthermore, in deep waters the wave amplitude, a , is usually much smaller than the water depth h and, as a consequence, the model equations are linear (Airy theory).

As the water waves propagate to the coast, the water depth h decreases and the wave propagation becomes influenced by it. Also, nonlinear effects become important. In shallow waters, where the wave propagation is dominated by the water depth, the wave celerity is given by $c \approx \sqrt{gh}$, which is independent of the wave period (*i.e.*, *non dispersive*). An important physical property of shallow waters is that the horizontal velocity profile is nearly uniform in the vertical. Nonlinear Shallow Waters Equations (NSWEs), which are vertically integrated, exploit this property and are valid for non dispersive conditions and for arbitrary amplitudes of the wave.

It is accepted that shallow water conditions correspond to $kh \lesssim 0.3$, with $k = 2\pi/\lambda$ the wave number and λ the wave length, while $kh \gtrsim 3$ corresponds to deep waters (Dean and Dalrymple, 1984). In intermediate waters (namely $0.3 \lesssim kh \lesssim 3$) nonlinearity and dispersion coexist and neither Airy theory

nor NSWE can represent the physics. To overcome this problem, two main perturbation approaches are found (Dingemans, 1997). On the one hand, Stokes theory departs from the fully dispersive linear Airy theory to incorporate weakly nonlinear effects. On the other, Boussinesq-Type Equations (BTEs) depart from NSWs and include weakly dispersive effects. This work is focused on BTEs.

Being a_0 , h_0 and k_0 characteristic values for wave amplitude, water depth and wave number respectively, the dimensionless parameters

$$\epsilon \equiv \frac{a_0}{h_0}, \quad \text{and} \quad \mu \equiv k_0 h_0, \quad (1)$$

represent nonlinear and dispersive effects respectively. The NSWs can represent fully nonlinear waves for the nondispersive case. The original BTEs by Peregrine (1967) included all the nonlinear non dispersive terms (NSWs) plus the weakly nonlinear and weakly dispersive terms $\mathcal{O}(\epsilon^1 \mu^2)$, but disregarded the highly nonlinear and weakly dispersive terms $\mathcal{O}(\epsilon^2 \mu^2, \epsilon^3 \mu^2)$. The inclusion of the highly nonlinear and weakly dispersive terms $\mathcal{O}(\epsilon^2 \mu^2, \epsilon^3 \mu^2)$ was done, *e.g.*, by Green and Naghdi (1976) and Wei et al. (1995).

The equations by Peregrine (1967) were derived for the depth averaged horizontal velocity, $\bar{\mathbf{u}}$, and give *good* linear dispersive performance, *i.e.*, errors below 1% relative to Airy's celerity, up to $kh \lesssim 1.1$. To improve the range of applicability, several approaches are found in the literature. Two of them are higher order and multilayer models: higher order models include terms $\mathcal{O}(\mu^4)$ or higher (Gobbi et al., 2000), while the multilayer models split the flow into several layers, applying low order models into each one (Lynett and Liu, 2004). These two kind of models increase the numerical complexity for

44 they include fifth order derivatives or more unknowns.

45 Based on the method of Agnon et al. (1999), Madsen et al. (2002) devel-
46 oped a fully nonlinear model, which is accurate in very deep water ($kh \lesssim 40$
47 for the linear case). Their model requires more differential equations to be
48 solved, compared to other higher order models such as that by Gobbi et al.
49 (2000), and the highest order of derivatives in the model is also fifth. Madsen
50 et al. (2003) presented a simplified version of their original model, where the
51 highest order of derivatives is reduced to three. The range of application is
52 also reduced to $kh \lesssim 10$.

53 Using a low order monolayer model, Nwogu (1993) improved the linear
54 dispersive performance up to $kh \lesssim 3.3$ by using the horizontal velocity $\mathbf{u}_\alpha$ at
55 $z = z_\alpha$ instead of the depth averaged velocity proposed by Peregrine (1967).
56 In fact, the above mentioned models by Wei et al. (1995) and Lynett and
57 Liu (2004), amongst other, follow this idea to improve the linear dispersion
58 performance.

59 Following the track of low order monolayer BTEs, Madsen and Schaffer
60 (1998) modified the model equations by Wei et al. (1995) by introducing new
61 terms which included free coefficients. While the equations remain exact up
62 to $\mathcal{O}(\mu^2)$, similar to those by Wei et al. (1995), for the proposed coefficients
63 they obtained errors in linear dispersion below 1% for $kh \lesssim 6.2$.

64 Although the improvements in linear dispersion, *i.e.*, in the representation
65 of the wave celerity, provided by Nwogu (1993) or Madsen and Schaffer (1998)
66 are substantial, it is not generally so with the linear shoaling, *i.e.*, with the
67 representation of the wave amplitude: the linear shoaling by the equations
68 by Nwogu (1993) and Madsen and Schaffer (1998) is fair (1% error in wave

amplitude) only up to $kh \approx 0.78$ and $kh \approx 0.82$ respectively, and at $kh = 2$ the errors are already above 7.4% in both cases.

Other approaches have considered the improvement of linear properties to arbitrary depths which are to mention. Beji and Nadaoka (1999) followed a different approach also for narrow banded wave trains, whereas Karambas and Memos (2009) reached fully dispersion (*i.e.*, for arbitrary depths and arbitrary ranges of frequencies). In both cases, however, the models do not allow fully nonlinearities in weakly dispersive conditions.

Departing from Madsen and Schaffer's equations and using an optimization approach, Galan et al. (2012) reduced the errors to 0.3% *both* in linear dispersion and shoaling up to $kh \approx 5$. Further, Galan et al. (2012) equations also include new terms $\mathcal{O}(\epsilon^1 \mu^4)$ to improve the weakly nonlinear and highly dispersive properties.

All the above works consider that the free coefficients introduced are constant. In the present work we consider that these coefficients are functions of the water depth. In this way we will be able to improve the model properties up to deeper waters. As a counterpart, we will require that the wave train travelling to the coast is, in deep waters, narrow banded in frequencies. Narrow banded wave trains are associated to long fetchs (swells), and hence its usefulness. Further, in weakly dispersive conditions the equations will remain fully nonlinear.

This work can be considered as an extension of that by Lee et al. (2003) for the propagation of monochromatic waves in deep waters. The work by Lee et al. departed from Wei et al. (1995) equations, having only one free parameter and only monochromatic waves could be represented (strickly speaking,

only linear dispersion could be well represented). Because we have more free parameters available, we will be able to propagate waves within a range of frequencies.

