Parametric invariance and the Pioneer anomaly

Antonio F. Rañada* A. Tiemblo†

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Abstract

It is usually assumed that the t parameter in the equations of dynamics can be identified with the indication of the pointer of a clock. Things are not so simple, however. In fact, since the equations of motion can be written in terms of t but also of $t' = f(t)$, f being any well behaved function, any one of those infinite parametric times t' is as good as the Newtonian one to study classical dynamics in Hamiltonian form. Here we show that, as a consequence of parametric invariance, the relation between the mathematical parametric time t in the equations of dynamics and the physical dynamical time σ that is measured with a particular clock (which is a dynamical system) requires the characterization of the clock that is used in order to achieve a complete treatment of dynamical systems. These two kinds of time, therefore, must be carefully distinguished. Furthermore, we show that not all the dynamical clock-times are necessarily equivalent and that the observational fingerprint of this non-equivalence has, curiously, the same form as that of the Pioneer anomaly, a still unexplained phenomenon.

*Facultad de Física, Universidad Complutense, 28040 Madrid, Spain. Email: afr@fis.ucm.es

†Instituto de Matemáticas y Física Fundamental, Consejo Superior de Investigaciones Científicas, Serrano 113b, 28006 Madrid, Spain. E-mail: Tiemblo@imaff.cfmac.csic.es

1 Introduction

The main problem of dynamics is probably to understand in depth the role and meaning of the term "time". Two kinds of time are used in physics. On one side, the parametric time t, just an auxiliary mathematical element which, strictly speaking, is not observable since any other time of the form $t' = f(t)$, f being any well behaved function, serves equally to describe the motion of a dynamical system. On the other the time measured with particular clocks, say σ, which are dynamical systems obeying the laws of physics. The latter is a dynamical variable, for instance the angle of a pointer, and deserves therefore to be qualified as dynamical. The consequences of the existence of these two kinds of time, parametric and dynamical, open an intriguing and probably promising field of research.

Here we show that the dynamical time $\sigma(t)$ measured by a clock σ, can be obtained as the solution of the equation of motion that characterizes the clock, of the form $\mathrm{d}\sigma/\mathrm{d}t = u(t)$, where $u(t)$ denotes here the "march" of the clock σ with respect to the parametric time t. While $\sigma(t)$ has a real dynamical character, t is just a mathematical parameter, which (i) has a purely auxiliary role to write the action and obtain the equations of motion, (ii) lacks any physical or dynamical nature, iii) is not observable and (iv) it is a symbol that describes the evolutive character of the reality. It seems evident, moreover, that this last point affects the deepest foundations of dynamics, so that it is not surprising that it had aroused a variety of different thought provoking ideas.

We adopt in this work a pragmatic position founded on the principle of parametric invariance of classical and relativistic dynamics. Note that this principle is the essential criterion to understand what the equations of motion really mean. In fact, with complete independence of the interpretation of the parameter t, this invariance guarantees that the equations of motion have the same (Hamiltonian) form for any arbitrary election of the time $f(t)$. To elect the function $f(t)$ is, evidently, to choose the gauge, which in practice means to adopt a clock. We wish to underscore that the expressions "dynamical time" and "time measured by a clock" are here dynamically equivalent. It is important to insist that a clock is a dynamical system and that the time it measures is a dynamical variable, for example the motion of a celestial body, as a planet or a pulsar, or the oscillations of atomic systems.

The differences between parametric and dynamical times could have significant consequences, since two dynamical clock-times, say σ_1 and σ_2, are not necessarily equivalent, so that there could be different times accelerating with respect to one another, *i. e.* they have different marches. The consequences of these arguments could be important; we just mention here two cases in which they could shed some light. First is the meaning of the cosmic time. Second, the fourth Heisenberg relation which requires that the time be a dynamical variable.

In order to write the equations of motion of a system in terms of really observable and dynamical quantities, what is done is to compare two motions, one of the system and the other of a standard clock. This requires the use of two principles. The first is the parametric invariance under transformations $t \to t' = f(t)$, an important property of classical and relativistic theories, the other is a principle of coherence, *i.e.,* that the equations of motion of both the system and the clock be described by the same physical theory. We will come again to this point.

2 Parametric invariance in classical dynamics

Though the parametric time is a fundamental concept in classical dynamics, as said before, it has a non-dynamical character because it is not a dynamical variable. As a consequence, there is no canonical momentum conjugate to t. Common wisdom assumes that this non-dynamical t is measured with a clock, but this assumption must be submitted to a rigorous analysis since this would be a prescription alien to the theory. Note that it is always possible to synchronize two clocks at a certain initial time t_0, but what cannot be assured is that they will keep ticking at the same rate. This raises the question whether the equations of motion of dynamics depend or not on the march of the clocks, which implies the need to establish a parametric invariance principle. There exists a scheme in which this problem can be solved by means of the introduction of the idea of a dynamical time [1, 2]. In fact, the theory so constructed is parametrically invariant, with respect to a new non-observable auxiliary parameter t, as happens also in general relativity.

