Stable soliton pairs in optical transmission lines and fiber lasers

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Optical-fiber transmission of pulses can be modeled with the complex Ginzburg–Landau equation. We find novel stable soliton pairs and trains, which are relevant in this case, and analyze them. We suggest that the distance between the pulses and the phase difference between them is defined by energy and momentum balance equations, rather than by equations of standard perturbation theory. We present a two-dimensional phase plane (interaction plane) for analyzing the stability properties and general dynamics of two-soliton solutions of the Complex Ginzburg–Landau equation. © 1998 Optical Society of America
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1. INTRODUCTION

Optical transmission lines^{1–6} and fiber lasers^{7–15} can be described, to a good approximation, with the well-known complex Ginzburg–Landau equation (CGLE). Single-pulse transmission and generation have been considered in detail within this framework. However, the problem of the interaction between two neighboring pulses is crucial for the transmission of information.^{16–19} Ideally, the pulses should not interact at all. It has been shown that, for the nonlinear Schrödinger equation (NLSE), two solitons have no force between them when the phase difference between the two pulses is $\pi/2$.²⁰ Although such solitons do not experience an interaction force initially, repulsion between the pulses takes place at later stages of propagation, owing to the creation of two solitons with different frequencies (hence velocities). On the other hand, trains of solitons in quadrature (i.e., where there is a $\pi/2$ phase difference between any two neighbors) can propagate without change.²¹ Nevertheless, there are inherent difficulties with this arrangement too. For example, slight deviations in the amplitudes may change the phase difference between the pulses and shorten the distance of noninteractive propagation.

The problem is even more complicated because real systems are not described by the pure (integrable) NLSE. However, real systems can be described by Hamiltonian generalizations of the NLSE. In the case of Hamiltonian systems, the interaction between the pulses becomes inelastic.²² This makes the two-soliton solutions of the perturbed NLSE unstable, even when stationary two-soliton solutions do exist.²² Energy exchange between the pulses is one of the mechanisms that makes these solutions unstable. The criterion for this instability has been found in Ref. 23. In principle, owing to this criterion, there could be systems in which energy exchange is forbidden, but they are beyond the range of dynamical

systems considered in practice. Moreover, the energy exchange mechanism is not the only one that causes instability.

The situation changes completely for nonconservative systems. Each pulse has its own internal balance of energy, which maintains its constant amplitude. All solutions then are a result of a double balance: between nonlinearity and dispersion and also between gain and loss. Interaction between the solitons in these systems has been studied by Brand and Deissler²⁴ and Alexander and Jones.²⁵ Bound states of two solitons in these systems were analyzed by Malomed.²⁶ Using a simplified version²⁷ of perturbation analysis for soliton interaction,^{28,29} he showed that stationary solutions in the form of bound states of two solitons, which are in-phase or out-of-phase, may exist. We also confirm that they do exist. However, careful numerical analysis in the two-dimensional phase space, which took into account the fact that the phase difference between the two solitons is an independent variable, showed that these types of soliton bound state are unstable.³⁰

In this work we report the discovery of stable two-soliton and multisoliton solutions of the quintic CGLE with a $\pi/2$ phase difference between them. We use the two-dimensional phase space (distance–phase difference) to analyze the dynamics of the two-soliton system, and show that this phase space adequately describes the system. It allows us to find the bound states, to analyze their stability, and to investigate their global dynamics.

2. STATEMENT OF THE PROBLEM

In the simplest situation, where a single transverse (or temporal) coordinate is retained in the analysis, the quintic CGLE reads (see, e.g., Ref. 8)

$$i\psi_\xi + \frac{D}{2}\psi_{\tau\tau} + |\psi|^2\psi = i\delta\psi + i\epsilon|\psi|^2\psi + i\beta\psi_{\tau\tau} + i\mu|\psi|^4\psi - \nu|\psi|^4\psi, \quad (1)$$

where τ is the retarded time, ξ is the propagation distance, ψ is the complex envelope of the electric field, δ is the linear gain or loss, β describes spectral filtering ($\beta > 0$), $D = \pm 1$ is the chromatic dispersion coefficient, ϵ accounts for nonlinear gain or absorption processes, μ represents a higher-order correction to the nonlinear amplification or absorption, and ν is a possible higher-order correction term to the intensity-dependent refractive index.

