A -manifolds admitting a functorial connection

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Abstract

A connection for a large class of G -structures is defined by means of a generic condition on the adjoint bundle and it is compared with the standard connections for classical G -structures. A characterization of the (not necessarily commutative) finite-dimensional $\mathbb{R}$ -algebras A for which $GL(r, A)$ -structures admit such a connection, is obtained.

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1 Introduction

Although A -structures were originally intended to provide a general setting for many important G -structures, questions on A -analyticity and A -differentiability (cf. [7], [23]) have posed interesting problems in analysis of hypercomplex variables, notably in the quaternionic case. The development of A -manifolds has been also influenced by the theory of supermanifolds, graded manifolds, gauge theories (e.g., see [6], [11], [12], [17]), and, in general, by the structures on manifolds modelled over a finite-dimensional ground $\mathbb{R}$ -algebra—instead of the real or complex fields—as introduced in the general survey [21] (also see [9]).

Below, a common definition of a connection for the class of G -structures satisfying: i) $\mathfrak{g}^{(1)} = 0$ and ii) $\mathfrak{g}$ is invariant under trasposition ($\mathfrak{g}$ being the Lie algebra of G), is proposed and then, such a connection is compared with the connections appearing in the literature for the classical geometric G -structures. In the case of A -structures, the algebras A for which $\mathfrak{gl}(r, A)$ satisfies the previous conditions i) and ii), are classified.

2 A -manifolds

Let A be a (not necessarily commutative) n -dimensional $\mathbb{R}$ -algebra, let $\mathfrak{gl}(r, A)$ be the $\mathbb{R}$ -algebra of $r \times r$ matrices with entries in A , and let $GL(r, A)$ be the group of invertible matrices in $\mathfrak{gl}(r, A)$ endowed with its Lie-group structure of dimension $\dim GL(r, A) = r^2 n$, whose Lie algebra can be identified to $\mathfrak{gl}(r, A)$ itself with the standard matrix Lie bracket. As $\mathfrak{gl}(r, A)$ can be identified to right A -linear mappings $\Lambda: A^r \rightarrow A^r$ there exists a linear representation $\lambda^r: GL(r, A) \rightarrow GL(nr, \mathbb{R})$.

Let M be an m -dimensional C^∞ manifold, $m = nr$. An *almost A -structure* (resp. an *A -structure*) on M is a $GL(r, A)$ -structure (resp. an integrable $GL(r, A)$ -structure) on M . This definition covers a large class of interesting geometric structures on manifolds, such as integrable almost tangent structures ($A = \mathbb{R}[t]/(t^2)$), complex and paracomplex structures ($A = \mathbb{R}[t]/(t^2 \pm 1)$), hypercomplex and quaternionic-like structures ($A = \mathbb{R}[i, j, k]$), Yano's structures ($A = \mathbb{R}[t]/(t^3 + t)$), etc.

Proposition 2.1. *Let M be an m -dimensional C^∞ manifold, $m = nr$. There is a natural bijection between almost A -structures on M and free right A -module structures on the tangent spaces $T_x M$, $\forall x \in M$. The manifold M admits an A -structure if and only if, with the natural identification $T_z A^r \cong A^r$, $z \in A^r$, there exists an atlas $\{\varphi_i: U_i \rightarrow A^r\}_{i \in I}$ such that the mapping*

$$(\varphi_i \circ \varphi_j^{-1})_*: T_{\varphi_j(x)} A^r \rightarrow T_{\varphi_i(x)} A^r$$

is right A -linear for every $x \in U_i \cap U_j$.

Proof. Assume M admits a $GL(r, A)$ -structure $\pi: P \rightarrow M$, i.e., P is a subbundle of the bundle of linear frames $\pi: FM \rightarrow M$. A linear frame of M at $x \in M$ can be identified to a $\mathbb{R}$ -linear isomorphism $u: A^r \rightarrow T_x(M)$. Once a frame $u \in P_x = \pi^{-1}(x)$ has been fixed, we can define a right free A -module structure on $T_x(M)$ by setting $X \cdot a = u(u^{-1}(X) \cdot a)$, $\forall a \in A$, $\forall X \in T_x(M)$, and the definition does not depend on the chosen frame, as any other linear frame $v \in P_x$ can uniquely be written as $v = u \circ \Lambda$, $\Lambda \in GL(r, A)$, and we thus have

$$\begin{aligned} v(v^{-1}(X) \cdot a) &= (u \circ \Lambda)(\Lambda^{-1}(u^{-1}(X) \cdot a)) \\ &= u(u^{-1}(X) \cdot a). \end{aligned}$$

Conversely, if the fibres of the tangent bundle TM are endowed with a structure of right free A -module, then we can define a $GL(r, A)$ -structure $\pi: P \rightarrow M$ as the linear frames $u: A^r \rightarrow T_x M$ which are right A -linear with respect to that structure. Moreover, if an atlas exists satisfying the condition in the statement, then we can define a right A -module structure on $T_x M$ by simply choosing an index i such that $x \in U_i$ and transporting the right A -module structure of A^r via $(\varphi_i)_*$. According to the first part of the proof we have thus defined a $GL(r, A)$ -structure on M , which is integrable as it follows from its very definition that this structure is locally isomorphic to the standard $GL(r, A)$ -structure on A^r . The converse is similar. $\square$

3 Functorial connections

Let G be a closed subgroup of $GL(m, \mathbb{R})$, $m = \dim M$. The G -structures over M are in bijection with the sections of the quotient bundle $\pi: FM/G \rightarrow M$; e.g., see [15, I, Propositions 5.5 & 5.6]. The bijection $s \leftrightarrow P_s$ between sections and G -structures is given by $P_s = \{u \in FM : u \cdot G = s(\pi(u))\}$. For each diffeomorphism $\varphi: M \rightarrow M'$, let $\tilde{\varphi}: FM \rightarrow FM'$ be the $GL(m, \mathbb{R})$ -principal bundle isomorphism given by (cf. [15, VI, p. 226]): $\tilde{\varphi}(X_1, \dots, X_m) = (\varphi_* X_1, \dots, \varphi_* X_m)$, for every linear frame $(X_1, \dots, X_m) \in F_x M$. By passing to the quotient, $\tilde{\varphi}$ induces a diffeomorphism $\bar{\varphi}: FM/G \rightarrow FM'/G$, such that $\bar{\pi}' \circ \bar{\varphi} = \varphi \circ \bar{\pi}$, and for every G -structure $s: M \rightarrow FM/G$ over M we can define a G -structure over M' —the G -structure obtained by transporting s to M' via φ —by setting $\varphi \cdot s = \bar{\varphi} \circ s \circ \varphi^{-1}$.