In order to do that we replace the standard action $S = \int [p\,\dot{q} - H(p,q)]\mathrm{d}t$, by the alternative expression

$$S = \int \{\Pi(t)\dot{\sigma}_0(t) + p\,(t)\dot{q}(t) - u(t)[\Pi(t) + H(p\,(t), q(t))]\}\mathrm{d}t\,, \qquad (1)$$

(overdot means derivation with respect to the auxiliary parameter t), where $\sigma_0(t)$, $\Pi(t)$ are conjugate dynamical variables, as we will see, that describe the behavior of a clock, and Π_u, the momentum conjugate to $u(t)$, weakly vanishes.

The corresponding Hamiltonian is $\hat{H} = u[\Pi + H(p,q)] + \lambda\,\Pi_u$ where λ is a Lagrange multiplier. The stability of the weak condition $\Pi_u = 0$ implies the following first class constraint

$$\Pi + H(p,q) = 0, \tag{2}$$

which induces the following reparametrization transformations $\delta\sigma_0 = \alpha(t)$, $\delta q = \alpha(t)\dot{q}$ and $\delta p = \alpha(t)\dot{p}$ with $\alpha(t)$ being an arbitrary function.

The transformations induced by Π_u allow then to interpret $u(t)$ as an arbitrary function, so that the Hamiltonian becomes

$$H^E = u[\Pi + H(p,q)]\,, \tag{3}$$

Though this Hamiltonian reduces to a first class constraint, it contains a very realistic dynamical evolution, given by the Hamiltonian equations

$$\dot{q} = u\frac{\partial H}{\partial p}, \quad \dot{p} = -u\frac{\partial H}{\partial q}, \quad u = \dot{\sigma}_0, \quad \dot{\Pi} = -\dot{H} = 0. \tag{4}$$

It follows that

$$\frac{\mathrm{d}q}{\mathrm{d}\sigma_0} = \frac{\partial H}{\partial p}, \quad \frac{\mathrm{d}p}{\mathrm{d}\sigma_0} = -\frac{\partial H}{\partial q}, \quad u = \frac{\mathrm{d}\sigma_0}{\mathrm{d}t}, \quad \frac{\mathrm{d}H}{\mathrm{d}\sigma_0} = 0\,, \tag{5}$$

equations that are full of dynamical significance. The first two are the canonical equations of motion with the dynamical time σ_0, which from now on will be called "standard dynamical time" (do not mistake this for the "standard clock in general relativity), as the time variable in such a way that the evolution becomes a correlation between dynamical variables. The third one can be interpreted as the equation of motion (*i.e.,* the "march") of σ_0 with respect to the parameter t. Notice that the total Hamiltonian $\hat{H} = u[\Pi + H(p,q)] + \lambda\,\Pi_u$ is the sum of two terms, describing, respectively, the physical system and a clock. The equation of motion of the second term, $H_{\text{clock}} = u\Pi + \lambda\Pi_u$, is precisely that of a clock with arbitrary match $u = \mathrm{d}\sigma_0/\mathrm{d}t$.

Since this theory is invariant under reparametrization, we may fix, for instance, the gauge by the condition $\sigma_0 = t$ (*i.e.,* $u = 1$), so that we recover the ordinary canonical formalism with t being the Newtonian time. Notice that the choice of gauge means in fact to choose a clock.

The observations are performed in practice with real clocks, which are dynamical systems, each one with a dynamical variable that is a well behaved increasing function of t and can therefore be identified with a dynamical clock-time $\sigma(t)$, which can be used to fix the reparametrization gauge. As long as the observations make use of only the standard dynamical time σ_0, the scheme is nothing else than the Hamiltonian equations. This may not occur, however, if a real clock $\sigma(t)$ with a different march is involved. In the latter case, the motion equations are (5), but with σ and σ_0 instead of σ_0 and t, respectively,

$$\frac{\mathrm{d}q}{\mathrm{d}\sigma} = \frac{\partial H}{\partial p}\,; \qquad \frac{\mathrm{d}p}{\mathrm{d}\sigma} = -\frac{\partial H}{\partial q}\,; \qquad u = \frac{\mathrm{d}\sigma}{\mathrm{d}\sigma_0}\,, \tag{6}$$

which describe the physics of a system in operationally realistic terms. This means that they do not refer to any unobservable parametric time but to σ, which is the time really observed by a real clock. The novelty is here the presence of the third equation (6), which is the dynamic equation of the second clock with respect to σ_0. The important fact for our purposes is that classical dynamics can be formulated as a parametrically invariant theory. Consequently, the above mentioned prescription is now just the gauge fixing of the parametric invariance.