2. Governing equations

The fully nonlinear BTEs by Galan et al. (2012), hereafter G12, are

$$\begin{aligned}
X - X_* + \nabla \cdot [d_{1\alpha} h^2 \nabla X + d_{2\alpha} h^3 \nabla Y] \\
+ \nabla \cdot \left[\left(c_{1\alpha} h - \frac{\eta}{2} \right) \eta \nabla X + \left(c_{2\alpha} h^2 - \frac{\eta^2}{6} \right) \eta \nabla Y \right] \\
+ (\delta - \delta_h) \nabla \cdot [h^2 \nabla (X - X_*)] + \delta_h \nabla^2 [h^2 (X - X_*)] \\
+ \delta_\epsilon \nabla \cdot [h \eta \nabla (X - X_*)] = 0, \quad (2a)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{Z} - \mathbf{Z}_* + c_{1\alpha} h \nabla \nabla \cdot (h \mathbf{Z}) + c_{2\alpha} h^2 \nabla \nabla \cdot \mathbf{Z} - \nabla \left[\eta \nabla \cdot (h \mathbf{Z}) + \frac{\eta^2}{2} \nabla \cdot \mathbf{Z} \right] \\
+ \nabla \left[(c_{1\alpha} h - \eta) \mathbf{u} \cdot \nabla X + \left(c_{2\alpha} h^2 - \frac{\eta^2}{2} \right) \mathbf{u} \cdot \nabla Y + \frac{(X + \eta Y)^2}{2} \right] \\
+ (\gamma - \gamma_h) h^2 \nabla \nabla \cdot (\mathbf{Z} - \mathbf{Z}_*) + \gamma_h h \nabla \nabla \cdot (h (\mathbf{Z} - \mathbf{Z}_*)) \\
- \gamma_\epsilon \nabla [\eta \nabla \cdot (h (\mathbf{Z} - \mathbf{Z}_*))] = 0, \quad (2b)
\end{aligned}$$

with η the free surface elevation, $\mathbf{u}$ the horizontal velocity evaluated at $z =$

$z_\alpha = \alpha h$, $Y \equiv \nabla \cdot \mathbf{u}$ and

	W95	M98	G12
α	-0.53096	-0.54122	-0.54217
δ	—	-0.03917	-0.02409
γ	—	-0.01052	-0.00492
δ_h	—	-0.14453	-0.15530
γ_h	—	-0.02153	-0.07897
δ_ϵ	—	—	-0.36052
γ_ϵ	—	—	0.13169

Table 1: Constant coefficients for Wei et al. (1995), Madsen and Schaffer (1998) and Galan et al. (2012), denoted respectively as W95, M98 and G12.

$$\mathbf{X} \equiv \nabla \cdot (h\mathbf{u}), \quad \mathbf{Z} \equiv \mathbf{u}_t, \quad (3a)$$

$$\mathbf{X}_* \equiv -\eta_t - \nabla \cdot (\eta\mathbf{u}), \quad \mathbf{Z}_* \equiv -\frac{1}{2}\nabla (\mathbf{u} \cdot \mathbf{u}) - g\nabla\eta, \quad (3b)$$

with g is the gravity acceleration. In equations (2)

$$c_{1\alpha} \equiv \alpha, \quad c_{2\alpha} \equiv \frac{\alpha^2}{2}, \quad d_{1\alpha} \equiv \alpha + \frac{1}{2}, \quad d_{2\alpha} \equiv \frac{\alpha^2}{2} - \frac{1}{6}, \quad (4)$$

where α is a free coefficient, as well as δ , γ , δ_h , γ_h , δ_ϵ and γ_ϵ . Table 1 shows the values by Galan et al. (2012), hereafter “G12”, and also the ones required to recover the equations by Madsen and Schaffer (1998) and Wei et al. (1995).

The equations (2) are obtained using an asymptotic expansion in kh and are exact up to $\mathcal{O}((kh)^2)$. No limitations are imposed on the nonlinearity, so that they can represent fully nonlinear waves up to order $\mathcal{O}((kh)^2)$. For $kh \rightarrow 0$ they become independent of the coefficients and tend to the exact

110 shallow water equations. The weighting coefficients influence the behavior of
 111 the equations only in deeper waters. Being more specific, the linear dispersion
 112 is influenced only by α , δ and γ , coefficients δ_h and γ_h only influence the linear
 113 shoaling and the coefficients δ_ϵ and γ_ϵ affect only the nonlinear performance.
 114 All seven coefficients have been chosen so as to improve the linear and weakly
 115 nonlinear performance in deeper waters.

116 As shown by G12 for constant coefficients, the linear dispersion relation-
 117 ship embedded in the above equations (2) is

$$\left\{ \frac{c_{\text{bte}}^2}{gh} = \right\} \frac{\omega^2}{gk_{\text{bte}}^2 h} = \frac{1 - (d_\alpha + \gamma + \delta) (k_{\text{bte}} h)^2 + (d_\alpha + \delta) \gamma (k_{\text{bte}} h)^4}{1 - (c_\alpha + \gamma + \delta) (k_{\text{bte}} h)^2 + (c_\alpha + \gamma) \delta (k_{\text{bte}} h)^4}, \quad (5)$$

118 where c_{bte} wave celerity corresponding to these BTEs, k_{bte} the corresponding
 119 wave number, ω the wave angular frequency, $c_\alpha \equiv c_{\alpha,1} + c_{\alpha,2} = \alpha^2/2 + \alpha$ and
 120 $d_\alpha \equiv d_{\alpha,1} + d_{\alpha,2}$. The exact Airy dispersion expression is

$$\left\{ \frac{c_{\text{Airy}}^2}{gh} = \right\} \frac{\omega^2}{gk_{\text{Airy}}^2 h} = \frac{\tanh(k_{\text{Airy}} h)}{k_{\text{Airy}} h}. \quad (6)$$

121 For given values of gravity acceleration g , water depth h , wave angular
 122 frequency ω and the three coefficients α , δ and γ , the values of k_{bte} and k_{Airy}
 123 obtained from the equations (5) and (6) are different in general, thus giving
 124 an error in the wave celerity (linear dispersion). Figure 1 shows the error in
 125 linear dispersion, defined as

$$\varepsilon_c \equiv \frac{c_{\text{bte}}}{c_{\text{Airy}}} - 1 \left\{ = \frac{k_{\text{Airy}}}{k_{\text{bte}}} - 1 \right\}, \quad (7)$$

126 as a function of the dimensionless group $\kappa \equiv \omega^2 h / g$. This group, κ , can be
 127 used as a k -independent alternative to $\xi \equiv kh$ to evaluate whether deep or

128 shallow waters hold (Nwogu, 1993). It has the advantage of not introducing
 129 k , which is different depending on whether equation (5) or (6) are used. For
 130 Airy theory $\kappa = \xi \tanh \xi$, and therefore, $\kappa \approx \xi$ for $\xi \gtrsim 3$.