Let $CM \rightarrow M$ be the bundle of linear connections on M , which is an affine bundle modelled over the vector bundle $\otimes^2 T^*M \otimes TM$. The global sections of CM are identified to the linear connections on M .

Let $p^k: J^k E \rightarrow M$ be the bundle of k -jets of local sections of a fibred manifold $p: E \rightarrow M$ with projections $p_k^l: J^l E \rightarrow J^k E$ for $l \geq k$.

A precise definition of a functorial connection is as follows (cf. [19]):

A *functorial connection* on G -structures over M is a presheaf morphism $s \mapsto \Gamma(s)$, from the presheaf of sections of $FM/G \rightarrow M$ into the presheaf of sections of $CM \rightarrow M$, such that,

1. If $s: U \rightarrow FM$ is a section on an open subset $U \subseteq M$, then $\Gamma(s)$ is a linear connection on U adapted to the G -structure defined by s ; i.e., $\Gamma(s)$ is reducible to the subbundle P_s .
2. Let $U' \subseteq M'$ be an open subset in another manifold M' . If $\varphi: U \rightarrow U'$ is a diffeomorphism, then $\Gamma(\varphi \cdot s) = \varphi \cdot \Gamma(s)$, where the right-hand side stands for the direct image of $\Gamma(s)$ by φ (cf. [15, II.Proposition 6.1]).
3. Γ factors smoothly through $J^k(FM/G)$ for some integer k , i.e., the value of the section $\Gamma(s)$ of $C(U) \rightarrow U$ at a point $x \in U$ depends only on $j_x^k s$, and the induced mapping $\Gamma^k: J^k(FM/G) \rightarrow CM$, $\Gamma^k(j_x^k s) = \Gamma(s)(x)$, is smooth.

We recall that the *first prolongation* $\mathfrak{g}^{(1)}$ of a Lie subalgebra $\mathfrak{g} \subseteq \mathfrak{gl}(m, \mathbb{R})$ is the kernel of the antisymmetrization (or Spencer) operator $\mathcal{A}$; namely,

$$\begin{aligned} \mathcal{A}(L)(u, v) &= L(u)v - L(v)u, \\ \forall u, v \in \mathbb{R}^m, \forall L \in (\mathbb{R}^m)^* \otimes \mathfrak{g} &\cong \text{Hom}(\mathbb{R}^m, \mathfrak{g}). \end{aligned}$$

Hence the following exact sequence holds:

$$0 \rightarrow \mathfrak{g}^{(1)} \rightarrow (\mathbb{R}^m)^* \otimes \mathfrak{g} \xrightarrow{\mathcal{A}} \wedge^2 (\mathbb{R}^m)^* \otimes \mathbb{R}^m.$$

Let $G \subseteq GL(m, \mathbb{R})$ be a Lie subgroup with Lie algebra $\mathfrak{g} \subseteq \mathfrak{gl}(m, \mathbb{R})$. In [22, Theorems 1.1 & 1.2] the following criterion is proved: If the first prolongation

of $\mathfrak{g}$ vanishes, i.e., $\mathfrak{g}^{(1)} = 0$, and $\mathfrak{g}$ is further invariant under transposition, then any G -structure admits a functorial connection.

As is well known, the *adjoint bundle* of an arbitrary principal G -bundle $\pi: P \rightarrow M$ is the bundle associated to P under the adjoint representation of G on its Lie algebra $\mathfrak{g}$, i.e., $\text{ad } P = (P \times \mathfrak{g})/G$, where the action of G on $P \times \mathfrak{g}$ is given by $(u, A) \cdot g = (u \cdot g, \text{Ad}_{g^{-1}}(A))$, for all $g \in G$, $u \in P$, and $A \in \mathfrak{g}$. In particular, if G is a closed subgroup on $GL(m, \mathbb{R})$, then $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}(m, \mathbb{R}) = (\mathbb{R}^m)^* \otimes \mathbb{R}^m$, and for every G -structure $P \subset FM$ the adjoint bundle is a subbundle of the bundle of endomorphisms of the tangent bundle, $T^*M \otimes TM = \text{End}(TM)$.

Bearing this in mind, the connection γ functorially associated to P is the only principal connection on P such that,

$$(1) \quad \text{trace}(\sigma \circ i_X \text{Tor}_{\nabla^\gamma}) = 0, \quad \forall \sigma \in \Gamma(M, \text{ad } P), \quad \forall X \in \mathfrak{X}(M),$$

where ∇^γ denotes the covariant derivative attached to γ (cf. [22, Theorem 1.1]).

Theorem 3.1. *In the classical settings, the connection defined by (1) can be identified as follows:*

- (i) *Let $\pi: F_g M \rightarrow M$ be the bundle of orthonormal frames over an m -dimensional Riemannian manifold (M, g) . The Levi-Civita connection of g is the only connection on $F_g M$ satisfying the condition (1).*
- (ii) *Let $\pi: F_{g,J} M \rightarrow M$ be the bundle of unitary frames over a $2m$ -dimensional almost Hermitian manifold (M, g, J) . The connection γ on $F_{g,J} M$ satisfying the condition (1) is completely determined by the following properties:*

$$(2) \quad \left. \begin{aligned} g(\text{Tor}_{\nabla^\gamma}(Z, V), \bar{U}) &= g(\text{Tor}_{\nabla^\gamma}(Z, \bar{U}), V) \\ g(\text{Tor}_{\nabla^\gamma}(\bar{Z}, V), \bar{U}) &= g(\text{Tor}_{\nabla^\gamma}(\bar{Z}, \bar{U}), V) \end{aligned} \right\} \quad \forall Z, U, V \in T_x^{1,0} M.$$

This connection coincides with the canonical connection corresponding to $t = \frac{1}{3}$ in the sense of [8] (also see [5], [18]).

