Chapter 1

Cumulant Dynamics of Strongly Interacting Ultracold Gases

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We discuss the cumulant approach to the non-equilibrium dynamics of strongly-interacting ultracold atomic gases. After a general introduction, we derive a non-Markovian, nonlinear Schrödinger equation for a Bose-Einstein Condensate, and a non-Markovian Boltzmann equation for the one-body density matrix of a thermal Bose gas. We apply these equations to the dynamics of Feshbach molecule production.

1.1. Introduction

In this chapter we explain the theory of cumulants¹ in the context of ultracold gases, focussing on the association of Feshbach molecules.² This process depends on tuning the strength of atomic interactions with external electromagnetic fields.³ Dynamical studies then require the ability to account for strong interactions and their effect on relevant observables, such as the kinetic energy distribution of the gas. Consequently, it is necessary to go beyond approaches such as the GPE, which is valid for a dilute zero-temperature BEC with $na^3 \ll 1$ (in 3D), where n is the local density of the gas and a is the s -wave scattering length.

The dynamics of molecule formation have been studied using a variety of techniques. For isolated pairs of atoms in an optical lattice, molecule production can be accurately calculated from the two-body physics.^{4,5} Zero-temperature condensates have been considered using a two-component mean-field approach with

contact interactions,^{6–9} and with microscopic quantum dynamics approaches.^{10–13} Loosely trapped thermal gases have been studied by assuming the trap to approximate a continuum, and summing over the free-bound transition probability density.^{10,14,15} Such approaches neglect the depletion and rethermalisation of the continuum of unbound states. These effects were studied by Williams *et al.*,^{16,17} who used the Keldysh non-equilibrium Green's function formalism¹⁸ to derive generalised Boltzmann equations for the resonance level and unbound pairs.

Our approach is motivated by kinetic theory, like that of Williams *et al.*, but further relaxes the assumptions regarding the rate of equilibration of the gas. The enabling ingredients are the cumulant expansion for the dynamical equation of motion for an observable, and a means of truncating the resulting series of equations at the desired order. In the following section we present and explain our formalism, deriving the equations necessary for studying the dynamics of both condensed and thermal gases. We also discuss the convenient separable potential used to model the interactions between atoms. After considering the range of validity of the theory, we present example calculations of molecule association.

1.2. Methodology

We start from the general Hamiltonian for a gas of bosonic atoms interacting pairwise via a potential $\hat{U}$:

$$\hat{H} = \sum_{ij} \hat{a}_i^\dagger \langle i | \hat{h}_0 | j \rangle \hat{a}_j + \frac{1}{2} \sum_{i_1 i_2 i_3 i_4} \langle i_1 i_2 | \hat{U} | i_3 i_4 \rangle \hat{a}_{i_1}^\dagger \hat{a}_{i_2}^\dagger \hat{a}_{i_3} \hat{a}_{i_4}. \quad (1.1)$$

Here, $\hat{h}_0$ is the one-body Hamiltonian of an atom, the indices refer to an arbitrary set of basis states, and the field operators have bosonic commutation relations: $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$, and $[\hat{a}_i, \hat{a}_j] = 0$. For the situation considered here, $\hat{h}_0$ and $\hat{U}$ are generally magnetic field dependent, and the basis states will span the momenta and internal states of the atoms. In dynamical calculations, we will consider the variation of the magnetic field in time. The dynamical equation for the expectation value of an operator $\hat{O}$ at time t , $\langle \hat{O} \rangle_t$, is given by

$$i\hbar \frac{\partial}{\partial t} \langle \hat{O} \rangle_t = \langle [\hat{O}, \hat{H}] \rangle_t. \quad (1.2)$$

Cumulants, denoted $\langle \hat{O}_i \rangle^c$, may be defined recursively in terms of expectation values.¹⁹ The first three of these are

$$\langle \hat{O}_1 \rangle = \langle \hat{O}_1 \rangle^c, \quad (1.3)$$

$$\langle \hat{O}_1 \hat{O}_2 \rangle = \langle \hat{O}_1 \hat{O}_2 \rangle^c + \langle \hat{O}_1 \rangle^c \langle \hat{O}_2 \rangle^c, \quad (1.4)$$

$$\begin{aligned} \langle \hat{O}_1 \hat{O}_2 \hat{O}_3 \rangle &= \langle \hat{O}_1 \hat{O}_2 \hat{O}_3 \rangle^c + \langle \hat{O}_1 \rangle^c \langle \hat{O}_2 \hat{O}_3 \rangle^c + \langle \hat{O}_2 \rangle^c \langle \hat{O}_1 \hat{O}_3 \rangle^c \\ &\quad + \langle \hat{O}_3 \rangle^c \langle \hat{O}_1 \hat{O}_2 \rangle^c + \langle \hat{O}_1 \rangle^c \langle \hat{O}_2 \rangle^c \langle \hat{O}_3 \rangle^c. \end{aligned} \quad (1.5)$$