Figure 1: Errors ε_c and ε_s for G12 (full lines), M98 (dashed) and W95 (dash-dotted).
 Shoaling errors, ε_s , are denoted with symbols.

131 Figure 1 also includes the error in the representation of wave amplitude
 132 assuming mild slope conditions. This error is defined as (Chen and Liu, 1995)

$$\varepsilon_s \equiv \exp \left(\int_0^h \frac{\alpha_{\eta, \text{Airy}} - \alpha_{\eta, \text{bte}}}{h_*} dh_* \right) - 1, \quad (8)$$

133 where $\alpha_{\eta, \text{Airy}} (\kappa_* \equiv \omega^2 h_* / g)$ and $\alpha_{\eta, \text{bte}} (\kappa_*, \alpha, \delta, \gamma, \delta_h, \gamma_h)$ are the shoaling gra-
 134 dients for Airy's and above BTEs (Galan et al., 2012; Madsen and Sorensen,
 135 1992). The error above defined is the relative error in the wave amplitude
 136 for a linear propagation over mild slopes from κ to the shore, and it has been
 137 shown to be the proper error to be used (Chen and Liu, 1995; Lee et al.,
 138 2003; Galan et al., 2012).

139 From Figure 1, the coefficients proposed by Galan et al. provide a better
 140 performance compared to the other sets both in linear dispersion (ε_c) and,
 141 specially, in linear shoaling (ε_s). The G12 coefficients in Table 1 were found
 142 so as to improve the performance for any κ up to $\kappa_{\max} = 5$ obtaining $|\varepsilon_c, \varepsilon_s| <$
 143 0.3%, and the corresponding sets for wider ranges (*i.e.*, up to deeper waters)
 144 such as $\kappa_{\max} = 10$ or $\kappa_{\max} = 20$ were also provided.

145 The above results have been here slightly improved, as shown in Table 2
 146 for different values of $\kappa_{\max}$. In a problem where, *e.g.*, the maximum values

	$\kappa_{\max} = 5$	$\kappa_{\max} = 10$	$\kappa_{\max} = 20$	$\kappa_{\max} = 40$	$\kappa_{\max} = 60$
α	-0.55247	-0.59441	-0.59412	-0.58723	-0.57057
δ	-0.01597	-0.05730	-0.03277	-0.02044	-0.02249
γ	-0.00014	-0.03277	-0.00407	-0.00176	-0.00181
δ_h	-0.09701	0.09615	0.03143	0.01593	0.02425
γ_h	-0.05526	0.01070	0.00201	0.00054	0.00073
$\varepsilon_c = \varepsilon_s$	0.170%	1.60%	3.72%	7.46%	16.8%

Table 2: Constant coefficients and errors for different $\kappa_{\max}$.

of κ are expected to be above 10 and below 20, the coefficients for $\kappa_{\max} = 20$ should be used: in that case, the errors in wave celerity and shoaling will be below 3.72%.

From Table 2, the wider the range (*i.e.*, the deeper waters we consider), the higher the error. This is a natural consequence of the perturbative nature of the BTEs. By construction, using constant coefficients in the BTEs, from equation (5) one gets $c \propto \sqrt{gh}$ as kh increases (deep waters), so that one could never obtain the desired result $c = g/\omega$ provided by the Airy theory in deep waters. To circumvent this problem, we will consider here that the coefficients are functions of h .

3. Coefficients functions of water depth h

Here we will consider that the coefficients are functions of the water depth h . Thinking in a dimensional way, the coefficients α , δ , γ , δ_h and γ_h will be functions of gravity g , local water depth h and the limits of the angular frequencies in deep waters, $\omega_{\min}$ and $\omega_{\max}$. Applying dimensional analysis,

162 *e.g.*, for α , we get

$$\alpha = f(g, h, \omega_{\min}, \omega_{\max}) = f\left(\kappa_{\max} \equiv \frac{\omega_{\max}^2 h}{g}, \varrho \equiv \frac{\omega_{\min}}{\omega_{\max}}\right),$$

163 where f stand for “function of”.

164 In the analytical approach in Section 4 the $\omega_{\min}$ and $\omega_{\max}$ are replaced by
165 a single frequency ω_0 , and therefore

$$\alpha = f(g, h, \omega_0) = f\left(\kappa_0 \equiv \frac{\omega_0^2 h}{g}\right).$$

166 Here, for the sake of clarity we will work in dimensional form. However,
167 the results for the coefficients and errors, which are all of them dimensionless,
168 will be presented as funtions of the above groups κ_j and ϱ .

169 Because the coefficients are functions of the water depth, the analysis of
170 the properties of the equations is slightly richer than in the case of constant
171 coefficients. Now, for instance, the one dimensional linearized equations over
172 mild slopes, which are the ones used to analyze the linear dispersion and
173 shoaling properties (Dingemans, 1997), read

$$\begin{aligned} \frac{\partial \eta}{\partial t} + h \left[\frac{\partial u_\alpha}{\partial x} + h \left(l_1 \frac{\partial^3 \eta}{\partial t \partial x^2} + l_2 h \frac{\partial^3 u_\alpha}{\partial x^3} \right) \right] \\ + \frac{dh}{dx} \left[u_\alpha + h \left(s_1 \frac{\partial^2 \eta}{\partial t \partial x} + s_2 h \frac{\partial^2 u_\alpha}{\partial x^2} \right) \right] + (\alpha + 1) \frac{\partial \alpha}{\partial x} h^3 \frac{\partial^2 u_\alpha}{\partial x^2} = 0, \quad (9a) \end{aligned}$$

174 and

$$\frac{\partial u_\alpha}{\partial t} + g \frac{\partial \eta}{\partial x} + h^2 \left[gl_3 \frac{\partial^3 \eta}{\partial x^3} + l_4 \frac{\partial^3 u_\alpha}{\partial t \partial x^2} \right] + h \frac{\partial h}{\partial x} \left[gs_3 \frac{\partial^2 \eta}{\partial x^2} + s_4 \frac{\partial^2 u_\alpha}{\partial t \partial x} \right] = 0, \quad (9b)$$

