Reply to “Comment on ‘Fermionic entanglement ambiguity in noninertial frames’ ”

Miguel Montero and Eduardo Martín-Martínez

Instituto de Física Fundamental, Consejo Superior de Investigaciones Científicas, Serrano 113-B, E-28006 Madrid, Spain

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We show that, contrary to the claims of the preceding Comment [Brádler and Jáuregui, *Phys. Rev. A* **85**, 016301 (2012)], our original work [*Phys. Rev. A* **83**, 062323 (2011)] is devoid of flaws or mistakes. We show that the criticism comes from a misunderstanding of part of our work.

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I. INTRODUCTION

In Ref. [1] we showed that some previous results on fermionic entanglement in noninertial frames were not physical due to the arbitrary endowment of a tensor product structure on the fermionic field before computing entanglement measures. We found that this scheme of endowing a tensor product structure to the Fock space may still yield physical results if a particular tensor product structure (different from that of the literature) was chosen.

In Ref. [2], Brádler and Jáuregui claim that our work is flawed. Their criticisms seem to reflect that they understood our work as claiming that there is a real ambiguity in the physical results instead of reflecting a wrong mathematical approach when studying fermionic entanglement in noninertial frames. This is not our claim, and in this Reply we intend to show explicitly where the misunderstandings lie. We will also show that the formalism in Ref. [1] is exactly equivalent to the derivation provided in Ref. [2].

However, after considering the Comment, we feel that this crucial point was not clear enough in our original paper: Ref. [1] does not say that there is an ambiguity in the way entanglement is defined in fermionic systems, but rather it is in the way it had been computed by some previous works on field entanglement in noninertial frames.

In addition, it is true that for some simple examples of the phenomenon we wanted to show, we chose field states not compliant with the fermionic superselection rule. This was regarded by Ref. [2] as a fatal flaw in our work. In this Reply we show that it was only was an academic (and unfortunate) choice of examples for the sake of simplicity. Indeed, the particular states used in these examples are irrelevant for the results of Ref. [1], so we give here similar examples, but now compliant with the rule showing that this fact has no impact on our claims.

In Sec. II we explicitly show the equivalence of the formalism in Ref. [2] with the one used in Ref. [1], both giving exactly the same results. In Sec. III we discuss the fermionic superselection rule as applied to our states and review the negativity as a valid entanglement measure for fermionic systems, contrasting our views with those of Ref. [2]. Finally, we summarize our conclusions in Sec. IV.

II. TENSOR PRODUCT STRUCTURES AND ENTANGLEMENT

We will show that the final results of Ref. [1] (see Sec. IV of that work) are consistent with the fermionic nature of the field.

The calculations with the formalism of Ref. [1] in Ref. [2] are consistent as well, contrary to the claims in that reference.

First, the authors of Ref. [2] correctly assert that there is no natural tensor product structure in a multipartite fermionic system, as this would conflict with the canonical anticommutation relations. Reference [2] considers that these different results appeared only because we did not respect the canonical anticommutation relations.

However, this is not the case. The fact that there is no natural tensor product structure in a fermionic system is a well-known problem and especially manifests in quantum information theory when partitions of fermionic systems have to be defined.

Not even the braided tensor product suggested in Ref. [2] is exempt from these issues when the problem of separability comes forth. As a concrete example, consider the fermionic four-mode state

$$|\Psi\rangle = \frac{1}{2}(|1_a 1_c\rangle + |1_a 1_d\rangle + |1_b 1_c\rangle + |1_b 1_d\rangle) = \left[\frac{1}{\sqrt{2}}(|1_a\rangle + |1_b\rangle) \right] \tilde{\otimes} \left[\frac{1}{\sqrt{2}}(|1_c\rangle + |1_d\rangle) \right], \quad (1)$$

which factors with respect to the braided tensor product when expressed in the basis

$$|1_a 1_c\rangle = a^\dagger \tilde{\otimes} c^\dagger |0\rangle, \quad |1_b 1_c\rangle = b^\dagger \tilde{\otimes} c^\dagger |0\rangle, \\ |1_a 1_d\rangle = a^\dagger \tilde{\otimes} d^\dagger |0\rangle, \quad |1_b 1_d\rangle = b^\dagger \tilde{\otimes} d^\dagger |0\rangle. \quad (2)$$