- (iii) *Let $\pi: F_{I,J} M \rightarrow M$ be the bundle of linear frames adapted to a hypercomplex structure $(I, J, K = I \circ J)$ on a $4m$ -dimensional manifold M . The connection γ on $F_{I,J} M$ satisfying the condition (1) coincides with the Obata connection (cf. [2], [4]).*

Proof. (i) As the Levi-Civita connection is the unique symmetric connection on $F_g M$ (e.g., see [15, IV, Theorem 2.2]), it will suffice to prove that, in the present case, the condition (1) implies $\text{Tor}_{\nabla^\gamma} = 0$. An endomorphism $\sigma: TM \rightarrow TM$ defines a section of the adjoint bundle if and only if, $g(\sigma(x)X, Y) + g(X, \sigma(x)Y) = 0$ for all $x \in M$ and all $X, Y \in T_x M$. Let $(U; x^1, \dots, x^m)$ be a coordinate system centered at x such that $(X_1, \dots, X_m)$, $X_i = (\partial/\partial x^i)_x$, $1 \leq i \leq m$, is an orthonormal basis for $T_x M$, i.e., $g(X_i, X_j) = \delta_{ij}$, with dual coframe $(w^1, \dots, w^m)$, $w^i = (X_i)^\flat$. Evaluating (1) at x and letting $\sigma(x) = w^b \otimes X_a - w^a \otimes X_b$, $X = X_c$, one obtains,

$$(3) \quad \Gamma_{ca}^b(x) - \Gamma_{ac}^b(x) = \Gamma_{cb}^a(x) - \Gamma_{bc}^a(x), \quad \forall a, b, c = 1, \dots, m,$$

where Γ_{bc}^a are the components of γ with respect to $(x^1, \dots, x^m)$. If $a = c$, then from (3) one deduces $\Gamma_{ab}^a(x) = \Gamma_{ba}^a(x)$. If $a \neq c$ but $b \in \{a, c\}$, then from (3) and the previous formulas one also deduces $\Gamma_{ac}^b(x) = \Gamma_{ca}^b(x)$ for $a = b$ or $a = c$. Finally, if the indices a, b, c are pairwise distinct, then letting $b \leftrightarrow c$, $a \mapsto b \mapsto c \mapsto a$ in (3) one obtains $\Gamma_{ba}^c(x) - \Gamma_{ab}^c(x) = \Gamma_{bc}^a(x) - \Gamma_{cb}^a(x)$ and $\Gamma_{ab}^c(x) - \Gamma_{ba}^c(x) = \Gamma_{ac}^b(x) - \Gamma_{ca}^b(x)$, respectively, thus proving that $\Gamma_{bc}^a(x) = \Gamma_{cb}^a(x)$.

(ii) Let $(U; x^1, \dots, x^m, y^1, \dots, y^m)$ be a coordinate system centered at $x \in M$ such that $(X_1, \dots, X_m, Y_1, \dots, Y_m)$, $X_j = (\partial/\partial x^j)_x$, $Y_j = (\partial/\partial y^j)_x$, $1 \leq j \leq m$, is a unitary basis for $T_x M$; i.e., $JX_j = Y_j$, $JY_j = -X_j$, $g(X_j, X_k) = \delta_{jk}$, $g(X_j, Y_k) = 0$, $g(Y_j, Y_k) = \delta_{jk}$.

We also write $Z_j = (\partial/\partial z^j)_x = \frac{1}{2}(X_j - iY_j)$, $\bar{Z}_j = (\partial/\partial \bar{z}^j)_x = \frac{1}{2}(X_j + iY_j)$.

An endomorphism $\sigma: T^c M \rightarrow T^c M$ defines a section of the adjoint bundle if and only if σ is anti-Hermitian, i.e.,

$$\begin{aligned} \sigma(x) &= c_j^h (dz^j)_x \otimes Z_h + \bar{c}_j^h (d\bar{z}^j)_x \otimes \bar{Z}_h \\ &= a_j^h \{ (dz^j)_x \otimes Z_h + (d\bar{z}^j)_x \otimes \bar{Z}_h \} \\ &\quad + ib_j^h \{ (dz^j)_x \otimes Z_h - (d\bar{z}^j)_x \otimes \bar{Z}_h \}, \end{aligned}$$

where $c_j^h = a_j^h + ib_j^h \in \mathbb{C}$, and $(a_j^h)_{h,j=1}^m$ (resp. $(b_j^h)_{h,j=1}^m$) is a skew-symmetric (resp. symmetric) matrix. Hence we can confine ourselves to impose the condition (1) at the point x for the standard basis

$$\begin{aligned} \sigma_j'^h(x) &= (dz^j)_x \otimes Z_h + (d\bar{z}^j)_x \otimes \bar{Z}_h - (dz^h)_x \otimes Z_j - (d\bar{z}^h)_x \otimes \bar{Z}_j, \quad h < j, \\ \sigma_j''^h(x) &= (dz^j)_x \otimes Z_h - (d\bar{z}^j)_x \otimes \bar{Z}_h + (dz^h)_x \otimes Z_j - (d\bar{z}^h)_x \otimes \bar{Z}_j, \quad h \leq j, \end{aligned}$$

and for $X = Z_a$ or $X = \bar{Z}_a$. By using the standard notations (e.g., see [15, IX, §5, p. 155]), in imposing such conditions the following equations are obtained:

$$(4) \quad \Gamma_{ah}^j(x) - \Gamma_{ha}^j(x) = \Gamma_{a\bar{j}}^{\bar{h}}(x) - \Gamma_{\bar{j}a}^{\bar{h}}(x),$$

$$(5) \quad \Gamma_{a\bar{j}}^{\bar{h}}(x) - \Gamma_{\bar{j}a}^{\bar{h}}(x) = \Gamma_{ah}^j(x) - \Gamma_{ha}^j(x),$$

for $a, h, j = 1, \dots, m$, which are equivalent to the equations (2).