3 The relativistic particle

Before going into this section let us summarize the arguments of the previous one. Starting from a Hamiltonian theory with n degrees of freedom, we introduced a new one, in such a way that the motion equations become correlations between dynamical variables only (5). Nevertheless the new theory has a first class constraint (reparametrizations) that allows us to fix arbitrarily the value of σ_0. The only way to do that is to choose a new dynamical system, *i.e.,* a real clock such as the Earth's motion or any other, with a well behaved dynamical variable $\sigma(t)$ as appears in (6). So in practice to fix the gauge of the symmetry under reparametrizations is to choose a clock. In other words, we measure a motion using another one as a standard. It must be underscored that, to completely formulate the equations of a dynamical system, the chosen clock must be specified.

It may be illustrative to verify that the kinematics of the free particle in special relativity follows the same scheme. The parametric invariant action $\mathcal{S} = mc\int \mathrm{d}s$, corresponds to the Lagrangian (overdot means derivative with respect to an arbitrary time)

$$\mathcal{L} = -mc\sqrt{\dot{x}_0^2 - \dot{x}_1^2 - \dot{x}_2^2 - \dot{x}_3^2}, \tag{7}$$

where the four coordinates are dynamical variables all of them.

It is easy to see that there is a first class primary constraint of the form

$$p_0^2 - p_1^2 - p_2^2 - p_3^2 = m^2c^2, \tag{8}$$

that expresses the evident parametric invariance of the action. Due to the existence of this primary constraint not all the time derivatives of the coordinates can be obtained in terms of the momenta. Choosing then $\dot{x}_0$ as an arbitrary function of t, the Hamiltonian becomes

$$H^E = \dot{x}_0\left(p_0 + \sqrt{p_1^2 + p_2^2 + p_3^2 + m^2c^2}\right), \tag{9}$$

which reproduces (3) with p_0 playing the role of Π and the square root being $H(p,q)$. It must be stressed thus that $x_0(t)$ plays the same role as $\sigma_0(t)$ introduced in the previous case.

Let us take now the motion of a particle in a general metric tensor $g_{\alpha\beta}$, so that $\mathrm{d}s = \sqrt{g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta}$. The Lagrangian is

$$\mathcal{L} = -mc\sqrt{g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}. \tag{10}$$

Note that, since the motion is geodesic, the components of the metric tensor are not dynamical variables but prescribed functions of the coordinates. Following the same procedure as in the previous case, the primary constraint is now $g^{\alpha\beta}p_\alpha p_\beta = m^2c^2$. Thus the Hamiltonian becomes

$$H = \dot{x}^0[p_0 - (N\sqrt{p_ip^i + m^2c^2} + p_iN^i)], \tag{11}$$

where $i = 1, 2, 3$, N is the lapse, N_i the shift and the Latin indices are raised and lowered with the three-dimensional metric. As we see, the situation is the same as in previous cases, x^0 playing the same role as the dynamical time.

It must be underlined that all the previous Hamiltonians are, in fact, first class constraints. They generate, however, well defined dynamical evolutions (see (5)–(6)). Notice that they contain two terms that describe i) the dynamical system which is studied and ii) a particular clock.

The case of the particle in a gravitational field $g_{\alpha\beta}$ illustrates the difference between the spatial coordinates x^i and the temporal one x^0 since the former can be chosen arbitrarily while the latter needs an additional dynamical system (a real clock) in order for it to be fixed so that probably a (3+1)-spacetime is closer to the reality than a 4-spacetime.

4 The Einstein–Hilbert action

General relativity was constructed to be a parametrically invariant theory from its very foundation, as happens with any other diff-invariant theory. Its essential difference from the previous examples is that, in the former cases, the dynamical variables are the coordinates, defined in a non-dynamical metric. Conversely, in the latter, the dynamical variables are the components of the metric tensor, while the coordinates are auxiliary objects with no dynamical nature. Accordingly to our previous statements, we will take from now on a (3+1)-spacetime. In the ADM scheme [3] the Hamiltonian becomes

$$H^E = \int \mathrm{d}^3x[N\mathcal{H}(q_{ij},\, \pi^{ij}) + N_i\chi^i(q_{ij},\, \pi^{ij})], \tag{12}$$

where N and N_i are the lapse and the shift, respectively, q_{ij} the 3-metric and π^{ij} its canonically conjugate momentum. The absence of time derivatives of N and N_i determines the presence of primary first class constraints, which implies in turn that N and N_i are arbitrary functions. The secondary first class constraints $\mathcal{H} = 0$ and $\chi_i = 0$ fix the subspace in which the motion takes place. If one fixes $N_i = 0$, the Hamiltonian becomes $H = \int \mathrm{d}^3x\, N\mathcal{H}$.