A. Discrete Set of Solutions

It is well known that bright-soliton solutions of the CGLE form a discrete set,^{31,32} i.e., given a value to the parameters of the equation, the amplitude and the width of the soliton are fixed. There can be a multiplicity of solutions for each set of parameters,^{31,33,34} but each solution has a fixed amplitude and phase profile. This fact means that the solitons of the CGLE differ qualitatively from the solitons of Hamiltonian (including integrable) systems, in which all bright-soliton solutions are always members of a family of solutions with variable amplitude. There are exceptions to this rule when there exists a certain relation between the parameters, where new symmetries appear,³¹ and for dark-soliton-type solutions of the cubic CGLE,^{32,35} but these are very special cases. Soliton solutions form a discrete set even in the limit when dissipative terms are small,³⁶ and the solutions are essentially perturbed NLSE solitons. In the latter case, small dissipative terms introduce restrictions on the soliton amplitude and width.

The physical reason for the above fact is that, in contrast to the NLSE and its Hamiltonian generalizations, the solitons of the CGLE arise as a result of a balance between the nonlinearity and dispersion on the one hand, and a balance between the gain and loss on the other hand. Each of these balances independently would define a family of solutions, but imposing both simultaneously gives a fixed solution.

B. Two-Dimensional Phase Space

In what follows, we deal with plain solitons, in contrast to composite solitons.³⁴ The latter are more complicated objects consisting of a plain soliton and two fronts. As basic solutions, plain solitons exist in a wider range of parameters than other types of pulselike solutions. The fact that the soliton parameters are fixed implies that during the interaction of two solitons, basically only two parameters may change: their separation, ρ , and the phase difference between them, ϕ . This means, in turn, that the phase space of the problem is truly two dimensional and that we may analyze the bound states formed of two solitons, their stability, and their global dynamics in this two-dimensional phase space. The possibility of this reduction of the number of degrees of freedom is a unique feature of systems with gain and loss. We note that this is not the case for nonintegrable Hamiltonian systems in

which the amplitudes of the solitons can also change, and therefore more sources of instability of the bound states appear.²²

C. Interaction Plane

This two-dimensional phase space has been used intuitively in Ref. 30 for analyzing the bound states formed from two individual CGLE solitons. The arguments in the previous subsection show that this plane is adequate for analyzing the problem of pulse interaction and that there are not more degrees of freedom involved. The two balance equations considered below make the interaction plane a powerful tool for analyzing soliton pairs. Using this tool, we were able to find the two-soliton solutions and to prove that they are stable solutions of the CGLE. We shall call this two-dimensional phase space the interaction plane. All bound-state solutions, their stability, and their dynamics can be visualized on this plane.

3. BALANCE EQUATIONS

The CGLE has no known conserved quantities. Instead, the rate of change of energy with respect to ξ is²²

$$\frac{d}{d\xi} \int_{-\infty}^{\infty} |\psi|^2 d\tau = F[\psi], \quad (2)$$

where the functional $F[\psi]$ is given by

$$F[\psi] = 2 \int_{-\infty}^{\infty} [\delta|\psi|^2 + \epsilon|\psi|^4 + \mu|\psi|^6 - \beta|\psi_\tau|^2] d\tau. \quad (3)$$

Similarly, the rate of change of momentum is defined by

$$\frac{i}{2} \frac{d}{d\xi} \int_{-\infty}^{\infty} (\psi_\tau \psi^* - \psi_\tau^* \psi) d\tau = J[\psi], \quad (4)$$

where the real functional $J[\psi]$ is given by

$$J[\psi] = i \int_{-\infty}^{\infty} [(\delta + \epsilon|\psi|^2 + \mu|\psi|^4)(\psi_\tau \psi^* - \psi_\tau^* \psi) + \beta(\psi_\tau \psi_{\tau\tau}^* - \psi_\tau^* \psi_{\tau\tau})] d\tau. \quad (5)$$

By definition, this functional is the force acting on a soliton along the τ axis.