Following the approach of Ref. [20], we define^a the mean field $\phi_i(t) = \langle \hat{a}_i \rangle_t^c$, the one-body density matrix $n'_{ij}(t) = \langle \hat{a}_j^\dagger \hat{a}_i \rangle_t^c$, and the pair function $m'_{ij}(t) = \langle \hat{a}_j \hat{a}_i \rangle_t^c$. Cumulants of order $(k + l) > 2$ are denoted

$$\Gamma_{i_1 i_2 \dots i_k; j_1 j_2 \dots j_l}^{(k, l)}(t) = \langle \hat{a}_{j_1}^\dagger \hat{a}_{j_2}^\dagger \dots \hat{a}_{j_l}^\dagger \hat{a}_{i_1} \hat{a}_{i_2} \dots \hat{a}_{i_k} \rangle_t^c. \quad (1.6)$$

The form of the general Hamiltonian in Eq. (1.1) shows that calculating the dynamics for the expectation value of an operator produces an infinite recursion of dynamical equations for higher order expectation values (see also Chapters ??, ??, ??). There is no guarantee that the expectation values of higher order terms are smaller—in fact, the opposite is sometimes true. By contrast, cumulants of order n add the n th order interactions around the interaction-free evolution already included in the $(n - 1)$ th order, and so get smaller as n increases. A cumulant expansion should therefore be the most stable technique for studying the dynamics of systems which stay reasonably close to equilibrium. Furthermore, in the thermal equilibrium of a grand canonical ensemble, all cumulants above second order vanish for an ideal gas.²¹ Consequently, cumulants can act as a measure of the deviation of the system from its interaction-free equilibrium.

A method of truncation is required to close the equations at order n , *i.e.* to ensure that no higher-order terms remain. We calculate the free evolution of products of $n + 1$ and $n + 2$ operators, neglecting any terms beyond their respective orders.²⁰ Because the Hamiltonian of Eq. (1.1) contains only pairwise interactions, the resulting equations have a formal solution involving only two-body dynamics and cumulants of order up to n , provided we can neglect the initial values of the cumulants of order $n + 1$ and $n + 2$. This allows us to close the original equations at n th order. We note that the two-body dynamics must first be solved and used as an input to the cumulant equations.

^aNote that this notation is different to that used in Ref. [20].

1.2.1. Mean-field evolution

Applying Eq. (1.2) to $\phi_i(t)$, we normal-order the expectation values and perform a cumulant expansion, obtaining

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \phi_i(t) = & \sum_j \langle i | \hat{h}_0 | j \rangle \phi_j(t) \\ & + \sum_{i_1 i_2 i_3} \langle i_1 i | \hat{U} | i_2 i_3 \rangle [\Gamma_{i_2 i_3; i_1}^{(2,1)}(t) + \phi_{i_1}^*(t) m'_{i_2 i_3}(t) + n'_{i_2 i_1}(t) \phi_{i_3}(t) \\ & + n'_{i_3 i_1}(t) \phi_{i_2}(t) + \phi_{i_1}^*(t) \phi_{i_2}(t) \phi_{i_3}(t)]. \end{aligned} \quad (1.7)$$

We assume that pairwise correlations and the number of thermal atoms are negligible at the initial time, t_i . This corresponds to setting $m'_{ij}(t_i) = n'_{ij}(t_i) = 0$.

The first order cumulant approach includes the exact evolution of the expectation value of one operator, and the free evolution of products of two or three operators. At this level of approximation, the one-body density matrix remains zero at all times. The pair function, by contrast, provides a non-Markovian term in the dynamical equation for the mean-field. For the simplest case, a 3D homogeneous Bose gas in a single internal state, we obtain the following nonlinear, non-Markovian Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} \phi(t) = \hat{h}_0 \phi(t) - \phi^*(t) \int_{t_i}^{\infty} dt' \phi^2(t') \frac{\partial}{\partial t'} h(t, t'), \quad (1.8)$$