From this expression one could reproduce the same process followed before in the case of ordinary analytical dynamics. To interpret the dynamics described by a Hamiltonian such as (12) it suffices, maintaining $N_i = 0$, to consider the meaning of N, defined as $\mathrm{d}\tau/\mathrm{d}t$ where $\mathrm{d}\tau = \sqrt{g_{00}}\,\mathrm{d}t$ is the proper time distance between two shells of the foliation. Note that N is an arbitrary dynamical variable which plays thus the same role as $\dot{x}^0$ and $\dot{\sigma}_0$ in the previous cases: all of them are derivatives with respect to the parametric time. We emphasize, therefore, that the dynamical time coincides with the proper time. Nevertheless, as is suitable to general relativity, the dynamical time is just a local time.

The be more precise, the three cases just considered in the previous sections put in evidence the need to introduce the concept of "dynamical time variable" that we have used in this work. It can be defined as a function of the parametric time which is physically realizable, obviously by means of a dynamical system, *i. e. a clock*.

The choice of a physical clock is then a most relevant question. For instance, the definition of a standard clock, *i. e.* a clock measuring proper time, is a very complex problem that can be solved in a Weyl manifold using light rays and free falling particles [4]. The clock must comply with some obvious conditions. It must be a dynamical system, the solution of its equation of motion $\sigma(t)$ being a well behaved and monotonously increasing function of the parametric time t, as for instance the number of cycles of an harmonic oscillator or of the Earth rotation. Strictly speaking the fixing of a gauge is a mathematical question, though physically relevant since it is equivalent to the choice of a clock. It must be underscored that the complete description of a dynamical system needs to specify the clock which is used. This is a very important problem, specially for cosmological models.

It must be underscored that the previous arguments imply that the parametric invariance is the main characteristic of classical dynamics. *I.e.*, this invariance states that the equations of motion are independent of the clock used to observe the trajectory. Otherwise said, it is a way to restrict to the time variable the principle of general covariance of relativistic physics.

Let us see what would happen if parametric invariance is not taken into account. For this purpose and in order to understand general relativity, simplified models have been proposed to obtain valuable information in areas such as quantum gravity or cosmology. The usual strategy is to kill some degrees of freedom. There is a way, however, to achieve the same result but going in the opposite direction, *i.e.,* adding degrees of freedom. This is the case of the Husain–Kuchař model [5], which lacks the Hamiltonian (scalar constraint) in such a way that the number of degrees of freedom per space point grows from 2 to 3. In such a theory, parametric invariance would be, in principle, absent. The price to be paid then is that the four dimensional metrics that can be constructed seem to be degenerate. Without discussing this point here, it is important to state that we have shown that the Husain–Kuchař model is a particular case of a more general theory (see [6] for details) that includes a scalar constraint and a dynamical time variable which, in fact, contains non nondegenerate metrics.

5 A principle of coherence

As was pointed out at the end of Section 1, when two clocks are involved the question of their coherence must be considered. There is no problem if the dynamics of both the system and the clock are governed by the same physical theory. This is because any discrepancy between two clocks must be solvable, from the theoretical point of view, in the frame of the theory itself. For instance, the effect of the tides on the Earth's rotation modifies the value of the day, an effect that can be calculated by taking into account the gravitation involved in the Earth–Moon system.

This requirement of coherence, which guarantees that the equation of motion of the clock (*i.e.*, its march) is given by the same theory as that of the dynamical system, cannot be maintained when the clock and the system obey two different theories. This is the case when atomic clocks are used in the study of systems governed by classical dynamics. Nevertheless since we lack a quantum gravity theory, the equation of motion of the atomic clocks $\sigma_2(t)$ cannot be determined *a priori* and, consequently, it is not possible to compare it with the equation of motion $\sigma_1(t)$ of a classical clock. The only way to do so relies necessarily on empirical methods. Note that if it is found that the two marches are different, this does not necessarily imply a violation of parametric invariance.

The previous considerations certainly clarify the role of the clocks and the meaning of the word "time". The two main kinds of clocks used in physics are the astronomical and the atomic ones, which are dynamical systems based on classical and quantum physics, respectively. The solar system taken as a clock gives the ephemeris time while the vibrations of quantum systems measure the atomic one. Current wisdom assumes implicitly that these two types of clocks give the same time but, as explained before, this is not necessarily so. Indeed there is no *a priori* reason to postulate that two clocks beat at the same rate if they are based on two different theories, such as gravitation and quantum physics which are not only different but, what's more, all efforts to unify them have failed up to now.