We are interested in stationary solutions. In this case, the energy and momentum do not change, and the corresponding solutions must satisfy the set of two equations

$$F[\psi] = 0, \quad (6)$$

$$J[\psi] = 0. \quad (7)$$

The first equation indicates the necessary balance that must exist between loss and gain for any stationary solution. The second equation guarantees the balance between the transverse forces acting on solitons. Trivially, $J[\psi] = 0$ for any symmetric [$\psi(\tau) = \pm\psi(-\tau)$] solution. What is not so obvious is that $J[\psi]$ may also be zero for nonsymmetric solutions. These solutions may have non-zero velocity and nonzero momentum.

4. BOUND STATES OF TWO SOLITONS

Given a set of values for the equation coefficients, we call the corresponding plain soliton solution $\psi_0(\tau)$. The bound solution of two plain solitons is well approximated by

$$\psi(\tau) = \psi_0(\tau - \rho/2) + \psi_0(\tau + \rho/2)\exp(i\phi), \quad (8)$$

where the values of the distance between the solitons, ρ , and the relative center-to-center phase between them, ϕ , are those that satisfy Eqs. (6) and (7), i.e.,

$$F[\psi_0, \rho, \phi] = 0, \quad (9)$$

$$J[\psi_0, \rho, \phi] = 0. \quad (10)$$

For very rough estimates, these calculations can be done with simple trial functions for $\psi_0(\tau)$. However, having at hand exact numerical plain-soliton solutions, we used them to find the zeros of Eqs. (9) and (10) numerically. The results of these simulations are given in Figs. 1 and 2. Figure 1 shows F and J versus ρ at $\phi = \pi/2$, with $\psi_0(\tau)$ being the stable plain soliton for those parameters written in the figure. Both functionals converge to zero as ρ goes to infinity, and there are also certain finite separations where they vanish. Figure 1

Fig. 1. Functionals (a) $F[\psi_0, \rho, \phi]$ and (b) $J[\psi_0, \rho, \phi]$ in terms of the separation ρ for ϕ fixed at $\pi/2$. The parameters of the simulation are shown in the figure. The inset in (b) shows the functional J on a different scale to reveal an additional zero of $J[\psi_0, \rho, \phi]$.

Fig. 2. Zeros of $F[\psi_0, \rho, \phi]$ (solid curves) and $J[\psi_0, \rho, \phi]$ (dotted curves) on the interaction plane. The points of intersection of the solid curves with the dotted curves correspond to the bound states of two solitons. They are shown as filled circles. The parameters of the simulation are shown in the figure.

shows that, for the given set of equation coefficients and phase difference (ϕ), Eq. (9) has one zero and Eq. (10) has two zeros.

The whole set of zeros of these functionals is presented on the interaction plane in Fig. 2. We have found the zeros of F and J in the interval $0.4 < \rho < 4$, where bound states may exist. The separation must be of the same order as the width of a single soliton. Smaller ρ correspond to the merging of the solitons, while at larger ρ the interaction between the solitons is extremely weak. The solid curves in Fig. 2 show the locus of points where $F[\psi] = 0$, while the dotted curves show those where $J[\psi] = 0$. It is seen, from this figure, that the functional $F[\psi]$ has two zeros in the interval $-\pi/2 < \phi < \pi/2$ but only one zero in the interval $\pi/2 < \phi < 3\pi/2$. (Here we have not included the obvious zero at $\rho = \infty$.)