175 where l_j and s_j are functions of α , δ , γ , δ_h and γ_h . In equation (9a), the term

$$(\alpha + 1) \frac{\partial \alpha}{\partial x} h^3 \frac{\partial^2 u_\alpha}{\partial x^2} = \frac{\partial h}{\partial x} (\alpha + 1) \beta_\alpha h^2 \frac{\partial^2 u_\alpha}{\partial x^2}, \quad \beta_\alpha \equiv h \frac{\partial \alpha}{\partial h},$$

176 is new and would cancel if α was constant. This term has the same structure
 177 than that corresponding to s_2 in equation (9a), is order $\mathcal{O}(\partial h / \partial x)$ and affects
 178 only the shoaling. The analysis of the linear dispersion and shoaling can be
 179 done following the usual procedures, and it is avoided here for clarity in the
 180 presentation. From this analysis one gets that the linear dispersion is not
 181 affected by the derivatives of the coefficients, so that equation (5) remains
 182 valid. Further, the shoaling analysis, and in particular the shoaling gradient
 183 $\alpha_{\eta, \text{bte}}$, is affected by β_α , β_δ and β_γ , which are defined as

$$\beta_a \equiv h \frac{\partial a}{\partial h},$$

184 which happen to be order one.

185 4. An analytical approach

186 Consider first the deep-water propagation of monochromatic waves with
 187 an angular frequency $\omega = \omega_0$. In deep waters nonlinear effects are negligi-
 188 ble and, hence, monochromatic waves remain monochromatic. In fact, the
 189 main feature to be captured by any model equations are wave celerity and
 190 amplitude.

191 Equation (5), which, as mentioned, remains valid for variable coefficients,
 192 and equation (6) can be understood as $k_{\text{bte}} = f_{\text{bte}}(\alpha, \delta, \gamma, g, h, \omega)$ and $k_{\text{Airy}} =$
 193 $f_{\text{Airy}}(g, h, \omega)$. Therefore, imposing the linear dispersion to be exact, *i.e.*,
 194 $c_{\text{bte}} = c_{\text{Airy}}$, which is equivalent to impose $k_{\text{bte}} = k_{\text{Airy}}$, gives the condition

$$\{f_c \equiv\} f_{\text{bte}}(\alpha, \delta, \gamma, g, h, \omega = \omega_0) - f_{\text{Airy}}(g, h, \omega = \omega_0) = 0. \quad (10)$$

For arbitrary values of g , h and ω_0 , the above condition can be satisfied in an infinite number of ways since we have three free coefficients. However, considering, *e.g.*, $\delta = \gamma = 0$ we can obtain α (or c_α) biunivocally. Recalling that $d_\alpha = c_\alpha + 1/3$, we get

$$c_\alpha = \frac{k_0 h - (k_0 h)^3/3 - \tanh(k_0 h)}{(k_0 h)^2 (k_0 h - \tanh(k_0 h))}, \quad (11)$$

where $k_0 h$ is obtained from $\kappa_0 \equiv \omega_0^2 h/g$ since $\kappa_0 = k_0 h \tanh(k_0 h)$. From c_α we recover α as $\alpha = -1 + \sqrt{1 + 2c_\alpha}$.

4.1. Linear dispersion in deep waters

The above condition (11) was already obtained by Lee et al. (2003) departing from BTEs with only one free parameter (α , since $\delta = \gamma = 0$ are not present in their approach). Taking advantage of the fact that we have three free coefficients for linear dispersion (α , δ and γ) we will now improve the dispersion performance in a neighbourhood of $\omega = \omega_0$. Instead of imposing $f_c = 0$, in order to improve the performance around ω_0 (and to increase the number of equations up to the number of unknowns, three) we consider here

$$f_c(\omega = \omega_0) = \frac{df_c}{d\omega}(\omega = \omega_0) = \frac{d^2 f_c}{d\omega^2}(\omega = \omega_0) = 0. \quad (12)$$

In this way we get a system of three equations for our three unknowns c_α (*i.e.*, α), δ and γ . The analytical solutions of the above equations are shown in Appendix A. In fact, there are four different sets of solutions. The first solution, denoted “+&+” in the appendix, has values similar to those in

213 Table 1 for M98 and G12. The other three solutions have shown to present
 214 numerical stability problems and are disregarded. In any case, the functions
 215 α , δ and γ turn out to be functions of the dimensionless group $\kappa_0 \equiv \omega_0^2 h/g$:
 216 this fact has been anticipated through dimensional analysis.

217 The consequences of imposing the conditions (12) are illustrated in Figure
 218 2 for $\omega_0 = 1 \text{ s}^{-1}$ and considering four different water depths h –the values of α ,
 219 δ and γ are different at each water depth h since $\kappa_0 = \omega_0^2 h/g$ changes–. The
 220 error ε_c always cancels at $\omega = \omega_0$ and, since the first and second derivatives
 221 are null, the error is kept small around ω_0 . In fact, for $h = 250 \text{ m}$, 500 m
 222 and 1000 m the errors behave similarly and are below 1% for $0.83 \text{ s}^{-1} \leq \omega \leq$
 223 1.20 s^{-1} . For $h = 50 \text{ m}$, *i.e.*, in shallower waters, the error behaves, naturally,
 224 better: in this case the error is below 1% for $0 \leq \omega \leq 1.32 \text{ s}^{-1}$. The solution
 225 “+&+” in Appendix A is considered to build Figure 2.

Figure 2: Illustration of the consequences of imposing $f_c = \partial f_c / \partial \omega = \partial^2 f_c / \partial \omega^2 = 0$
 at $\omega = \omega_0 = 1 \text{ s}^{-1}$.

226 For a given ω_0 , Figure 3 shows the range frequencies ω that can be prop-
 227 agated with some given errors (5%, 1% and 0.1%) as a function of h using
 228 variable coefficients α , δ and γ obtained above. The results are presented
 229 showing the ranges ω/ω_0 as a function of $\kappa_0 \equiv \omega_0^2 h/g$. We recognize the con-
 230 venient fact that the curves tend to be horizontal as $h \rightarrow \infty$, so that the same
 231 range of frequencies can be propagated up to arbitrary deep waters. From
 232 Figure 3, using the coefficients as functions of κ_0 , shown in Appendix A, one
 233 can propagate in arbitrary deep waters waves the range $0.71 \omega_0 \leq \omega \leq 1.39 \omega_0$

234 with error $\varepsilon_c < 5\%$, the range $0.83 \omega_0 \leq \omega \leq 1.20 \omega_0$ with $\varepsilon_c < 1\%$ (as already
 235 stated), and the range $0.92 \omega_0 \leq \omega \leq 1.09 \kappa_0$ with $\varepsilon_c < 0.1\%$.

Figure 3: Range of application for the coefficients correspondig to a given $\kappa = \kappa_0$.
 The errors are ε_c .