It is important to understand in depth what is really done in practice in order to calculate, for instance, the trajectory of a spaceship. The starting point is a theory which is diff-invariant (and parametrically invariant therefore) as general relativity is. In order to make concrete calculations about concrete systems, what is made is to choose a metric well adapted to the particular problem, in the case of celestial mechanics, for instance, the post-Newtonian one which depends on a t-variable usually called "coordinate time". In this way, the dynamical system under study becomes a clock that assigns a coordinate time to each point of the trajectory. Because of practical reasons this coordinate time must be compared with another one chosen as a standard. What is done in practice is to identify this coordinate time with the so-called "ephemeris time". This is a coherent election since the "ephemeris time" provides a clock associated to the motion of celestial bodies, also calculated in post-Newtonian approximation, based therefore on the same theory. This is a particularly neat example in which coordinate time and ephemeris time are associated to the same clock. Consequently a value of the ephemeris time determines a point in the trajectory of the spaceship and conversely a point of the trajectory defines a value of the ephemeris time.

It must be emphasized that to assume that classical and atomic clocks tick at the same rate is a strong hypothesis, which is however commonly accepted probably because of its simplicity and because it has been observationally difficult to disprove it up to now. It is, therefore, justified to explore the possibility that they could accelerate with respect to one another, specially since, as shown here, this is not in conflict we any physical law or principle.

6 Looking for observational evidence

Once accepted the possibility that the atomic and ephemeris (or coordinate) clocks could be non-equivalent, we must look for some observational footprint of this eventual new phenomenon. We denote the two times as $t_{\rm atom}$ and $t_{\rm ephe}$. Is is clear that their difference is either nil or very small, otherwise an unexpected new effect should have been detected by now. Let us admit that it is non-nil. Because of the continuous improvement of measurement devices during the last decades, an observational test of the relative acceleration between these two clock-times might already be available, although we could be unaware of this possibility. What's more, the effect could have been observed by now but without being properly interpreted.

A thought provoking case could be a spaceship receding from the Sun. Since its trajectory is calculated with standard gravity theories in a post-Newtonian metric [9], section IV A, that use ephemeris time but it is measured with devices based on quantum physics that use atomic time, some anomaly could be observed. In fact the theory gives the ship's trajectory as a certain function parametrized by astronomical time $\mathbf{r} = \mathbf{r}(t_{\rm ephe})$ but the observations see the same three-dimensional trajectory, although parameterized by atomic time and given by a different function $\mathbf{r}' = \mathbf{r}'(t_{\rm atom})$. The two times are related as $\mathbf{r}'(t_{\rm atom}) = \mathbf{r}(t_{\rm ephe})$ (they are examples of the clock-times $\sigma_2(t)$ and $\sigma_1(t)$ mentioned in section 5). It is clear that they can be synchronized at any initial time so that $t_{\rm astr,\,0} = t_{\rm atom,\,0} = t_0$, but they will start to desynchronize progressively afterwards as

$$\mathrm{d}t_{\rm atom} = [1 + a(t - t_0)]\, \mathrm{d}t_{\rm ephe}, \qquad \text{with} \qquad a = \frac{\mathrm{d}^2 t_{\rm atom}}{\mathrm{d}t_{\rm ephe}^2}, \tag{13}$$

where the small inverse time a is the relative acceleration of $t_{\rm atom}$ and $t_{\rm ephe}$, and $u = \mathrm{d}t_{\rm atom}/\mathrm{d}t_{\rm ephe} = 1 + a(t - t_0)$ the march of $t_{\rm atom}$ with respect to $t_{\rm ephe}$. Note that it is not necessary, at first order, to specify which one of the two times is t and that the instant t_0, at which the two times are synchronized, is completely arbitrary.

Defining the velocities of a spaceship with with the two times as $v_{\rm atom} = \mathrm{d}\ell/\mathrm{d}t_{\rm atom}$ and $v_{\rm ephe} = \mathrm{d}\ell/\mathrm{d}t_{\rm ephe}$, it follows that

$$v_{\rm atom} = \frac{v_{\rm ephe}}{u}, \qquad\qquad \frac{\Delta v}{v} = -a(t - t_0), \tag{14}$$

with $\Delta v = v_{\rm atom} - v_{\rm ephe}$. As could have been expected, the observational fingerprint of the relative acceleration of the two clock-times would be a discrepancy between the expected and observed speeds of a mobile.

Evidently the value of the speed of light in empty space depends on which clock-time is used. It is a fundamental constant only if measured with atomic clock-time. It must be so since the periods of the atomic oscillations are obviously constant with respect to $t_{\rm atom}$, in fact they are its basic units (see [2, 7] where the details are explained).