Furthermore, by imposing $\nabla J(x) = 0$ (e.g., see [15, IX, §5, (9)]) one has $\Gamma_{b\bar{c}}^a(x) = 0$, $\Gamma_{\bar{b}c}^a(x) = 0$, $\Gamma_{bc}^{\bar{a}}(x) = 0$, and $\Gamma_{\bar{b}\bar{c}}^{\bar{a}}(x) = 0$. Therefore, the equations (4) and (5) transform respectively into the following:

$$(6) \quad \Gamma_{a\bar{c}}^{\bar{b}}(x) = \Gamma_{ab}^c(x) - \Gamma_{ba}^c(x),$$

$$(7) \quad \Gamma_{\bar{a}b}^{\bar{c}}(x) = \Gamma_{\bar{a}\bar{c}}^{\bar{b}}(x) - \Gamma_{\bar{c}\bar{a}}^{\bar{b}}(x).$$

Hence $\Gamma_{a\bar{c}}^{\bar{b}}(x) = -\Gamma_{\bar{b}\bar{c}}^{\bar{a}}(x)$. Moreover, from (6) and (7) with the same notations

as in [8], the following formulas hold:

$$\begin{aligned}\mathfrak{b}(\mathrm{Tor}_{\nabla\gamma})(Z_i, Z_j, Z_k) &= 0, \\ \mathfrak{b}(\mathrm{Tor}_{\nabla\gamma})(Z_i, Z_j, \bar{Z}_k) &= \frac{1}{3}\Gamma_{j\bar{k}}^{\bar{i}}(x), \\ \mathfrak{b}\left((\mathrm{Tor}_{\nabla\gamma})^{1,1}\right)(Z_i, Z_j, Z_k) &= 0, \\ \mathfrak{b}\left((\mathrm{Tor}_{\nabla\gamma})^{1,1}\right)(Z_i, Z_j, \bar{Z}_k) &= \frac{2}{3}\Gamma_{j\bar{k}}^{\bar{i}}(x),\end{aligned}$$

thus proving that $\mathfrak{b}((\mathrm{Tor}_{\nabla\gamma})^{1,1}) = 2\mathfrak{b}(\mathrm{Tor}_{\nabla\gamma})$.

In addition, denoting as in [8, Lemme 3] by $(\mathrm{Tor}_{\nabla\gamma})_s^{1,1}$ the projection of $(\mathrm{Tor}_{\nabla\gamma})^{1,1}$ onto the subspace of elements satisfying the zero cyclic sum property and by $(\mathrm{Tor}_{\nabla\gamma})_a^{1,1}$ the component in its orthogonal subspace, the formula [8, (1.4.3)] leads one to the following equation:

$$(\mathrm{Tor}_{\nabla\gamma})_a^{1,1} = \frac{3}{2}(\mathfrak{b}(\mathrm{Tor}_{\nabla\gamma}) + \mathfrak{M}(\mathfrak{b}(\mathrm{Tor}_{\nabla\gamma}))),$$

and then,

$$\begin{aligned}(\mathrm{Tor}_{\nabla\gamma})_a^{1,1}(Z_i, Z_j, Z_k) &= 0 = (\mathrm{Tor}_{\nabla\gamma})^{1,1}(Z_i, Z_j, Z_k), \\ (\mathrm{Tor}_{\nabla\gamma})_a^{1,1}(Z_i, Z_j, \bar{Z}_k) &= \Gamma_{j\bar{k}}^{\bar{i}}(x) = (\mathrm{Tor}_{\nabla\gamma})^{1,1}(Z_i, Z_j, \bar{Z}_k), \\ (\mathrm{Tor}_{\nabla\gamma})_a^{1,1}(Z_i, \bar{Z}_j, Z_k) &= -\Gamma_{k\bar{j}}^{\bar{i}}(x) = (\mathrm{Tor}_{\nabla\gamma})^{1,1}(Z_i, \bar{Z}_j, Z_k), \\ (\mathrm{Tor}_{\nabla\gamma})_a^{1,1}(\bar{Z}_i, Z_j, Z_k) &= 0 = (\mathrm{Tor}_{\nabla\gamma})^{1,1}(\bar{Z}_i, Z_j, Z_k).\end{aligned}$$

Hence $(\mathrm{Tor}_{\nabla\gamma})_s^{1,1} = 0$.

Finally, in order to obtain the value of the parameter t , we use the second formula in [8, (2.3.5)], from which it follows:

$$(d^c F)^+ = 3\left(\mathfrak{b}(\mathrm{Tor}_{\nabla\gamma})^+ - 2\mathfrak{b}((\mathrm{Tor}_{\nabla\gamma})_a^{1,1})\right).$$

As the components $(3, 0)$ and $(0, 3)$ of $\mathfrak{b}(\mathrm{Tor}_{\nabla\gamma})$ both vanish, one obtains

$$\mathfrak{b}(\mathrm{Tor}_{\nabla\gamma})^+ = \mathfrak{b}(\mathrm{Tor}_{\nabla\gamma}).$$