where $h(t, t') = (2\pi\hbar)^3 \langle 0 | \hat{U}(t) \hat{\mathcal{U}}_{2B}(t, t') | 0 \rangle \theta(t - t')$, where θ denotes the step function. Here, $\hat{\mathcal{U}}_{2B}(t, t')$ is the two-body evolution operator of the relative motion, defined by

$$i\hbar \frac{\partial}{\partial t} \hat{\mathcal{U}}_{2B}(t, t') = \hat{H}_{2B}(t) \hat{\mathcal{U}}_{2B}(t, t'),$$

with $\hat{\mathcal{U}}_{2B}(t, t) = 1$. We note that $\hat{\mathcal{U}}_{2B}(t, t')$ and the two-body Hamiltonian $\hat{H}_{2B}(t)$ are dependent on the time-varying magnetic field $B(t)$.

1.2.2. One-body density matrix evolution

We now consider the evolution of the one-body density matrix for a thermal gas in the second-order cumulant approach. In the absence of a mean field, we are able to neglect the expectation values of all operators that are not number conserving.

The cumulant expansion of Eq. (1.2) for this case then gives

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} n'_{ij} = & \sum_{i_1} (\langle i|\hat{h}_0|i_1\rangle n'_{i_1j} - \langle i_1|\hat{h}_0|j\rangle n'_{ii_1}) \\ & + \sum_{i_1 i_2 i_3} \langle ii_1|\hat{U}|i_2 i_3\rangle (\Gamma_{i_2 i_3; j i_1}^{(2,2)} + n'_{i_2 j} n'_{i_3 i_1} + n'_{i_3 j} n'_{i_2 i_1}) \\ & - \sum_{i_1 i_2 i_3} \langle i_1 i_2|\hat{U}|j i_3\rangle (\Gamma_{i_3 i_1; i_2}^{(2,2)} + n'_{i_3 i_1} n'_{i_2} + n'_{i_1} n'_{i_3 i_2}), \end{aligned} \quad (1.9)$$

where for brevity we did not write out the t -dependence of each n' .

At this order of approximation, we need only the free evolution of the two-body density matrix, $\Gamma_{i_1 i_2; j_1 j_2}^{(2,2)}(t)$, to close the equation. This can be formally solved and substituted into Eq. (1.9), again providing a non-Markovian term. We set $\Gamma_{i_1 i_2; j_1 j_2}^{(2,2)}(t_i) = 0$, corresponding to no two-body correlations being present, and we again limit ourselves to a 3D homogeneous gas in a single internal atomic state. This gives the non-Markovian Boltzmann equation (NMBE):²²

$$\begin{aligned} \frac{\partial}{\partial t} n'(p, t) = & \frac{4}{\hbar} n'(p, t_i) \int_0^\infty dq q^2 n'(q, t_i) \mathcal{K}_1(p, q, t, t_i) \\ & + 2 \int_0^\infty dp_1 \int_0^\infty dp_2 p_1^2 p_2^2 n'(p_1, t_i) n'(p_2, t_i) \mathcal{K}(p, p_1, p_2, t, t_i) \\ & + \frac{4}{\hbar} \int_{t_i}^t dt' \int_0^\infty dq q^2 \mathcal{K}_1(p, q, t, t') \frac{\partial}{\partial t'} [n'(p, t') n'(q, t')] \\ & + 2 \int_{t_i}^t dt' \int_0^\infty dp_1 \int_0^\infty dp_2 p_1^2 p_2^2 \mathcal{K}(p, p_1, p_2, t, t') \frac{\partial}{\partial t'} [n'(p_1, t') n'(p_2, t')]. \end{aligned} \quad (1.10)$$

Here, $n'(p, t)$ is the reduced one-body density matrix for a homogeneous gas, depending only on the absolute value of the momentum, p , and normalised such that $\int d\mathbf{p} n'(p, t) = (2\pi\hbar)^3 n$, where n is the total density of the system. The kernel for the terms of Eq. (1.10) with a double momentum integral is given by

$$\begin{aligned} \mathcal{K}(p, p_1, p_2, t, t') = & 2\pi \int_0^\pi d\theta_{p_2} \sin \theta_{p_2} \int_0^\pi d\theta_p \sin \theta_p \int_0^{2\pi} d\phi_p \\ & \times \frac{\partial}{\partial t} \left| \frac{1}{\hbar} \int_{t'}^t d\tau \exp \left[i \frac{p_{\text{cm}}^2}{2m_{\text{red}}\hbar} (\tau - t') \right] \langle \mathbf{p}_{\text{cm}} | \hat{U}(\tau) \hat{\mathcal{U}}_{2B}(\tau, t') | \mathbf{p}_r \rangle \right|^2. \end{aligned}$$