This state can be also expressed in the basis

$$|1_a 1_c\rangle' = a^\dagger \tilde{\otimes} c^\dagger |0\rangle, \quad |1_b 1_c\rangle' = b^\dagger \tilde{\otimes} c^\dagger |0\rangle, \\ |1_a 1_d\rangle' = a^\dagger \tilde{\otimes} d^\dagger |0\rangle, \quad |1_b 1_d\rangle' = -d^\dagger \tilde{\otimes} b^\dagger |0\rangle, \quad (3)$$

which differs in one sign due to the antisymmetry properties of fermionic systems (we have permuted the labels of the first and second subsystems). Then it reads

$$|\Psi\rangle = \frac{1}{2}(|1_a 1_c\rangle + |1_a 1_d\rangle + |1_b 1_c\rangle - |1_b 1_d\rangle), \quad (4)$$

which is not factorizable (in fact, it is maximally entangled with respect to the basis defined by the braided product). Of course we are not extracting any conclusions about the physical meaning of this factorizability; we are only showing that problems with multipartite fermionic tensorial structures are present even if we use the formalism in Ref. [2] and even if the state has no superselection rule issues.

However, we are not claiming that the formalism used in Ref. [2] is incorrect; on the contrary, we claim that it is equivalent to what we do in Ref. [1]. In that paper we simply

use an approach that is different from that in Ref. [2], which is equally valid since all the physical conclusions are the same. To show that, we now proceed to describe our approach in detail from a somewhat more mathematical viewpoint than that used in Ref. [1]; to do so, we start with some definitions.

We shall consider a particular set of $m + n$ fermionic modes. The annihilation operators of these modes will be labeled a_i and c_j , with $i = \{1, \dots, n\}$ and $j = \{1, \dots, m\}$. These modes span a 2^{m+n} -dimensional fermionic Fock space when acting on their vacuum state $|0\rangle_a \tilde{\otimes} |0\rangle_c$. We can define the usual particle number basis as follows:

$$\begin{aligned} & \underbrace{|1 \dots 1\rangle}_{m+n}{}_{a_1, \dots, a_n c_1, \dots, c_m} \\ &= (a_1)^\dagger \dots (a_n)^\dagger (c_1)^\dagger \dots (c_m)^\dagger |0\rangle_a \tilde{\otimes} |0\rangle_c \end{aligned} \quad (5)$$

and for any basis elements with a 0 instead of a 1, we just remove the corresponding operator on the right-hand side.

We also consider a field state ρ over this Hilbert space. This state will be uniquely determined by its matrix elements in the basis described above, so that

$$\rho = \sum_{k_a, l_a, k_c, l_c} \rho_{k_a, l_a, k_c, l_c} |k_a k_c\rangle \langle l_a l_c|, \quad (6)$$

where k_a and k_c are strings of 0's and 1's of length n and l_a and l_c are strings of 0's and 1's of length m . Here, it is worth noting that

$$(|l\rangle_{a_1, \dots, a_n c_1, \dots, c_m})^\dagger = \langle l|_{c_m, \dots, c_1 a_n, \dots, a_1}. \quad (7)$$

Note the reordering of the mode labels in Eq. (7). Finally, we introduce a mapping from this fermionic Fock space to a qubit Hilbert space of $m + n$ modes.

Definition. Let a_i^{bos} and c_j^{bos} be two sets of bosonic operators acting on their vacuum state $|0\rangle$. For each fermionic state ρ , we define $\mathcal{Q}(\rho)$ as the state formally obtained as follows. First, write down ρ in the basis Eq. (5) [i.e., as in Eq. (6)]. Then, formally replace each a_i with a_i^{bos} , c_j with c_j^{bos} , and the vacuum state $|0\rangle$ with $|0\rangle$.

We will denote the inverse mapping by $\mathcal{Q}^{-1}$. The restriction of this mapping to the subspace spanned by a^{bos} modes on $|0\rangle$ will be denoted by $\mathcal{Q}^{-1}|_a$.

Note that this definition relies heavily on the use of the basis defined by Eq. (5). We can define analogous mappings that are different from $\mathcal{Q}$ and that rely on different bases obtained, e.g., by permuting the operators in Eq. (5). This way we obtain a family of different mappings, each of them associated with an operator ordering. The mapping $\mathcal{Q}$ corresponds to what we call the physical ordering in Ref. [1].