Furthermore, from the following equalities:

$$\mathfrak{b}((\mathrm{Tor}_{\nabla\gamma})_a^{1,1}) = \mathfrak{b}((\mathrm{Tor}_{\nabla\gamma})^{1,1}) = 2\mathfrak{b}(\mathrm{Tor}_{\nabla\gamma}),$$

we obtain $(d^c F)^+ = 3(\mathfrak{b}(\mathrm{Tor}_{\nabla\gamma}) - 4\mathfrak{b}(\mathrm{Tor}_{\nabla\gamma})) = -9\mathfrak{b}(\mathrm{Tor}_{\nabla\gamma})$. Hence, the second formula in [8, (2.5.1)] yields

$$\mathfrak{b}(\mathrm{Tor}_{\nabla\gamma}) = \frac{2t-1}{3}(-9\mathfrak{b}(\mathrm{Tor}_{\nabla\gamma})) = 3(1-2t)\mathfrak{b}(\mathrm{Tor}_{\nabla\gamma}).$$

(iii) Here, we follow [1], [2], [3], [4]. Let $\leq$ be the ordering in the set of indices $\tilde{i} \in \{i, i', i'', i''' : 1 \leq i \leq m\}$ given by the following rules: $i < j' < k'' < l'''$,

$a' < b' \iff a'' < b'' \iff a''' < b''' \iff a < b$. Let $(x^{\tilde{a}})$ be a coordinate system centred at $x \in M$ such that the basis $v_{\tilde{a}} = (\partial/\partial x^{\tilde{a}})_x$ is adapted to I, J, K , i.e., $v_{a'} = I(v_a)$, $v_{a''} = J(v_a)$, $v_{a'''} = K(v_a)$. Hence, the matricial expression of the operators I, J and K at x (writing $I(v_{\tilde{a}}) = I_{\tilde{a}}^{\tilde{b}} v_{\tilde{b}}$, and similarly for J, K) are

$$\text{Matrix of } I_x = \begin{pmatrix} O & -\text{id}_m & O & O \\ \text{id}_m & O & O & O \\ O & O & O & -\text{id}_m \\ O & O & \text{id}_m & O \end{pmatrix},$$

$$\text{Matrix of } J_x = \begin{pmatrix} O & O & -\text{id}_m & O \\ O & O & O & \text{id}_m \\ \text{id}_m & O & O & O \\ O & -\text{id}_m & O & O \end{pmatrix},$$

$$\text{Matrix of } K_x = \begin{pmatrix} O & O & O & -\text{id}_m \\ O & O & -\text{id}_m & O \\ O & \text{id}_m & O & O \\ \text{id}_m & O & O & O \end{pmatrix}.$$

Let

$$\begin{aligned} (\text{Tor}_{\nabla^{\gamma_{\text{Ob}}}})_x &= \sum_{\tilde{a} < \tilde{b}} \tau_{\tilde{a}\tilde{b}}^{\tilde{c}} v_{\tilde{c}} \otimes v^{\tilde{a}} \wedge v^{\tilde{b}} \\ &= \frac{1}{12} \{N(I)_x + N(J)_x + N(K)_x\}, \end{aligned}$$

be the torsion of the Obata connection with $\tau_{\tilde{a}\tilde{b}}^{\tilde{c}} = (\Gamma^{\gamma_{\text{Ob}}})_{\tilde{a}\tilde{b}}^{\tilde{c}}(x) - (\Gamma^{\gamma_{\text{Ob}}})_{\tilde{b}\tilde{a}}^{\tilde{c}}(x)$, $N(I), N(J), N(K)$ being the Nijenhuis tensor field of I, J, K , respectively. The local expression of $N(I)$ is (cf. [15, IX, p. 124]),

$$N_{\tilde{a}\tilde{b}}^{\tilde{c}}(I)(x) = 2 \left\{ I_{\tilde{a}}^{\tilde{h}}(x) \frac{\partial I_{\tilde{b}}^{\tilde{c}}}{\partial x^{\tilde{h}}}(x) - I_{\tilde{b}}^{\tilde{h}}(x) \frac{\partial I_{\tilde{a}}^{\tilde{c}}}{\partial x^{\tilde{h}}}(x) - I_{\tilde{h}}^{\tilde{c}}(x) \frac{\partial I_{\tilde{b}}^{\tilde{h}}}{\partial x^{\tilde{a}}}(x) + I_{\tilde{h}}^{\tilde{c}}(x) \frac{\partial I_{\tilde{a}}^{\tilde{h}}}{\partial x^{\tilde{b}}}(x) \right\},$$

and similarly for J and K . Differentiating the equations $I^2 = -\text{id}$, $J^2 = -\text{id}$, $IJ = K$, and $IJ = -JI$ at $x \in M$, yields the following set of constraints on the derivatives of I, J , and K :

$$\begin{aligned} 0 &= \frac{\partial I_{\tilde{h}}^{\tilde{b}}}{\partial x^{\tilde{c}}}(x) I_{\tilde{a}}^{\tilde{h}}(x) + \frac{\partial I_{\tilde{a}}^{\tilde{h}}}{\partial x^{\tilde{c}}}(x) I_{\tilde{h}}^{\tilde{b}}(x), \\ 0 &= \frac{\partial J_{\tilde{h}}^{\tilde{b}}}{\partial x^{\tilde{c}}}(x) J_{\tilde{a}}^{\tilde{h}}(x) + \frac{\partial J_{\tilde{a}}^{\tilde{h}}}{\partial x^{\tilde{c}}}(x) J_{\tilde{h}}^{\tilde{b}}(x), \\ \frac{\partial K_{\tilde{a}}^{\tilde{b}}}{\partial x^{\tilde{c}}}(x) &= \frac{\partial J_{\tilde{a}}^{\tilde{h}}}{\partial x^{\tilde{c}}}(x) I_{\tilde{h}}^{\tilde{b}}(x) + \frac{\partial I_{\tilde{h}}^{\tilde{b}}}{\partial x^{\tilde{c}}}(x) J_{\tilde{a}}^{\tilde{h}}(x), \\ 0 &= \frac{\partial I_{\tilde{h}}^{\tilde{b}}}{\partial x^{\tilde{c}}}(x) J_{\tilde{a}}^{\tilde{h}}(x) + \frac{\partial J_{\tilde{a}}^{\tilde{h}}}{\partial x^{\tilde{c}}}(x) I_{\tilde{h}}^{\tilde{b}}(x) + \frac{\partial J_{\tilde{h}}^{\tilde{b}}}{\partial x^{\tilde{c}}}(x) I_{\tilde{a}}^{\tilde{h}}(x) + \frac{\partial I_{\tilde{a}}^{\tilde{h}}}{\partial x^{\tilde{c}}}(x) J_{\tilde{h}}^{\tilde{b}}(x). \end{aligned}$$