Here, m_{red} is the reduced mass, and we must evaluate angular integrals over the vectors $\mathbf{p}_r = (\mathbf{p}_1 - \mathbf{p}_2)/2$ and $\mathbf{p}_{\text{cm}} = \mathbf{p} - (\mathbf{p}_1 + \mathbf{p}_2)/2$, with the subscripts on the angles θ and ϕ indicating the momentum to which they belong. For a homogeneous system, translational invariance reduces the number of non-trivial angular

integrals from six to three. The kernel for the terms of Eq. (1.10) with a single momentum integral is given by

$$\mathcal{K}_1(p, q, t, t') = \exp\left[i\frac{q_r^2}{2m_{\text{red}}\hbar}(t - t')\right] \int d\Omega_{\mathbf{q}_r} \text{Im} \left[\langle \mathbf{q}_r | \hat{U}(t) \hat{\mathcal{U}}_{2B}(t, t') | \mathbf{q}_r \rangle \right],$$

where $\mathbf{q}_r = (\mathbf{p} - \mathbf{q})/2$. In the limit of long times and Markovian interactions, Eq. (1.10) reduces to the well-known Boltzmann equation.²²

For implementation, it is possible to perform all the angular integrals in advance and store the kernels $\mathcal{K}(p, p_1, p_2, t, t')$ and $\mathcal{K}_1(p, q, t, t')$, leaving only the integrals over time and the moduli of the momenta to be performed during the dynamical calculation. Once the kernel has been calculated for a particular magnetic field variation sequence, it can be used for several different input values of the temperature and density of the gas.

1.2.3. Form of two-body potential

Feshbach resonances are created by the coupling of a colliding pair to a near-degenerate bound state (see Chapter ??). This can be achieved in gases of ultracold alkali atoms using the Zeeman effect to tune the energy of a molecular state near a collision threshold.³ The s -wave scattering length can then be varied according to the formula²³

$$a(B) = a_{\text{bg}} \left(1 - \frac{\Delta B}{B - B_0} \right). \quad (1.11)$$

Here, a_{bg} is the background scattering length, and ΔB is the width of the resonance occurring at $B = B_0$. As the magnetic field is varied across the resonance from negative to positive scattering length, a molecular bound state can be stabilised.²

In our dynamical calculations, we are not concerned with the details of the internal atomic states which make the resonance possible, and furthermore constrain ourselves to small binding energies. We also note the separation of scale between the de Broglie wavelength of the atoms and the characteristic length of the interatomic potential. Consequently, we are able to replace $\hat{U}(t)$ with a separable potential of the form

$$\hat{U}(t) \rightarrow \hat{U}_{\text{sep}}(t) = |\chi\rangle \Xi(t) \langle \chi|. \quad (1.12)$$

Here, we give the form factor $|\chi\rangle$ a Gaussian profile in momentum,^{11,24}

$$\langle \mathbf{p} | \chi \rangle = (2\pi\hbar)^{-3/2} \exp \left[-p^2 \sigma^2 / (2\hbar^2) \right],$$

where the length σ gives the range of the potential. Resonance enhanced properties are reproduced by varying $\Xi(t)$, which accounts for the magnetic-field dependence of $\hat{U}_{\text{sep}}(t)$ through its dependence on the time-varying scattering length

$a(B(t))$:²⁵

$$\frac{1}{\Xi(t)} = \langle \chi | G_0(0) | \chi \rangle + \frac{m(2\pi\hbar)^3 |\langle \chi | 0 \rangle|^2}{4\pi\hbar^2} \frac{1}{a(B(t))}.$$

Here, $\hat{G}_0(z) = (E + i0 - \hat{H}_0)^{-1}$ is the free Green's function, $\hat{H}_0 = -\hbar^2 \nabla^2 / (2m_{\text{red}})$ is the free Hamiltonian of the relative motion, and '+i0' indicates that the physical energy E is approached from the upper half of the complex plane.