The following argument is important and must be underscored. Let a group of astronomers, using atomic clocks, start an observation at time $t_{\rm atom} = t_0$, without being conscious of the acceleration of $t_{\rm atom}$ and $t_{\rm ephe}$ with respect to one another. Since they think that the two times are the same one, the ephemeris time of the beginning of the experiment will implicitly be fixed as $t_{\rm ephe,\,0} = t_{\rm atom,\,0} = t_0$. However, if the two times are not strictly equivalent, it will be impossible to maintain the synchronization, so that they

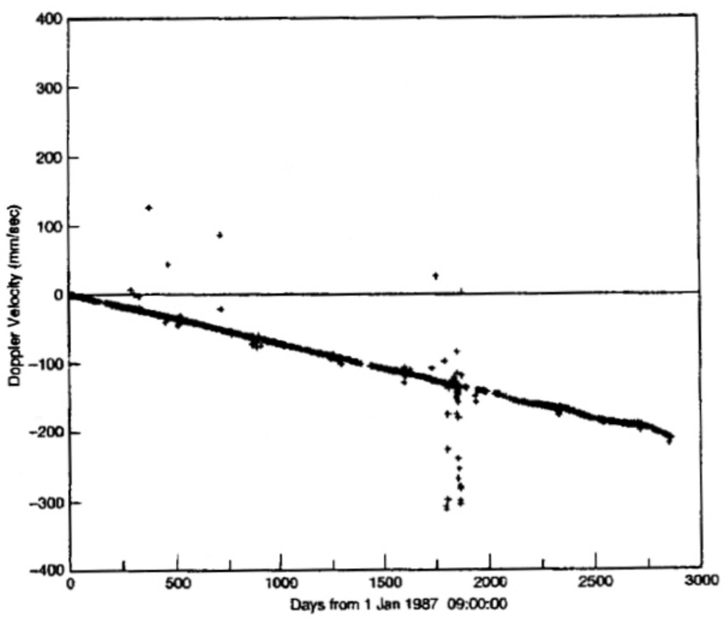

Figure 1: The Pioneer anomaly: plot of the extra velocity of the spaceship (mm/s) *versus* time (days) for near 3,000 days starting on $t_0 = 1$ Jan 1987 09:00:00. It can be seen that the extra velocity vanishes at t_0. Taken from [8], Anderson *et al.*, Phys. Rev. Lett. **81**, 2858 (1998).

will begin to separate progressively afterwards. A diagram of the values of $v_{\rm ephe} - v_{\rm expected}$ $(= 0)$ and $v_{\rm atom} - v_{\rm expected}$ versus $t_{\rm atom}$ will show two diverging lines from the initial time t_0, an instant in which the anomaly vanishes, just as those appearing in the Figure 1, taken from reference [8] (remember that the initial time t_0 is arbitrary).

Note that (13)–(14) imply that if $a < 0$, then $v_{\rm atom} > v_{\rm ephe}$ while if $a > 0$, then $v_{\rm atom} < v_{\rm ephe}$ (assuming $t > t_0$). In the latter case, the ship would seem to lag behind the position predicted by gravity theories.

In fact quite a similar lag has already been observed and has even a name: the Pioneer anomaly. Surprisingly, it remains unexplained more than thirty years after being discovered by Anderson *et al.* in 1980 in spite of many efforts to account for it [8, 9, 10]. What is important for the purpose of this work is that the observational fingerprint of the anomaly has the same form as the second equation (14). What Anderson *et al.* found is that the frequencies of the two-way signals to and from the Pioneer 10 spaceship included an unexpected Doppler residual which did not correspond to any known motion of the ship. They were able to measure the value $a = (5.84 \pm 0.88) \times 10^{-18}$ s^{-1}, although using the inverse time $a_{\rm t} = a/2$, and suggested that $a_{\rm t}$ could be "like a non-homogeneity of time" or a "clock acceleration" [8], a term that suggests a reparametrization of the form $t \to f(t)$, *i.e.* a parametric transformation. But they did not explain acceleration with respect to what, nor did they develop any theoretical analysis of this idea, assuming at first that $2a_{\rm t}$ was just the measure of a real Doppler effect. However it was soon understood that this interpretation is neither compatible with the equivalence principle nor with the cartography of the solar system. For several years it was thought that systematics would be the most probable explanation of the anomaly (see the conclusions of [9]) but no error was found in spite of several different mathematical analyses of the data, including independent ones [11]-[19]. For a relation of the recent attempts to explain the anomaly, see [15], Section 2.3; [19], Section 2; or [10], Section 6. Up to now and more than thirty years after its discovery, the Pioneer anomaly remains without a generally accepted solution, even though it happens in our backyard, the solar system.

Some researchers put now their reliance on the thermal model which postulates that the anomalous acceleration of the Pioneers is just the reaction to the radiation emitted by the ships because of the heat produced by the Radioisotopes Thermoelectric Generators (RTG) inside. Note that the anomaly is due in this model to a real force. Recently two papers by F. Francisco *et al.* [20] and B. Rievers and C. Lämmerzahl [21] found numerical results that seem to indicate that the thermal model gives an explanation of the Pioneer anomaly.