Taking these constraints into account, it follows that the components $\tau_{\bar{a}\bar{b}}^{\bar{c}}$ of the torsion of the Obata connection fulfill the condition (1), which are written down as follows in the present case:

$$\begin{aligned} 0 &= \tau_{\bar{i}q}^p + \tau_{\bar{i}q'}^{p'} + \tau_{\bar{i}q''}^{p''} + \tau_{\bar{i}q'''}^{p'''} , & 0 &= \tau_{\bar{i}q'}^p - \tau_{\bar{i}q}^{p'} - \tau_{\bar{i}q'''}^{p''} + \tau_{\bar{i}q''}^{p'''} , \\ 0 &= \tau_{\bar{i}q''}^p + \tau_{\bar{i}q'''}^{p'} - \tau_{\bar{i}q}^{p''} - \tau_{\bar{i}q'}^{p'''} , & 0 &= \tau_{\bar{i}q'''}^p - \tau_{\bar{i}q''}^{p'} + \tau_{\bar{i}q'}^{p''} - \tau_{\bar{i}q}^{p'''} . \end{aligned}$$

□

Remark 3.2. The conclusion of item (i) in Theorem 3.1 also holds for a pseudo-Riemannian metric, and item (ii) holds for a para-Hermitian manifold with the same value of the parameter $t = \frac{1}{3}$. Both proofs are similar; see [13] for the para-Hermitian case.

As far as we know, the canonical connections in the Hermitian as well as in the para-Hermitian case, do not appear as distinguished canonical connections in the literature.

4 A criterion for the existence of a functorial connection

Below, we characterize the algebras $\mathfrak{gl}(r, A)$ satisfying the criterion given in [22, Theorems 1.1 & 1.2].

Theorem 4.1. *With the previous hypotheses and notations we have*

- (a) *The first prolongation of the algebra $\mathfrak{gl}(r, A)$ can be identified with the set $\mathcal{S}^r$ of systems (v_{ij}) , $1 \leq i \leq j \leq r$, $v_{ij} \in A^r$, such that,*

$$(8) \quad v_{ij} \cdot [a, b] = 0, \quad \forall a, b \in A, \quad 1 \leq i \leq j \leq r.$$

Accordingly, $\mathfrak{gl}(r, A)^{(1)}$ vanishes if and only if the condition $x \cdot [a, b] = 0$, $\forall a, b \in A$, implies $x = 0$.

- (b) *The subalgebra $\mathfrak{gl}(r, A) \subset \mathfrak{gl}(nr, \mathbb{R})$ is invariant under transposition if and only if A is a C^* -algebra.*

Hence the unique subalgebras $\mathfrak{gl}(r, A)$ invariant under transposition and further satisfying $\mathfrak{gl}(r, A)^{(1)} = 0$, correspond to the algebras A that are isomorphic to a finite direct sum of matrix algebras of the form

$$\text{Mat}_k(\mathbb{R}) \ (k \geq 2), \quad \text{Mat}_k(\mathbb{C}) \ (k \geq 2), \quad \text{Mat}_k(\mathbb{H}) \ (k \geq 1).$$

Proof. The elements of $\mathfrak{gl}(r, A)^{(1)}$ can be viewed as the symmetric $\mathbb{R}$ -bilinear mappings $B: A^r \times A^r \rightarrow A^r$ such that for every $v \in A^r$, $B(v, -) \in \mathfrak{gl}(r, A)$, or equivalently $B(v, -): A^r \rightarrow A^r$ is a right A -linear mapping. Let $(v_1, \dots, v_r)$ be

the standard basis of A^r . For every $1 \leq i \leq j \leq r$, we set $B(v_i, v_j) = v_{ij}$. Hence for all $a, b \in A$, we have

$$\begin{aligned} B(v_i \cdot a, v_j \cdot b) &= B(v_i \cdot a, v_j) \cdot b = B(v_j, v_i \cdot a) \cdot b = v_{ij} \cdot ab, \\ B(v_j \cdot b, v_i \cdot a) &= B(v_j \cdot b, v_i) \cdot a = B(v_i, v_j \cdot b) \cdot a = v_{ij} \cdot ba. \end{aligned}$$

Taking the symmetry of B into account, we conclude that the equation (8) holds. Accordingly we can define an $\mathbb{R}$ -linear mapping $\varphi: \mathfrak{gl}(r, A)^{(1)} \rightarrow \mathcal{S}^r$ by setting $\varphi(B) = (v_{ij})$, $1 \leq i \leq j \leq r$, which is injective since if $(e_1, \dots, e_n)$ is a basis of A as an $\mathbb{R}$ -vector space, then $(v_i \cdot e_h)$, $1 \leq i \leq r$, $1 \leq h \leq n$, is a basis of A^r over $\mathbb{R}$, and from the equation above we obtain, $B(v_i \cdot e_h, v_k \cdot e_j) = v_{ik} \cdot e_h e_j$, for $i, k = 1, \dots, r$, $1 \leq h, j \leq n$, with $v_{ik} = v_{ki}$ whenever $i > k$.