236 For a given range of frequencies $[\omega_{\min}, \omega_{\max}]$ and a given maximum depth
 237 h , the value of ω_0 that minimizes the error in the range, which is not neces-
 238 sarily the mean value $(\omega_{\min} + \omega_{\max})/2$, can be found. As already mentioned,
 239 in shallow waters, as it corresponds to BTEs, all frequencies are well repre-
 240 sented. This fact is clear from Figure 3: the range ω/ω_0 increases as $\kappa_0 \rightarrow 0$.
 241 For instance, for $\kappa_0 = 3$ the errors are below only 0.1% for any $\omega \lesssim 1.27\omega_0$,
 242 what is to say for any $\kappa = \omega^2 h/g \lesssim 1.27^2 \omega_0^2 h/g = 1.61 \kappa_0 \approx 4.83$.

243 4.2. Linear shoaling in deep waters

244 For a given frequency ω_0 , above we have found the values of α , δ and γ ,
 245 functions of h , so as to improve the linear dispersion performance around ω_0 .
 246 In fact, it has been seen that α , δ and γ are functions of $\kappa_0 \equiv \omega_0^2 h/g$.

247 A similar reasoning is followed to obtain the coefficients δ_h and γ_h in
 248 Appendix B. Now the focus is on shoaling error ε_s as defined in expression
 249 (8). Recall that, because we are considering variable coefficients, the shoaling
 250 gradient $\alpha_{\eta, \text{bte}}$ depends on α , δ , γ , δ_h and γ_h but also on β_α , β_δ and β_γ ,
 251 which are known from the the solution of α , δ and γ obtained from the
 252 linear dispersion analysis. As shown in Appendix B, there are four sets of
 253 solutions for δ_h and γ_h corresponding to the different sets obtained for α , δ
 254 and γ above. The solution of δ_h and γ_h corresponding to “+&+”, the only

one we are interested in, presents an infinite discontinuity at $\kappa_0 \approx 4.2$ (Figure B.13) and, therefore, this approach must be abandoned.

5. A global minimization approach

The analytical approach above has shown to yield useful results in determining α , δ and γ as functions of $\kappa_0 \equiv \omega_0^2 h/g$ to improve the linear dispersion performance for frequencies around ω_0 . However, it gives undesirable results for δ_h and γ_h as functions of $\omega_0^2 h/g$ when trying to improve the linear shoaling performance. Therefore, the above approach is of use if the only property to be well represented was wave celerity.

A different approach aimed to improve *both* linear dispersion and shoaling is proposed here. We consider water waves which in deep waters have frequencies within the range $\omega \in [\omega_{\min}, \omega_{\max}]$ propagating from a water depth $h_{\max}$ to the shore. We will find α , δ , γ , δ_h and γ_h at n values of h from $h_{\min}$ to $h_{\max}$, given by

$$h_j \equiv h_{\min} + (j - 1) \Delta h, \quad j = 1, \dots, n,$$

with

$$\Delta h \equiv \frac{h_{\max} - h_{\min}}{n - 1},$$

where the values of $h_{\min}$ and n are discussed later. From the above definitions $h_1 = h_{\min}$ and $h_n = h_{\max}$. For $h \leq h_1$ the coefficients will be constant and equal to those at h_1 while for any $h \in [h_j, h_{j+1}]$ with $j < n - 1$, we consider linear interpolations of the values at h_j and h_{j+1} .

274 For given values of $h_{\min}$, $h_{\max}$, $\omega_{\min}$, $\omega_{\max}$ and Δh we will get the values
 275 of the five coefficients at each h_j so as to minimize the error

$$\varepsilon \equiv \max_{0 \leq h \leq h_{\max}}^{\omega_{\min} \leq \omega \leq \omega_{\max}} \{|\varepsilon_c|, |\varepsilon_s|\},$$

276 where h can take any value from 0 to $h_{\max}$.

277 We note that, while ε_c is “local” –*i.e.*, it depends only on the coefficients
 278 at the water depth where the error is evaluated–, the error ε_s depends on all
 279 five coefficients evaluated at any depth below the local water depth. For this
 280 reason, the minimization of the five coefficients at all h_j must be performed
 281 at once.

282 Given $h_{\max}$ and $\omega_{\max}$, the maximum value of κ is $\kappa_{\max, \max} = \omega_{\max}^2 h_{\max} / g$.
 283 Although any possibility could be chosen, for illustrative purposes we con-
 284 sider $\kappa_{\max, \max} = \{20, 40, 60, 80, 100\}$. Besides, we consider $h_{\min}$ so that the
 285 minimum value of κ at this depth, which is $\omega_{\min}^2 h_{\min} / g$, equals 4. In this way
 286 we ensure that the coefficients are constant up to, at least, $\kappa = 4$.

287 Finally, $\Delta h = (h_{\max} - h_{\min}) / (n - 1)$ where n is chosen so that

$$\frac{\kappa_{\max, \max} - 4}{n - 1} = 4, \quad i.e., \quad n = \frac{\kappa_{\max, \max}}{4}.$$

288 According to the dimensional analysis, for a given $\kappa_{\max, \max}$, now the co-
 289 efficients will be functions of

$$\kappa_{\max, j} \equiv \frac{\omega_{\max}^2 h_j}{g}, \quad \text{and} \quad \varrho \equiv \frac{\omega_{\min}}{\omega_{\max}},$$

290 and the error ε will be a function of $\kappa_{\max, \max}$ and ϱ .

291 The results are shown in Table 3. We note that the minimization problem
 292 is complex ($5n$ unknowns and a non convex objective function ε) and the

	$\varrho = 0.8$	$\varrho = 0.9$	$\varrho = 1.0$
$\kappa_{\max, \max} = 20$	3.67%	2.67%	0.98%
$\kappa_{\max, \max} = 40$	6.60%	4.51%	0.98%
$\kappa_{\max, \max} = 60$	11.2%	6.60%	2.46%
$\kappa_{\max, \max} = 80$	12.1%	6.91%	2.86%
$\kappa_{\max, \max} = 100$	12.4%	6.91%	2.86%

Table 3: Errors.

results could probably be further improved. The table presents the errors $\varepsilon = f(\kappa_{\max, \max}, \varrho)$: the values of the coefficients α , δ , γ , δ_h and γ_h , functions of $\{\kappa_{\max, \max}, \kappa_{\max, j}, \varrho\}$ can be found at

<https://dl.dropbox.com/u/11753471/web/p110315.zip>

The general expectable trends in Table 3 are the same observed in Figure 3. First, the error diminishes as the ϱ decreases, *i.e.*, as the frequency range is diminished. Second, the error increases with $\kappa_{\max, \max}$, but it seems to tend to a finite error as $\kappa_{\max, \max}$ grows.