The map $\mathcal{Q}$ can be said to be equivalent to the endowment of a particular tensor product structure on the fermionic Fock space. This formal endowment has been widely used in the literature [3–13]. The fact that that fermions and bosons correspond to different group representations is not an obstacle for the definition of $\mathcal{Q}$. Of course, if care is not taken in the choice of the basis in Eq. (5), this procedure leads to unphysical results for reduced states and entanglement measures, among other things. Reference [1] studies when and how this procedure can be used to obtain correct, physical results. These results are the same as those that would be obtained with the braided formalism of Ref. [2].

Now, we are interested in tracing out the c modes. In the context of Ref. [1], this is because field modes in region II of Rindler space-time are causally disconnected from the observers. This is implemented by the partial trace, which may be defined as

$$\begin{aligned} \rho &\rightarrow \text{Tr}_c(\rho) \\ &\equiv \sum_{j_1, j_2, \dots, j_m \in \{0, 1\}} \langle 0|_c (c_m)^{j_m} \dots (c_1)^{j_1} \rho (c_1^\dagger)^{j_1} \dots (c_m^\dagger)^{j_m} |0\rangle_c. \end{aligned} \quad (8)$$

Note that Eq. (8) may give ambiguous results for states not complying with the superselection rule of Ref. [2], but it defines a unique reduced state for states within a given superselection sector.

We now prove that the same result is obtained through direct application of Eq. (8) or, equivalently, through the use of the $\mathcal{Q}$ mapping (i.e., the tensor product structure associated with the physical ordering), which is what we do in our paper.

Theorem. Let ρ be a fermionic state compliant with the fermionic superselection rule. Then the equality

$$\text{Tr}_c(\rho) = \mathcal{Q}^{-1}|_a \circ \text{Tr}_c^{\text{bos}}(\mathcal{Q}(\rho))$$

holds, where Tr_c^{bos} is the usual partial trace operation in a qubit system.

Proof. Since by virtue of the definition in Eq. (8) the fermionic partial trace is a linear operator, we can consider each of the terms in Eq. (6) independently. For each of these terms we have

$$\begin{aligned} & \text{Tr}_c |k_a k_c\rangle \langle l_a l_c| \\ &= \delta_{k_c, l_c} (c_m)^{j_m} \dots \langle 0|_c (c_1)^{j_1} (a_1^\dagger)^{i_1} \dots (a_n^\dagger)^{i_n} (c_1^\dagger)^{j_1} \dots (c_m^\dagger)^{j_m} \\ & \quad \times |0\rangle \langle 0| (c_m)^{j_m} \dots (c_1)^{j_1} (a_n)^{k_n} \dots (a_1)^{k_1} (c_1^\dagger)^{j_1} \dots (c_m^\dagger)^{j_m} |0\rangle_c \\ &= (a_1^\dagger)^{i_1} \dots (a_n^\dagger)^{i_n} |0\rangle \langle 0| (a_n)^{k_n} \dots (a_1)^{k_1} \\ &= \delta_{k_c, l_c} |k_a\rangle \langle l_a|, \end{aligned} \quad (9)$$

with the last equality holding because in the second line of Eq. (9) the number of a operators to the left and the number of a operators to the right of $|0\rangle \langle 0|$ are of the same parity, due to the superselection rule. Hence, both left and right operators give the same global sign when all the anticommutators are computed, giving no extra anticommutation signs.

The trace on $\mathcal{Q}(\rho)$ is also a linear operator, so we also consider the terms independently. In this case, no sign issues appear and the reduced state is simply $|k_a\rangle \langle l_a|_{\text{bos}}$, where the subscript reminds us that this is still a qubit state; finally, $\mathcal{Q}^{-1}|_a$ maps $|k_a\rangle \langle l_a|_{\text{bos}}$ to $|k_a\rangle \langle l_a|$, which ends the proof.

Note that this theorem implies that any operation being a function of the fermionic matrix elements only (for instance, to study separability or correlations) is bound to give the same results for $\text{Tr}_c(\rho)$ and $\text{Tr}_c^{\text{bos}} \mathcal{Q}(\rho)$. As in our previous paper we use only this kind of operation, our work is devoid of algebraic mistakes.

We have just proven that using $\mathcal{Q}$ guarantees the correctness of our results. As we mentioned before, we can also define other mappings that are analogous to $\mathcal{Q}$, but which take bases different from Eq. (5) as the starting point for their definition. For instance, we could rearrange the operators in Eq. (5) in an arbitrary way and then by a definition analogous to that in

Sec. II we would obtain different mappings, say, $\mathcal{D}_1$ and $\mathcal{D}_2$, among others.