Conversely, given a system $(v_{ij}) \in \mathcal{S}^r$ we can define a symmetric $\mathbb{R}$ -bilinear mapping $B: A^r \times A^r \rightarrow A^r$ by the previous formula and $B(v_i \cdot e_h, -)$ is right A -linear as for every $1 \leq l \leq n$, we have

$$\begin{aligned} B(v_i \cdot e_h, (v_k \cdot e_j) \cdot e_l) &= \sum_{p=1}^n \lambda_{jl}^p B(v_i \cdot e_h, v_k \cdot e_p) \\ &= \sum_{p=1}^n \lambda_{jl}^p v_{ik} \cdot e_h e_p \\ &= B(v_i \cdot e_h, v_k \cdot e_j) \cdot e_l, \end{aligned}$$

where the scalars $\lambda_{jl}^p \in \mathbb{R}$ are defined by $e_j \cdot e_l = \sum_{p=1}^n \lambda_{jl}^p e_p$, thus proving that φ is surjective.

Let $R_a^r: A^r \rightarrow A^r$ be the right translation by $a \in A$, i.e.,

$$R_a^r(x_1, \dots, x_r) = (x_1 a, \dots, x_r a).$$

Hence $R_a^r = \oplus^r R_a^1$ and $(R_a^r)^T = \oplus^r (R_a^1)^T$, where Λ^T denotes the transpose of the matrix $\Lambda \in \mathfrak{gl}(nr, \mathbb{R})$. Therefore $(R_a^r)^T$ is A -linear if and only if $(R_a^1)^T$ is. Accordingly, if $\mathfrak{gl}(r, A)$ is invariant by transposition, then for every $a \in A$ there exists a unique element $a^* \in A$ such that, $(R_a)^T = R_{a^*}$. The algebra A endowed with $a \mapsto a^*$ and the norm $a \mapsto \|a\| = \|R_a\|$ inherited from the standard norm in $\mathfrak{gl}(nr, \mathbb{R})$, determines a C^* -algebra structure on A ; e.g., see [10, Chapter 8].

We can thus deduce the final conclusion of the statement by simply applying the Wedderburn theorem for real C^* -algebras, e.g., see [10, Theorem 8.4].

In order to prove the converse of (b) we can confine ourselves to consider only the cases in which A is equal to $\text{Mat}_k(\mathbb{R})_{k \geq 2}$, $\text{Mat}_k(\mathbb{C})_{k \geq 2}$, or $\text{Mat}_k(\mathbb{H})_{k \geq 1}$, as transposition commutes with direct sums.

If $A = \text{Mat}_k(\mathbb{F})$, where $\mathbb{F}$ denotes either $\mathbb{R}$, or $\mathbb{C}$, or $\mathbb{H}$ and $\Lambda \in \mathfrak{gl}(r, A)$, then we set $\Lambda(v_i) = \sum_{h=1}^r v_h \cdot \Lambda_i^h$, where $\Lambda_i^h \in A$, $h, i = 1, \dots, r$.

Assume $\mathbb{F} = \mathbb{R}$ (hence $n = k^2$) and let $(E_b^a)_{a,b=1}^k$ be the standard basis for $A = \text{Mat}_k(\mathbb{R})$ as a vector $\mathbb{R}$ -space, i.e., E_b^a is the square matrix such that the entry at the a -th row and b -th column is 1 and all the other entries are zero.

The lexicographic order is considered on the basis $(v_h \cdot E_b^a)$, $1 \leq h \leq r$, $a, b = 1, \dots, k$, for $\mathfrak{gl}(nr, \mathbb{R})$ as a vector $\mathbb{R}$ -space; namely,

$$\begin{aligned} & v_1 \cdot E_1^1 < v_1 \cdot E_2^1 < \dots < v_1 \cdot E_k^1 < \dots \\ & < v_1 \cdot E_1^k < v_1 \cdot E_2^k < \dots < v_1 \cdot E_k^k < \dots \\ & \vdots \\ & < v_r \cdot E_1^1 < v_r \cdot E_2^1 < \dots < v_r \cdot E_k^1 < \dots \\ & < v_r \cdot E_1^k < v_r \cdot E_2^k < \dots < v_r \cdot E_k^k. \end{aligned}$$

Let $\tilde{\Lambda}$ be the matrix representing Λ in the embedding $\mathfrak{gl}(r, A) \subset \mathfrak{gl}(nr, \mathbb{R})$ with respect to the basis above. As the straightforward—but rather long—computation of the products $\Lambda_i^h E_b^a$ shows, the matrix $\tilde{\Lambda}$ is the $r \times r$ block matrix $\tilde{\Lambda} = (\lambda_{i,b}^{h,a} I_r)$, $h, i = 1, \dots, r$, $a, b = 1, \dots, k$, where I_r denotes the identity matrix of order r and $\lambda_{i,b}^{h,a}$ are the entries of Λ_i^h . Hence $\tilde{\Lambda}^T = \widetilde{\Lambda^T}$, the A -linear mapping Λ^T in $\mathfrak{gl}(r, A)$ being defined by $\Lambda^T(v_i) = \sum_{h=1}^r v_h \cdot (\Lambda_i^h)^T$, $1 \leq i \leq r$.

Next, assume $\mathbb{F} = \mathbb{C}$ (hence $n = 2k^2$) and let us put the following order on the basis $(v_h \cdot E_b^a, v_h \cdot E_b^a \mathbf{i})$, $1 \leq h \leq r$, $a, b = 1, \dots, k$, for $\mathfrak{gl}(nr, \mathbb{R})$:

$$\begin{aligned} & v_1 \cdot E_1^1 < v_1 \cdot E_1^1 \mathbf{i} < v_1 \cdot E_2^1 < v_1 \cdot E_2^1 \mathbf{i} < \dots < v_1 \cdot E_k^1 < v_1 \cdot E_k^1 \mathbf{i} < \dots \\ & < v_1 \cdot E_1^k < v_1 \cdot E_1^k \mathbf{i} < v_1 \cdot E_2^k < v_1 \cdot E_2^k \mathbf{i} < \dots < v_1 \cdot E_k^k < v_1 \cdot E_k^k \mathbf{i} < \dots \\ & \vdots \\ & < v_r \cdot E_1^1 < v_r \cdot E_1^1 \mathbf{i} < v_r \cdot E_2^1 < v_r \cdot E_2^1 \mathbf{i} < \dots < v_r \cdot E_k^1 < v_r \cdot E_k^1 \mathbf{i} < \dots \\ & < v_r \cdot E_1^k < v_r \cdot E_1^k \mathbf{i} < v_r \cdot E_2^k < v_r \cdot E_2^k \mathbf{i} < \dots < v_r \cdot E_k^k < v_r \cdot E_k^k \mathbf{i}. \end{aligned}$$