For each case in Table 3, the coefficients δ_ϵ and γ_ϵ , constant, have been established following the same procedure as that presented in Galan et al. (2012). The results are presented in the above link.

6. Numerical scheme and results

The numerical scheme considered to solve the model equations is the one presented by Galan et al. (2012). This scheme uses a fourth order accuracy finite differences discretization in space and a fourth order Runge-Kutta explicit scheme in time.

309 In this Section, three numerical examples are shown in order to demon-
 310 strate the capabilities of the proposed equations. The first case is the prop-
 311 agation of a bichromatic linear wave train over a submerged shoal in deep
 312 waters, the second is the simulation of one the experiments of the Dingen-
 313 mans bar and the third one considers one of the experiments by Trulsen
 314 et al. (2012) for irregular and nonlinear wave propagation.

315 6.1. Case 1: linear propagation over sloping bathymetry

316 A first example is meant to illustrate the linear performance of the equa-
 317 tions with variable coefficients in deep waters. We consider the propagation
 318 of a wave train composed by the sum of two monochromatic waves with
 319 amplitudes $a_1 = a_2 = 0.1$ m and periods $T_1 = 6.0$ s and $T_2 = 6.5$ s.

320 The bathymetry is a shoal given by

$$h \text{ (m)} = h_{\max} - (h_{\max} - h_{\min}) \exp \left(- \left(\frac{x - x_c}{800} \right)^2 \right),$$

321 with $h_{\max} = 300$ m and $h_{\min} = 150$ m respectively the maximum and mini-
 322 mum depths (see Figure 5, bottom panel). The top of the bump is located at
 323 $x = x_c = 4750$ m and the maximum slope, at $x = x_c \pm 800/\sqrt{2}$, is $\partial h/\partial x \approx$
 324 0.098 . In this case $\omega_{\min} = 2\pi/T_2 = 0.967 \text{ s}^{-1}$ and $\omega_{\max} = 2\pi/T_1 = 1.047 \text{ s}^{-1}$
 325 and, hence

$$\kappa_{\max, \max} = \frac{\omega_{\max}^2 h_{\max}}{g} \approx 33.54, \quad \varrho = \frac{\omega_{\min}}{\omega_{\max}} \approx 0.923,$$

326 and we consider the coefficients corresponding to $\kappa_{\max, \max} = 40$ and $\varrho = 0.9$,
 327 with errors bounded by 4.51%. The coefficients are provided at 10 different

$\kappa_{\max,j}$	h_j (m)	α	δ	γ	...
4.94	44.18	-0.580282	-0.019680	-0.000789	...
8.83	79.03	-0.579805	-0.020601	-0.001293	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
36.10	322.98	-0.578831	-0.020237	-0.001801	...
40.00	357.83	-0.577795	-0.020225	-0.001703	...

Table 4: Coefficient sets to be used depending on the maximum expected κ .

Figure 4: Linear coefficients for test case 1, corresponding to $\kappa_{\max,\max} = 40$ and $\varrho = 0.9$ (here expressed as functions of h_j)

values of $\kappa_{\max,j} = \omega_{\max}^2 h_j / g$ equally spaced from $\kappa_{\max,1} = 4.94$ to $\kappa_{\max,10} = \kappa_{\max,\max} = 40$. In table 4 some of them are presented as a function of h_j .

Linear interpolation gives the values of the coefficients at any of the grid points, x , imposing as well constant values corresponding to those at h_1 in points where $h \leq h_1$. This is shown graphically in Figure 4, where a constant initial length can be localized below $\kappa_{\max} = 4.94$ for all the free coefficients.

Table 5 summarizes the errors made in linear dispersion for the two considered frequencies at two discrete points: the first point at the beginning of the domain where the depth is maximum and the second point on the top of the bump. As shown, maximum error is 3.82% ($\leq 4.51\%$).

Figure 5 shows the propagation of the two different frequencies throughout the domain obtained by the numerical model together with the analytical envelope for the amplitude obtained by using the linear theory (which gives

wave component	$j = 1$	$j = 2$
a (m)	0.1	0.1
T (s)	6.0	6.5
at $h = 300$ m		
κ	33.54	28.57
λ_{Airy} (m)	56.21	65.97
λ_{bte} (m)	55.42	65.65
ε_c	-1.40%	-0.47%
at $h = 150$ m		
κ	16.77	14.29
λ_{Airy} (m)	56.21	65.97
λ_{bte} (m)	58.36	65.31
ε_c	3.82%	1.00%

Table 5: Coefficient sets to be used depending on the maximum expected κ .

Figure 5: Numerical results (continuous lines) obtained with linear coefficients corresponding to $\kappa_{\max,\max} = 40$ and $\varrho = 0.9$ and analytical envelope (discontinuous lines) for free surface elevation. Snapshot at time = 1000 s for the wave component of $T = 6.0$ s (top panel) and for the wave component of $T = 6.5$ s (middle panel). The bathymetry and the generation area is depicted in the bottom panel.

Figure 6: Time history for free surface elevation at two different locations. Numerical results (line) obtained with linear coefficients corresponding to $\kappa_{\max,\max} = 40$ and $\varrho = 0.9$. The analytical solution is displayed with stars.

341 nearly constant wave amplitude). For the numerical scheme we considered
 342 a mesh size of 1 m and a time step of 0.25 s, satisfying the CFL condition
 343 presented in the work by Galan et al. (2012). The numerically propagated
 344 amplitude has a maximum error of 0.9% for the wave with $T = 6.0$ s and
 345 4.31% for the one with $T = 6.5$ s (nearly unappreciable in the figure).

346 Figure 6 shows the time history for free surface elevation at two different
 347 locations (#A, with $x = 2500$ m, and #B, with $x = x_c = 4750$ m), one at the
 348 maximum depth and another one at the top of the shoal, compared with the
 349 analytical solution (in phase at #A). The results for linear dispersion (*i.e.*,
 350 wave celerity) compare well and are consistent with the expected results.

351 6.2. Case 2: non linear propagation over a bar

352 A second example is meant to show how the model equations can handle
 353 with the nonlinear behaviour of the wave as they reach shallow waters from

Figure 7: Linear coefficients for test case 2, corresponding to $\kappa_{\max, \max} = 20$ and $\varrho = 1.0$.