As we noted above, the functional $J[\psi]$ is zero for any symmetric function [$\psi(-\tau) = \pm\psi(\tau)$]. Hence it is zero everywhere on the horizontal axis of the interaction plane. This shows that every intersection of a solid curve with the horizontal axis corresponds to a bound state of two solitons. There are three examples of this type of bound state; in Fig. 2 they are labeled S_i , with $i = 1, 2, 3$. These solutions have a 0 or π phase difference between the component solitons. In addition, the functional $J[\psi]$ has zeros along two almost circular curves. The intersections of the outer circle with the solid curve (points F_1 and F_2) correspond to the new bound states with a $\pi/2$ phase difference between the solitons. Because of this phase difference, the phase profile of the solution is asymmetric (see Fig. 6 below).

We can see now that physically the existence of the two-soliton solutions is the result, first, of the balance between gain and loss and, second, of the balance between forces along the τ axis, which act on the soliton pair. They are also defined by gain and loss mechanisms. The

interplay between the two gives the actual distance and phase difference between the solitons in the bound state.

5. STABILITY AND GLOBAL DYNAMICS OF BOUND STATES

Numerically, the stability of all bound states can be studied by means of the same interaction plane. Moreover, the general dynamics of the interaction of two solitons can also be described with just this plane. Initial condition (8) in the form of two stable solitons with arbitrary separation (ρ) and phase difference between them (ϕ) will result in a trajectory on this plane. Bound states will be singular points of this plane. The type of the singular point defines the stability of the bound state.

Figure 3 shows an example of a numerical simulation of an interaction between the two solitons on the interaction plane. This figure indicates that, for the given set of parameters, there are five singular points. Within the accuracy of the method, these coincide with the solutions, which are found with the balance equations of the previous section. Discrepancies are very minor. They have their origin in the fact that the exact bound solutions differ slightly from those that are approximated by Eq. (8). Three of these singular points (S_1 , S_2 , and S_3) are saddles. In these states, the phase difference between the solitons is zero or π . Clearly, these are unstable bound states of two solitons. They were unstable for all the sets of parameters that we used in our numerical simulations.

In addition, there are two symmetrically located stable foci (F_1 and F_2), which correspond to stable bound states of two solitons in quadrature, i.e., the phase difference between them is $\pm\pi/2$. These are the bound states with

Fig. 3. Trajectories showing the evolution of two-soliton solutions on the interaction plane. The five singular points correspond to the five bound states depicted in Fig. 2. Only two of them (F_1 and F_2) are stable. The central part of the figure, where ρ is less than a single soliton width, does not describe a valid bound state. Trajectories converging to the center describe the merging of two solitons. The parameters used in the simulation are shown in the figure.

Fig. 4. (a) Stable propagation of a two-soliton solution and (b) its spectrum. The parameters of the simulation are the same as in Fig. 3.

asymmetric phase profiles found in the previous section. A consequence of this asymmetry is that the two-soliton solution moves with a constant velocity. The direction of motion depends on the sign of ϕ . This motion would also occur for any state corresponding to a singular point not on the horizontal axis. An example of stable propagation of a two-soliton bound state and its spectra at each value of ξ are given in Fig. 4. The spectrum is also asymmetric because of the phase asymmetry, as expected for this type of solution.²¹

Note that asymmetric bound states are not always stable. Changing the parameters in Eq. (1) may convert stable foci into centers (or elliptic points) and further into unstable foci. An example of the interaction plane for the two latter cases are given in Fig. 5. These results show that there is a certain region in the parameter space where stable soliton pairs do exist. Finding the boundaries of this region is an important problem to be solved. However, the full solution of this problem would require a vast amount of numerical simulations and still would leave open the question of the existence of several such regions. For example, we did not find such regions in the case of normal dispersion, although stable single pulses are known to exist.³⁷ More numerical and analytical work is required on this aspect.

6. PERTURBATIVE APPROACH

An attempt to apply a simplified perturbative approach²⁶ to the soliton interaction fails in the case of the CGLE. The main reason is that the more powerful mechanism of balance between the gain and the loss has a stronger ef-

fect on the bound state than the weak interaction between the tails of the solitons. This can be seen from the following.

First, the interaction of the quasi-linear tails of one soliton with the second soliton²⁶ cannot explain the existence of stable stationary states with the $\pi/2$ phase difference between the solitons. Second, this approach predicts periodically located zeros for the interaction potential. This is not the case for the interaction of two solitons in the case of CGLE, as we can see from Figs. 2 and 3.