What Figure 1 shows is a run of observations starting on $t_0 = 1$ January 1987 09:00:00. Note that the ship was continuously monitored from three stations, in California, Madrid and Canberra, so that it was continuously under the observation of at least one of them.

Two important facts are essential to understand the Pioneer riddle and must be underlined:

(i) Both before and after t_0, the signal from the ship observed at Earth showed always an anomalous blueshift, never a redshift, the observed Doppler residual being always negative, as it corresponds to an unexpected extra speed towards the Sun;

(ii) as is seen in Figure 1, the observed anomaly for the particular plotted run vanishes exactly at t_0, the initial instant of the run.

We will see now that, as a consequence of these two features, the anomaly cannot be due to any extra real force whatever its nature could be, either gravitational or electromagnetic. We show why in the following.

Let us assume that the cause of the anomaly was a constant real force producing a constant real acceleration. In that case and after t_0, the extra residual and the extra speed would be both negative and have an increasing modulus (see Figure 1). The horizontal line and the sloping one in the plot, that represent the two velocities of the Pioneer, v_{ephe} and v_{atom} minus $v_{\text{expected}}\,(=v_{\text{ephe}})$, would cut one another at a certain well determined time which, according to Figure 1, is the initial time of the run $t_0 = 1$ Jan 1987 09:00:00. At this time the anomaly would vanish since the two speeds would be equal.

This has an important consequence. As noted before and for times larger than t_0, the extra velocity and the Doppler residual would be negative so that the ship would seem to lag behind the expected position, as happens in the Pioneer anomaly. Before the time t_0, quite on the contrary, the extra velocity and the the Doppler residual would be positive, the ship seeming to advance ahead of the position calculated with gravitation theories (just prolong to the left the line representing the extra velocity, what can be made since we consider a real force). This would imply that before t_0 the direction of the extra velocity would be the contrary to the one which is observed. Moreover, how to explain that the extra velocity $\Delta v = v_{\text{atom}} - v_{\text{ephe}}$ vanishes at the initial instant of the run up to one second? And if other run was started one month later or 3.4 months before? The fact that the force is real implies, therefore, a contradiction with the observations.

However these problems do not arise in our model. If the observers don't consider the non-equivalence of t_{atom} and t_{ephe} and since they use atomic time for their observations, they would accept, unawares and implicitly, that both times are equal to t_0. This means that they are synchronized *de facto*. But since in fact they accelerate with respect to one another, the synchronization can not be maintained, so that the two times begin to diverge, as well as the values of the extra velocities. Furthermore, we could have an infinity of identical plots starting at different times t'_0, with the characteristic behavior of two non-equivalent clocks which are synchronized at an instant but which diverge progressively afterwards. This means that the plot in Figure 1 is easily understood with our model.

Note that, if the anomaly would be due to a force that depends on the shape and the internal characteristics of the spacecraft, different shapes or internal structures would imply different observed values of the anomalous acceleration. On the contrary, the effect would be universal if our model is right: the value of the inverse time a, defined as the acceleration of the atomic time with respect to the astronomical time, would be the same for all the crafts. The same can be said of their anomalous accelerations a_P. An example is given by the Pioneer 10 and its twin brother the Pioneer 11 for which a has very close values, their small difference being most probably due to the rotational Doppler effect [22]. However there are two other spaceships, Galileo and Ulysses, for which the effect was measured to give values close to the one of the Pioneers, in spite of the differences between the shapes and internal structures. More precisely, the value of the corresponding anomalous accelerations were measured to be $a_P = (8 \pm 3) \times 10^{-10}$ m/s^2 (Galileo) and $a_P = (12 \pm 3) \times 10^{-10}$ m/s^2 (Ulysses), to compare with $a_P = (8.74 \pm 1.33) \times 10^{-10}$ m/s^2 for the Pioneers [9]. The ships Galileo and Ulysses are not usually considered after 2002 because the effect was more difficult to measure, their trajectories being very different from those of the Pioneers (the best measurements are obtained when the crafts recede away from the Sun, while Galileo and Ulysses followed bounded orbits). Taking everything into account, it seems really difficult that these close values of a_P for Galileo and Ulysses could be explained by any model depending on the shape and internal characteristics of the spaceship.

Moreover, it has been argued by some experts that the dust through which the Pioneers traveled would surely stick to their surfaces, modifying their thermal properties and blurring the radiation pattern in a way that could not be predicted from Earth. Consequently the push of the radiation calculated from the designed plans of the ships could be different from the real one.