354 deep waters. For this purpose we consider the propagation of a monochro-
 355 matic wave with period $T = 2.857\text{ s}$ over a constant slope ($\approx 20/300$) from
 356 a maximum depth of 20 m (*i.e.*, $\kappa \approx 9.9$) to 0.86 m ($\kappa \approx 0.42$). At the end
 357 of the slope we introduce the bathymetry by Dingemans (1997) in order to
 358 compare the experimental results with those measured in laboratory at dif-
 359 ferent control gages. The bathymetry is shown in Figure 8 (top panel), while
 360 Dingemans bathymetry is shown as a zoom.

361 The wave amplitude generated in the experiment of Dingemans (case
 362 A) is $\eta_0 = 0.02\text{ m}$ over the depth of 0.86 m, so that, to propagate from
 363 deep water with an adequate amplitude, and based on the linear theory
 364 ($a^2 c_g = \text{constant}$, being a the wave amplitude and c_g the group celerity), we
 365 introduce an amplitude $\eta_0 = 0.0205\text{ m}$ in the generation deep zone.

366 For this test we have $\omega_{\min} = \omega_{\max} = 2\pi/T = 2.2\text{ s}^{-1}$ and, as anticipated

$$\kappa_{\max, \max} = \frac{\omega_{\max}^2 h_{\max}}{g} \approx 9.86,$$

367 so that we will consider the coefficients corresponding to $\kappa_{\max, \max} = 20$ and
 368 $\varrho = 1.0$ (monochromatic). Using this set of coefficients the linear dispersion
 369 and shoaling errors are below 0.98%, as shown in Table 3. The values for
 370 linear coefficients are shown in Figure 7 while nonlinear coefficients are $\delta_\epsilon =$
 371 -0.276780 and $\gamma_\epsilon = 0.135060$.

372 Figure 8 shows the time history comparison between numerical results

Figure 8: Dingemans’ experiments (case A). Numerical results (lines) and experimental data (stars) for the normalized free surface elevation.

and experimental data at 8 different gages (from #1 to #8). Section #1 has been used as control section, allowing to synchronize model and experimental time. As shown, the comparison between numerical and experimental results is good for all considered section.

6.3. Case 3: non linear irregular waves propagation

Finally, we present a numerical simulation one of the test presented by Trulsen et al. (2012). The laboratory experiments consist on the propagation of irregular waves travelling from a water depth $h_{\max} = 0.60\text{m}$ to $h_{\min} = 0.30\text{m}$ through a 6 meter long ramp (1:20). We consider the “case 1” in the original paper, the most demanding attending to their dispersive conditions. The significant wave height is around 0.06 m at $h = h_{\max}$, so that nonlinear effects are significant as the water depth decreases.

For the case under consideration, the Figure 9 shows the wave amplitudes corresponding to the angular frequencies composing the incident signal at $h_{\max} = 0.6\text{ m}$. We discretized the continuous signal with 240 frequencies. From the figure $\omega_{\min}/\omega_{\max} \approx 0.1 \ll 0.8$ and, therefore, the experiments are beyond the scope of the analysis for variable coefficients. The closest set of coefficients would be those for $\varrho = 0.8$ and $\kappa_{\max, \max} = 20$.

Figure 10 shows the errors at $h = 0.6\text{ m}$ in linear dispersion and shoaling using constant coefficients (those for $h \leq h_1$) corresponding to $\varrho = 0.8$ and $\kappa_{\max, \max} = 20$, which are $\alpha = -0.590334$, $\delta = -0.032415$, $\delta_h = 0.031415$,

Figure 9: Frequencies and amplitudes of each harmonic composing the incident signal.

Figure 10: Linear dispersion and shoaling errors as a function of ω for $h = 0.6$ m and the constant coefficients corresponding to $\varrho = 0.8$ and $\kappa_{\max, \max} = 20$.

394 $\gamma = -0.004324$ and $\gamma_h = 0.001212$. As depicted in the figure, the errors are
 395 $\lesssim 4\%$ for the whole range of frequencies. Obviously, the results are better at
 396 $h = 0.3$ m.

397 Figure 11 shows the comparison of the spectra at the different gages (ex-
 398 perimental and computed). The numerical results show fair agreement with
 399 the experimental data. Besides the above errors above 4%, it is to mention
 400 that in this experiment strong nonlinearities and strong dispersive conditions
 401 coincide. This is beyond the scope of low order Boussinesq-type equations,
 402 which can handle strong nonlinearities in weakly dispersive conditions. Also,
 403 according to Tucker and Pitt (2001), a statistical instability exists due to
 404 wave density spectrum estimation from a finite record (wave density spec-
 405 trum has been estimated by scanning), so the estimated spectrum could show
 406 differences when compared with the real one.

Figure 11: Normalized wave density spectrum at the 8 different gages. Full line represent data from Trulsen *et al.* (2012) (case 1) and points are results obtained by the proposed model propagating the incident spectrum.

407 7. Concluding remarks

408 The possibility of using variable coefficients (functions of the water depth)
409 in enhanced Boussinesq-type equations has been studied and presented. An
410 analytical approach is disregarded since it has shown to give infinite disconti-
411 nities in the solutions. Alternatively, the coefficients are numerically found
412 so as to optimize the linear performance in terms of dispersion and shoaling
413 over mild slopes. The results are presented in dimensionless general form.

414 The performance of the model is determined by the ratio between the
415 minimum to maximum deep water wave angular frequencies, $\varrho \equiv \omega_{\min}/\omega_{\max}$,
416 and a kh -type number, $\kappa_{\max,\max}$. The results are particularly interesting for
417 $\varrho \lesssim 1$, *i.e.*, for narrow banded swells approaching to the coast. For these
418 conditions, the wave can be propagated with small errors in linear dispersion
419 *and* shoaling up to very deep waters. The theoretical results are supported
420 by numerical simulations compared to analytical and experimental results.