The power of the separable potential lies in the way it simplifies calculation of the matrix elements required for Eqs. (1.8) and (1.10). In particular, we only need $\langle \chi | \hat{\mathcal{U}}_{2B}(t, t') | \mathbf{p} \rangle$, with $p \rightarrow 0$ for the BEC case. This may be found as a function of t from the Lippman-Schwinger equation,

$$\hat{\mathcal{U}}_{2B}(t, t') = \hat{\mathcal{U}}_0(t - t') + \frac{1}{i\hbar} \int_{t'}^t d\tau \hat{\mathcal{U}}_0(t - \tau) \hat{V}_{\text{sep}}(\tau) \hat{\mathcal{U}}_{2B}(\tau, t'). \quad (1.13)$$

Here, $\hat{\mathcal{U}}_0(t - t')$ represents the non-interacting evolution of the system, and satisfies $\hat{\mathcal{U}}_0(t - t') | \mathbf{p} \rangle = \exp[-ip^2(t - t') / (2m_{\text{red}}\hbar)] | \mathbf{p} \rangle$. A closed equation is obtained by projecting Eq. (1.13) onto $\langle \chi |$ on the left and $| \mathbf{p} \rangle$ on the right.

1.3. Validity issues

The first order cumulant expansion for the mean field of Eq. (1.8) is valid for an initially pure condensate. By contrast, the NMBE of Eq. (1.10) requires absence of a condensate, applying to a pure thermal gas. In principle, our cumulant approach could also be used to study the dynamics of molecule formation from a partially condensed gas with a significant thermal fraction, but we find this too computationally expensive in practice. Therefore, we limit ourselves to either a pure condensed or pure thermal cloud. The cumulant expansion allows us to handle the dynamics of strongly interacting gases very well, and so makes our approach ideal for studying problems such as Feshbach molecule association. We also note that our implementation of the NMBE has been limited to homogeneous gases, again due to computational requirements, while the first order approach can be readily applied to inhomogeneous systems.^{12,13}

We now give a few points of comparison between our cumulant approach and some other quantum kinetic theories that have been applied to molecule production. Our detailed and stable inclusion of the interactions allows us to go beyond mean-field approaches,⁶⁻⁹ which require the interactions to occur on a timescale much faster than the condensate evolution, and are inherently two-level. As noted above, the work of Williams *et al.*^{16,17} is also limited in this regard. Our cumulant expansion has greater stability for strongly interacting gases than many

techniques, e.g. the positive-P approach²⁶ which has been applied to molecular dissociation. The multiconfigurational time-dependent Hartree method has been applied to molecule production by Alon *et al.*²⁷ This method expands the many-body wavefunction in orbitals, and so should be able to handle a few-mode partially condensed gas more readily than our approach. However, it is not readily extended to thermal gases. Indeed, several kinetic theory approaches detailed in this book [! Nick - please add ref. handles !] are explicitly constructed with partially condensed gases in mind, and so are better suited for problems such as evaporative cooling and condensate formation than our approach, which is constructed with molecules and strong interactions as its focus.

1.4. Applications

In the context of ultracold gases, the theory of cumulants has been applied to colliding condensates,²⁰ the dynamics of quantum accelerator modes,²⁸ and spin-squeezing in two-component BECs.²⁹ It has also been applied to the dissociation of Feshbach molecules,³⁰ and their production by linear magnetic field ramps,¹¹ the Ramsey fringe experiments of Wieman's group,^{12,13} and resonantly modulated magnetic fields.¹⁰ The theory of cumulants has also been applied to the production of cold molecules by photoassociation.³¹ Here, we present results on the dynamics of Feshbach molecule production, and the appearance of saturation effects on the efficiency of this process.

1.4.1. Dynamics of Feshbach molecule production

To calculate the molecule production efficiency, it is necessary to calculate the amplitude of the bound-state wavefunction. For a BEC, this can be shown to be¹²

$$\phi_b(t) = -\frac{(2\pi\hbar)^{3/2}}{\sqrt{2}} \int_{t_i}^t d\tau \phi^2(\tau) \frac{\partial}{\partial \tau} [\langle \chi_b | \hat{\mathcal{U}}_{2B}(t, \tau) | 0 \rangle \theta(t - \tau)]. \quad (1.14)$$

Here, $|\chi_b\rangle$ is the bound state wavefunction, and the conversion efficiency is then given by $2|\phi_b(t)|^2/|\phi(0)|^2$.