7 Conclusions

It is very important to understand that the description of a dynamical system can not be considered complete without the explicit mention of the chosen physical clock, which means to fix the gauge of parametric invariance. This is specially true in cosmology problems. We emphasize that two stable, accurate and good but different clocks can be non-equivalent. By this we mean that, if they are based on different physical theories, the times they measure could accelerate with respect to one another. In this case, the relative march of the two clocks can only be determined by empirical methods. This happens in the case of atomic and astronomical times, $t_{\rm ephe}$ and $t_{\rm atom}$, which are based on classical gravity and quantum electromagnetism, respectively. In order to determine theoretically their relative march we would need a still undiscovered theory: quantum gravity. This could be stated by saying that the principle of parametric invariance has room for non-equivalent clock-times.

Although these arguments might seem rather formal, they are also of practical importance. In particular, this work proposes an explanation of the Pioneer anomaly that is a refinement of a previous one which is fully compatible with the cartography of the solar system [2, 23]. It is based on the non-equivalence of the atomic time and the astronomical time, which happens to have the same observational fingerprint as the anomaly. The inverse time a that characterizes the observations turns out to be the second derivative of $t_{\rm atom}$ with respect to $t_{\rm ephe}$.

The main characteristic properties of this model are that: (i) the Pioneer anomaly, described by Anderson *i. e.* as a clock acceleration, can be interpreted as a parametric transformation $t \to f(t)$; (ii) the Pioneer anomaly is not due to any force but to the desynchronization of the two times $t_{\rm atom}$ and $t_{\rm ephe}$ which mimics the effect of a force; (iii) at the initial time t_0 of any run of observations the two times are implicitly synchronized and the anomaly is not observed; and (iv) the observed anomaly is independent of the choice of t_0; In fact, to change it amounts only to a translation of the lines in Figure 1, to the right or to the left.

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References

[1] A. T. Hanson, T. Regge, and C. Teitelboim, Constrained Hamiltonian systems (Academia dei Lincei, Roma, 1976.)

[2] A. F. Rañada, and A. Tiemblo, Found. Phys. **38**, 458–469 (2008).

[3] R. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. **116**, 1322–1331 (1959).

[4] V. Perlick, Gene. Rel. and Grav. **19**, 1059–1073 (1987).

[5] V. Husain, and K. Kuchař, Phys. Rev. D **42**, 4070-4077 (1990).

[6] J. Barbero, A. Tiemblo, and R. Tresguerres, Phys. Rev. D **57**, 6104–6112 (1998).

[7] A. F. Rañada, Found. Phys. **34**, 1955-1971 (2004), and references therein.

[8] J. D. Anderson, Ph. A. Laing, E. L. Lau, A. S. Liu, M. M. Nieto, and S. G. Turyshev, Phys. Rev. Lett. **81**, 2858–2861 (1998).

[9] J. D. Anderson, Ph. A. Laing, E. L. Lau, A. S. Liu, M. M. Nieto, and S. G. Turyshev, Phys. Rev. D **65**, 082004/1–50 (2002).

[10] S. G. Turyshev, and V. T. Toth, arXiv:1001.3686v2 19 Aug 2010; Living Rev. Relativity, **13**, 4–169 (2010).

[11] C. B. Markwardt, arXiv:gr-qc/0208046v1 16 August 2002.

[12] A. Levy, B. Christoffe, P. Berio, G. Metris, J.-M. Courty, and S. Reynaud, Adv. Space Res. **43**, 1538–1544 (2009).

[13] Ø. Olsen, Astron. Astrophys. **463**, 393–397 (2007).

[14] V. T. Toth, Int. J. Mod. Phys. D **18**, 717-741 (2009).

[15] S. G. Turyshev, V. T. Toth, L. R. Kellog, E. L. Lau, and K. J. Lee, Int. J. Mod. Phys. D **15**, 1–56 (2006).

[16] V. T. Toth, and S. G. Turyshev, arXiv:0710.25561 14 Oct 2007.

[17] V. T. Toth, and S. G. Turyshev, arXiv:0901.4597v2 [physics.space-ph] 27 Feb 2009, and references therein.

[18] S. G. Turyshev, and V. T. Toth, Space Sci. Rev. **148**, 149–167 (2010).

[19] S. G. Turyshev, M. M. Nieto, and J. D. Anderson, in: G. A. Mamon, F. Combes, C. Deffayet, and B. Fort (eds.), arXiv:gr/qc/0510081v1 17 Oct 2005, EAS Publications Series **20**, 243–250 (2006).

[20] F. Francisco, O. Bertolami, P. J. S. Gil and J. Paramos, arXiv:1103.5222v1 [physics.space-ph] 27 Mar 2011.

[21] B. Rievers and C. Lämmerzahl, Annalen der Physik (Berlin) **523**, 439 (2011).

[22] J. P. Mbelek & Michalski, Int. J. Mod. Phys. D **13**, 865 (2004).

[23] A. F. Rañada, and A. Tiemblo, arXiv:0909.0912v1 [gr-qc] 4 Sep 2009.