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425 Appendix A. Dispersion: coefficients α , δ and γ

426 For given g , h and ω_0 , the solution of equations (12) is

$$\delta = (\varrho_1 \pm \sqrt{\varrho_1^2 - 4\varrho_2})/2, \quad (\text{A.1a})$$

$$\gamma = (\varrho_1 + 1/3 \pm \sqrt{\varrho_1^2 + 1/9 + 2\varrho_1/3 - 4\varrho_3})/2, \quad (\text{A.1b})$$

427 and $c_\alpha = \varrho_1 - \gamma - \delta$ so that, since $c_\alpha \equiv \alpha^2/2 + \alpha$, we can recover the coefficient
 428 α as $\alpha = -1 + \sqrt{1 + 2c_\alpha}$. Above

$$\varrho_1 = \frac{n_1}{3\xi_0^2 d}, \quad \varrho_2 = \frac{n_2}{3\xi_0^4 d}, \quad \varrho_3 = \frac{n_3}{3\xi_0^5 d}, \quad (\text{A.2})$$

429 with ξ_0 verifying $\xi_0 \tanh \xi_0 = \kappa_0 \{ \equiv \omega_0^2 h/g \}$ and

$$\begin{aligned} n_1 &\equiv 6 \{ 2s_0^2 \xi_0^2 + 5 \} t_0^2 + \{ 2s_0^2 \xi_0^4 + (-12s_0^2 + 1) \xi_0^2 - 6(7s_0^2 + 3) \} \xi_0 t_0 + \\ &\quad + \{ -s_0^2 \xi_0^2 + 6(2s_0^4 + 3s_0^2) \} \xi_0^2, \\ n_2 &\equiv 3 \{ 2s_0^2 \xi_0^2 + 15 \} t_0^2 + \{ 2s_0^2 \xi_0^4 - 3(2s_0^2 + 1) \xi_0^2 - 9(3s_0^2 + 7) \} \xi_0 t_0 + \\ &\quad + \{ 3s_0^2 \xi_0^2 + 3(2s_0^4 + 5s_0^2 + 8) \} \xi_0^2, \\ n_3 &\equiv 24t_0^3 + \{ 2s_0^2 \xi_0^4 + (6s_0^2 - 1) \xi_0^2 - 27 \} \xi_0 t_0^2 + \\ &\quad + \{ -7s_0^2 \xi_0^2 + 9(-3s_0^2 + 1) \} \xi_0^2 t_0 + \{ 2s_0^4 \xi_0^2 + 3(2s_0^4 + 5s_0^2) \} \xi_0^3, \\ d &\equiv \{ 2s_0^2 \xi_0^2 + 3 \} t_0^2 - \{ 2s_0^2 \xi_0^2 + (5s_0^2 + 1) \} \xi_0 t_0 + \{ 2s_0^4 + s_0^2 \} \xi_0^2, \end{aligned}$$

430 with $s_0 \equiv \text{sech } \xi_0$ and $t_0 \equiv \tanh \xi_0$.

431 The coefficients α , δ and γ are, thus, functions of $\kappa_0 \equiv \omega_0^2 h/g$. As $\kappa_0 \rightarrow 0$,
 432 $\kappa_0 \rightarrow \xi_0^2$ and $\varrho_1 \rightarrow -4/9$, $\varrho_2 \rightarrow 1/63$ and $\varrho_3 \rightarrow 1/945$, so that we recover the
 433 Padé $[4/4]$ approximation (Madsen and Schaffer, 1998; Gobbi et al., 2000). In
 434 equations (A.1), there are four possible combinations depending on the signs,
 435 equivalent to the four possible solutions discussed by Madsen and Schaffer
 436 (1998).

437 Figure A.12 shows the three functions α , δ and γ in all four cases. The
 438 coefficients are so that $\beta_\alpha \equiv h\partial\alpha/\partial h = \kappa_0\partial\alpha/\partial\kappa_0$, β_δ and β_γ are small.
 439 For “+&+” the values are similar to the values by M98 and G12 in Table 1.
 440 However, all four solutions give the same results in terms of linear dispersion.

Figure A.12: Coefficients α (full line), δ (dashed line), γ (dash-dotted line), which
 are functions of $\kappa_0 \equiv \omega_0^2 h/g$, depending on the signs considered in equations (A.1).
 For instance, the case “+&-” results from considering “+” in equation (A.1a) and
 “-” in equation (A.1b).

441 Appendix B. Shoaling: coefficients δ_h and γ_h

442 For given ω and h , the error in shoaling is (Chen and Liu, 1995)

$$\varepsilon_s = \exp \left(\int_0^h \frac{\alpha_{\eta,\text{Airy}} - \alpha_{\eta,\text{bte}}}{h_*} dh_* \right) - 1,$$

443 where here $\alpha_{\eta,\text{Airy}} = \alpha_{\eta,\text{Airy}}(\omega^2 h_*/g)$ and

$$\alpha_{\eta,\text{bte}} = \alpha_{\eta,\text{bte}}(\omega^2 h_*/g, \alpha, \delta, \gamma, \delta_h, \gamma_h, \beta_\alpha, \beta_\delta, \beta_\gamma),$$

444 are the shoaling gradients corresponding to Airy and BTEs. In the shoal-
 445 ing gradient $\alpha_{\eta,\text{bte}}$, the α , δ , γ and their corresponding β ’s are known from
 446 Appendix A.

447 Since we now have two (not three) free coefficients, δ_h and γ_h , we impose
 448 the two conditions, equivalent to the conditions (12) in the linear dispersion
 449 analysis,

$$\varepsilon_s(\omega = \omega_0) = \frac{\partial \varepsilon}{\partial \omega}(\omega = \omega_0) = 0,$$

450 at any h to obtain δ_h and γ_h as a function of h for the ω_0 used in the dispersion
 451 analysis. Defining $f_s \equiv \alpha_{\eta, \text{Airy}} - \alpha_{\eta, \text{bte}}$, the above is equivalent to impose, at
 452 any h ,

$$\int_0^h \frac{f_s}{h_*} dh_* = \int_0^h \frac{\partial f_s / \partial \omega}{h_*} dh_* = 0, \quad (\text{B.1})$$

453 always evaluated at $\omega = \omega_0$.

454 Given ω_0 , consider that we know the values of δ_h and γ_h to satisfy the
 455 conditions (B.1) up to some given depth $h - \Delta h$ (with Δh infinitesimal).
 456 Taking into account that equation (B.1) already holds at $h - \Delta h$, imposing
 457 it at h is simply

$$f_s(h, \omega = \omega_0) = \frac{\partial f_s}{\partial \omega}(h, \omega = \omega_0) = 0. \quad (\text{B.2})$$

458 The above nonlinear system has been solved using Newton's method to
 459 obtain δ_h and γ_h at any h and for a given ω_0 . Again, the resulting δ_h and γ_h
 460 are functions of $\kappa_0 \equiv \omega_0^2 h / g$. Depending on the solution considered for α , δ
 461 and γ (Figure A.12), Figure B.13 shows the resulting δ_h and γ_h .

Figure B.13: Coefficients δ_h (full line) and γ_h (dashed line), functions of $\kappa_0 \equiv \omega_0^2 h / g$,
 depending on the signs considered in equations (A.1) to obtain α , δ and γ .

462 From Figure B.13, the solutions “+&+” and “-&-” violate the condition
 463 of slow variations. In fact, they have discontinuities to the infinity.

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