Section II [more concretely, Eqs. (5) and (6) and their associated discussion], where Ref. [2] considers that there are algebraic mistakes, is just a simple proof by counterexample of the fact that, in general,

$$\mathcal{D}_1^{-1}|_a \circ \text{Tr}_c^{\text{bos}} \circ \mathcal{D}_1 \neq \mathcal{D}_2^{-1}|_a \circ \text{Tr}_c^{\text{bos}} \circ \mathcal{D}_2. \quad (10)$$

In other words, all Ref. [1] does is consider the class of all these mappings $\mathcal{D}_i$ (each identified by the fermionic operator ordering they come from) and analyze the entanglement properties of the qubit systems they give rise to. This does not lead to physical results in general, but it is what has been done before in the literature. We also provide a particular class of operator orderings (and their associated $\mathcal{D}_i$) that can be regarded as physical; when we perform the partial trace in the qubit state, the states obtained are exactly those obtained with the braided tensor product formalism in Ref. [2].

A simple proof of the validity of the formalism developed in Ref. [1] can be seen in Ref. [14], where it is shown that, in general, entanglement of fermionic systems must converge in the infinite acceleration limit. This work shows how the measures computed using nonphysical operator orderings do not comply with this requirement, while measures computed using the physical ordering [1] do.

Considering these various unphysical orderings is far from being a fruitless mathematical exercise; endowing a fermionic system with some particular tensor product structure may prove quite useful, for instance, when computing the field vacuum and excitations. This was the main reason for doing so, as can be seen in our previous paper as well as in a number of other works [3–12]. Even Ref. [13] (a seminal article cited in Ref. [2]) uses a definite tensor product structure defined on a fermionic Hilbert space to provide a separability criterion for fermionic systems.

We are not claiming, as Ref. [2] suggests, that endowing a tensor product structure a fermionic field becomes a bosonic one. This would indeed result in inconsistencies, as pointed out by Ref. [2]. However, this is not what we did in Ref. [1]. Rather, we established a correspondence between fermionic and bosonic fields and studied how physical results for the fermionic case may be obtained from the corresponding bosonic systems.

To sum up, what Ref. [2] discusses is just another formalism for studying the same issues, not unlike other approaches such as defining entanglement only through observables, as done in Ref. [15], or through observable-induced tensor products [16]. Furthermore, the endowment of tensor product structures on fermionic systems as a separability criterion is not new [13]. Our aim when we wrote the paper was to make it accessible for all the people who had written papers about fermionic entanglement in noninertial frames without taking proper care of the problems regarding the tensor product structure of fermionic systems. Hence, we tailored our language for such an audience.

We realize that, maybe, the choice of title and abstract in Ref. [1] was not very fortunate. Instead of calling it ambiguity in noninertial frames, which may suggest that we were *reporting* a real ambiguity, we should have emphasized

the fact that we were pointing out and correcting an issue present in previous literature on fermionic entanglement in noninertial frames.

III. SUPERSELECTION RULES AND ENTANGLEMENT MEASURES

The authors of Ref. [2] note that the state in Eq. (3) of Ref. [1], namely,

$$|\Psi\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle), \quad (11)$$

with

$$|00\rangle = |0\rangle, \quad |10\rangle = a^\dagger|0\rangle, \quad |01\rangle = b^\dagger|0\rangle, \quad |11\rangle = a^\dagger b^\dagger|0\rangle, \quad (12)$$

does not comply with the fermionic superselection rule. This is true, but irrelevant for the argument. As Ref. [17] explains, the issue of whether or not a certain field state can be formed in practice cannot be settled by reference to symmetry principles resulting in superselection rules (in the case at hand, these would be rotational and CP symmetries) since we can always enlarge the symmetry group to a new one that lacks these superselection rules [17].

In any case, we were not claiming to be able to construct such states. In Sec. II of Ref. [1] we were merely presenting an academic example and not studying physical field states yet; the state in Eq. (1) reflects the fact that a state (a legitimate element of the Fock space) may be regarded as separable or entangled if we map it to a qubit system, depending on the operator ordering chosen. In any case, Eq. (1) or the three-mode state [Eq. (9) of Ref. [2] and Eq. (8) of Ref. [1]] are just examples to illustrate the kind of manipulations that are employed in Ref. [1]. Although the very general family of states presented in Ref. [1] does not fulfill the superselection rule, all the particular physical field states studied in that work do comply with it. This is also the case of the state in Eq. (1), which is nothing but a reformulation of the example state in Eq. (11), but compliant with the superselection rule.