If $\lambda_{i,b}^{h,a} = \lambda_{0,i,b}^{h,a} + \lambda_{1,i,b}^{h,a} \mathbf{i}$, $\lambda_{0,i,b}^{h,a}, \lambda_{1,i,b}^{h,a} \in \mathbb{R}$, are the entries of Λ_i^h , and we set

$$\tilde{\lambda}_{i,b}^{h,a} = \begin{pmatrix} \lambda_{0,i,b}^{h,a} & -\lambda_{1,i,b}^{h,a} \\ \lambda_{1,i,b}^{h,a} & \lambda_{0,i,b}^{h,a} \end{pmatrix}, \quad \tilde{\Lambda}_{i,b}^{h,a} = \begin{pmatrix} \tilde{\lambda}_{i,b}^{h,a} & O_2 \\ O_2 & \tilde{\lambda}_{i,b}^{h,a} \end{pmatrix},$$

where O_2 denotes the zero matrix of size 2×2 , then $\tilde{\Lambda}$ is the $r \times r$ block matrix $\tilde{\Lambda} = (\tilde{\Lambda}_{i,b}^{h,a})$. Hence $\tilde{\Lambda}^T = \widetilde{\Lambda^*}$, where the A -linear mapping Λ^* in $\mathfrak{gl}(r, A)$ is given by,

$$(9) \quad \Lambda^*(v_i) = \sum_{h=1}^r v_h \cdot (\overline{\Lambda_i^h})^T, \quad 1 \leq i \leq r,$$

the bar denoting complex conjugation.

Finally, if $\mathbb{F} = \mathbb{H}$ (hence $n = 4k^2$), then the following order is put on the basis $(v_h \cdot E_b^a, v_h \cdot E_b^a \mathbf{i}, v_h \cdot E_b^a \mathbf{j}, v_h \cdot E_b^a \mathbf{k})$, $1 \leq h \leq r$, $a, b = 1, \dots, k$, for $\mathfrak{gl}(nr, \mathbb{R})$:

$$\dots < v_h \cdot E_b^a < v_h \cdot E_b^a \mathbf{i} < v_h \cdot E_b^a \mathbf{j} < v_h \cdot E_b^a \mathbf{k} < \dots$$

If $\lambda_{i,b}^{h,a} = \lambda_{0,i,b}^{h,a} + \lambda_{1,i,b}^{h,a}\mathbf{i} + \lambda_{2,i,b}^{h,a}\mathbf{j} + \lambda_{3,i,b}^{h,a}\mathbf{k}$, $\lambda_{c,i,b}^{h,a} \in \mathbb{R}$, $0 \leq c \leq 3$, are the entries of Λ_i^h , and we set

$$\tilde{\lambda}_{i,b}^{h,a} = \begin{pmatrix} \lambda_{0,i,b}^{h,a} & -\lambda_{1,i,b}^{h,a} & -\lambda_{2,i,b}^{h,a} & -\lambda_{3,i,b}^{h,a} \\ \lambda_{1,i,b}^{h,a} & \lambda_{0,i,b}^{h,a} & -\lambda_{3,i,b}^{h,a} & \lambda_{2,i,b}^{h,a} \\ \lambda_{2,i,b}^{h,a} & \lambda_{3,i,b}^{h,a} & \lambda_{0,i,b}^{h,a} & -\lambda_{1,i,b}^{h,a} \\ \lambda_{3,i,b}^{h,a} & -\lambda_{2,i,b}^{h,a} & \lambda_{1,i,b}^{h,a} & \lambda_{0,i,b}^{h,a} \end{pmatrix}, \quad \tilde{\Lambda}_{i,b}^{h,a} = \begin{pmatrix} \tilde{\lambda}_{i,b}^{h,a} & O_4 \\ O_4 & \tilde{\lambda}_{i,b}^{h,a} \end{pmatrix},$$

where O_4 denotes the zero matrix of size 4×4 , then $\tilde{\Lambda}$ is the $r \times r$ block matrix $\tilde{\Lambda} = (\tilde{\Lambda}_{i,b}^{h,a})$, and as in the previous case we obtain, $\tilde{\Lambda}^T = \tilde{\Lambda}^*$, where the A -linear mapping $\Lambda^* \in \mathfrak{gl}(r, A)$ is again defined by the formula (9) and the bar now denotes quaternionic conjugation. $\square$

Corollary 4.2. *Although every almost complex manifold admits almost complex linear connections with special properties (cf. [15, IX, Theorem 3.4]), the category of such manifolds does not admit any functorial connection as its associated algebra, i.e., $A = \mathbb{C}$, is not included in the list of Theorem 4.1.*

Remark 4.3. As a consequence of the property (a) in the previous theorem, it follows that the property (b) implies $\mathfrak{gl}(r, A)^{(1)} = 0$.

Remark 4.4. We should finally like to mention the weak connection of the results above with nonassociative algebras. If A is a nonassociative F -algebra, then there exists a natural one-to-one correspondence between left A -modules and left $\bar{A}$ -modules, $\bar{A}$ being the associative algebra attached to A (i.e., $\bar{A}$ is the quotient algebra of A by the bilateral ideal generated by the associators $(a, b, c) = (ab)c - a(bc)$ of any three elements in A , cf. [20, p. 13]). In fact, if N is a left A -module, then every associator (a, b, c) belongs to the kernel of the homomorphism $A \rightarrow \text{End}_F N$, $a \mapsto L_a$. Hence, N inherits a structure of left $\bar{A}$ -module. In practice, however, this result is not interesting, as the associative algebras attached to classical nonassociative algebras are the ground field or alike.

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