We show an example dynamical calculation of molecule production from a BEC in Fig. 1.1. We consider the resonant association experiments of Thompson *et al.*,³² where a near resonant magnetic field modulation associated molecules near the 15.5 mT resonance of ⁸⁵Rb. The conversion efficiency as a function of time shows oscillations with a frequency corresponding to the detuning of the RF from the resonant association frequency. Smaller-amplitude, higher-frequency oscillations are visible on top of the main curve, created in the simulation by

Fig. 1.1. Association of molecules from a pure ^{85}Rb BEC at a magnetic field of $B_{\text{av}} = 15.645$ mT, with a magnetic field modulation of duration t_{pulse} , amplitude $6.5 \mu\text{T}$, and a frequency (6.5 kHz) that closely corresponds to the molecular binding energy ($\hbar \times 5.9$ kHz). Different initial densities are labelled in cm^{-3} , with the result for the lowest density, $n = 10^{10} \text{ cm}^{-3}$, multiplied by 20 for clarity. The inset shows a schematic of the magnetic field variation. A 0.5 ms ramp from $B_i = B_{\text{av}} + 0.1$ mT to B_{av} is included, improving the assumption of an initially non-interacting gas. For further details, see Ref. [10].

assuming an initial state corresponding to a non-interacting gas. To reduce these smaller oscillations, we include a short linear ramp before the modulation, which provides some conversion efficiency even for zero pulse duration. The role of the mean-field shift in bringing the energy gap between a free atom pair and a Feshbach molecule close to the modulation frequency is clear from the conversion efficiency, which peaks at approximately 17% for $n = 1.5 \times 10^{12} \text{ cm}^{-3}$.

The conversion efficiency from a thermal gas is given by²²

$$\frac{2N_{\text{mol}}}{N} = \frac{2}{(2\pi\hbar)^3 n} \int d\mathbf{p}_1 d\mathbf{p}_2 \left[n'(p_1, t_i) n'(p_2, t_i) \left| \langle \chi_b | \hat{\mathcal{U}}_{2B}(t, t_i) | (\mathbf{p}_1 - \mathbf{p}_2)/2 \rangle \right|^2 + \int_{t_i}^t dt' \left| \langle \chi_b | \hat{\mathcal{U}}_{2B}(t, t') | (\mathbf{p}_1 - \mathbf{p}_2)/2 \rangle \right|^2 \frac{\partial}{\partial t'} [n'(p_1, t') n'(p_2, t')] \right], \quad (1.15)$$

where N_{mol} is the number of diatomic molecules produced. In the limit of short times and small molecule production efficiencies, this reduces to the expression:^{2,10}

$$\frac{2N_{\text{mol}}}{N} = 2n(2\pi\hbar)^3 \left(\frac{\beta}{2\pi m_{\text{red}}} \right)^{3/2} \int d\mathbf{p} \exp\left(-\beta \frac{p^2}{2m_{\text{red}}}\right) \left| \langle \chi_b | \hat{\mathcal{U}}_{2B}(t, t_i) | \chi_{\mathbf{p}}^{(+)} \rangle \right|^2, \quad (1.16)$$

Fig. 1.2. Association of Feshbach molecules using a 0.25 ms ramp from 15.25 mT to 15.75 mT in a thermal ^{85}Rb gas of temperature 20 nK. As a function of the initial density, we show the conversion efficiency calculated from the non-Markovian Boltzmann equation (diamonds, dashed line) and from a thermal average of the two-body transition probability density (squares, full line). For more details, see Ref. [22].

This equation can be interpreted as a thermal average over the transition probability density between bound and scattering states. Here, $\beta = (k_B T)^{-1}$, where T is the temperature of the gas, and we have substituted the full scattering state $\chi_p^{(+)}$ for the plane wave. This gives a better approximation to the initial state of the gas. The simpler result of Eq. (1.16) has been utilised in a number of studies.^{10,14,15}

In Fig. 1.2 we show the molecule production efficiency from a thermal gas as a function of density. Here, we have solved the non-Markovian Boltzmann equation for a linear ramp across the 15.5 mT resonance in ^{85}Rb , and extracted the molecular conversion efficiency using Eq. (1.15). Because of the difficulty of calculating the kernel, we solve for the dynamics of several initial atomic densities while keeping the same magnetic field variation. The higher densities deviate from the simpler result of Eq. (1.16), which does not account for depletion or rethermalisation of the continuum, and show the onset of saturation of the conversion efficiency, an inherently many-body effect.

1.4.2. Relevance to Other Systems

The non-Markovian Boltzmann equation we have presented here is applicable to other important problems in ultracold gases, such as thermalisation in the presence of strong interactions, or spin transport in a two-component Fermi gas.³³ In

close analogy to the results presented here, it can also be applied to the study of molecule production in mixed gases, and single-component Fermi gases with resonant p -wave interactions.

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‘cumulants’, ‘molecules’, ‘strong interactions’, ‘non-Markovian’, ‘Boltzmann equation’, ‘dynamics’, ‘Feshbach resonance’, ‘separable potential’

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