These issues solved, Ref. [2] indeed raised an important style flaw in Ref. [1]: We introduced states and families of states not compliant with the superselection rule when this was not necessary at all for our arguments or conclusions. In one case, the use of states violating the rule allowed us to present clearer examples of the phenomena under consideration; in the other, it was done only for the sake of generality, although the particular cases considered were compliant with the rule. However, the introduction of these states has led the discussion far from the main points raised in Ref. [1] and the resultant confusion outweighs the benefits obtained by far. If we were to write Ref. [1] anew, we definitely would not use the same examples.

After this point, the criticism in Ref. [2] broadens from Ref. [1] to a questioning of the entanglement measures employed in many previous works in field entanglement in noninertial frames. Reference [2] claims that, as a consequence of the superselection rule, some entanglement measures are ill defined. While it is true that these entanglement measures cannot be interpreted exactly in the same way as for bosonic fields, some points of this interpretation remain however: If a

fermionic bipartite field state ρ is separable, meaning that it may be written as

$$\rho = \sum_i p_i A_i B_i |0\rangle\langle 0| B_i A_i, \quad p_i > 0, \quad (13)$$

with A_i containing only creation operators of the first bipartition and B_i containing only operators of the second, then it has vanishing negativity. The converse statement also holds for (2×2) - and (2×3) -dimensional systems.

Furthermore, Ref. [2] asserts that the superselection rule invalidates the use of entanglement measures in fermionic fields except in a few specific cases. However, this criticism is relevant only as long as we consider field states with different superselection parities for each of the parties. If we consider the field state of Ref. [11],

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A |0\rangle_R + |1\rangle_A |1\rangle_R), \quad (14)$$

then it is true that a Fock space basis for any of the parties (Alice or Rob) must include states in different superselection sectors, such as $|0\rangle$ and $|1\rangle$. However, if following Ref. [1] we consider, e.g., a Dirac singlet state such as

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A |\downarrow\rangle_R + |\downarrow\rangle_A |\uparrow\rangle_R), \quad (15)$$

then the Fock space basis for any party is composed of states in only one superselection sector. In this case, entanglement measures such as negativity recover exactly the same meaning they would have for qubit systems since the superselection rule now becomes trivial.

Finally, Ref. [2] elaborates on the various benefits of studying field entanglement in noninertial frames with the tools of quantum Shannon theory. Though we acknowledge these benefits, we see it as a complementary tool rather than an invalidation of the use of entanglement measures in noninertial frames. Negativity (the measure we use in Ref. [1]), for instance, is a direct measure of the distillable entanglement present in the state under consideration, which may be purified by means of standard protocols [18].

It can be shown, however, that the qualitative results are not completely sensitive to the entanglement measure employed

either. For instance, Ref. [19] computes the entanglement of formation (which has a clear operational meaning) in the same setting we consider and there are no qualitative differences with our result: The behavior of the entanglement of formation is qualitatively the same as negativity. Independently of the many uses quantum Shannon theory may have, entanglement measures provide a clear and straightforward way to study quantum correlations between two parties; this, and nothing else, is what has been done in the numerous works on the subject.

IV. CONCLUSION

We have shown that there are no algebraic mistakes in Ref. [1]. We are not directly studying a fermionic system, but rather a qubit one obtained by means of a well-defined mapping. We simply characterize the set of these mappings and find the circumstances under which they give rise to physical results. The mere existence of Ref. [2], however, proves that we have failed in our attempt to transmit our ideas in a clear and concise way. It is our hope that this Comment may shed some light on the more confusing parts of Ref. [1].

It is true that we use states violating the superselection rule, but do so only for an academic example in which the parity of the fermionic states plays no role. The main results of Ref. [1] all comply with the superselection rule. Nevertheless, all of our results could (and should) have been presented with states compliant with the superselection rule, thus avoiding these concerns. Furthermore, although the interpretation of entanglement measures for fermionic systems may not be the same as for qubits in general, there are field states for which such an interpretation may be recovered (such a Dirac singlet state). Finally, though quantum Shannon theory may be a useful tool to study quantum information transmission, entanglement measures provide a clear and straightforward way to study quantum correlations between two parties, whether in a relativistic setting or not.

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