We should stress that the profile of each soliton is hardly modified at all by the interaction. Figure 6 shows the profile of a two-soliton solution and the initial condition, which consists of two solitons that are $\pi/2$ out of phase and are separated by a distance $\rho = 2.4$ (Eq. 8). In fact, the two different curves are totally indistinguishable on the scale of this plot. However, they are not identical. It takes a while to evolve to the bound state. The process of convergence from the initial condition to the final state of two-soliton solution can be seen clearly in Fig. 7. Both functionals, J [Fig. 7(a)] and F [Fig. 7(b)], oscillate with exponentially decaying amplitudes before converging to

Fig. 5. Trajectories showing the evolution of two-soliton solutions on the interaction plane. The two singular points (F_1 and F_2) are now (a) neutrally stable and (b) unstable. The parameters used in the simulation are shown in the figure.

Fig. 6. (a) Amplitude profile and (b) phase profile of the two-soliton solution. The initial condition, consisting of two solitons at a separation $\rho = 2.4$ and with a $\phi = \pi/2$ phase difference between them, is also given here. The two curves are indistinguishable on the scale of this plot. Note that the phases of the peaks in (b) differ by $\pi/2$.

zero, which then corresponds to the bound state. Figure 7(c) shows that the oscillations in energy ($\int_{-\infty}^{\infty} |\psi|^2 d\tau$) during this transition are really very small (<0.01 or less than $\approx 0.03\%$).

Clearly, the tails of each individual soliton can be described by the linearized version of the master equation, Eq. (1). It seems possible to improve the approximation by adding cubic terms to the linearized equation, to describe, for example, the central part of the solution in Fig. 6. The cubic equation has the advantage that all of its elementary solutions can be presented analytically. The exact solution of the cubic equation, in the form of a dark soliton, has been obtained by Nozaki and Bekki.³⁵ We have rederived this solution and obtained it in a simpler form than that given originally and in Ref. 32, where the formulas have been improved. This solution, which is given in the Appendix A, has an arbitrary parameter, the velocity V , which can be adjusted to our solution.

If this approach is applicable to the CGLE, then we would expect that the central part of the dark soliton of the cubic CGLE would accurately describe the central part of the two-soliton solution of the quintic CGLE. However, this does not happen. Figure 8 shows a comparison between the profiles of the two-soliton bound state of the quintic CGLE for the equation parameters written in the figure and the dark-soliton-type solution of the cubic solution, which moves at the same velocity, V

$= 0.00126$. [The remaining parameters of the dark (and virtually black) soliton solution are readily calculated to be $K = 0.0028$, $\Omega = -0.0013$, $b = 0.00025$, $d = -10.69$, $\kappa = 0.1824$, and $u = 6.3$.] Clearly the width of the dark soliton exceeds the distance between the two solitons by more than an order of magnitude when the two solutions have the same velocity. Note that the solution of Nozaki and Bekki is the only known solution that is suitable for describing the central part of the two-soliton solution of the quintic equation. In principle, there could exist another solution of the cubic equation for the same set of parameters, which would fit the soliton pair. However, its analytical expression and stability properties are not known.

The main conclusion from this analysis is that the energy balance is the main mechanism for the interaction of basic solutions. Equations (9) and (10) are necessary and sufficient to find the roots in the two-dimensional interaction plane and hence to find soliton pairs. Note that the energy balance is quite a general law, and that its validity

Fig. 7. Oscillations of the functionals (a) J and (b) F with (c) the energy in the process of convergence of the initial condition (8) into a bound state of two solitons. The parameters used are given in the plot for J .

Fig. 8. Comparison of the profiles of the two-soliton bound state of the quintic CGLE (solid curves) and the dark-soliton solution of the cubic equation (dotted curves). The inset shows the dark soliton in a much wider time window.

does not depend on the dissipative terms being small or large. This means that our approach can be applied and approximate analytic expressions can be used in our formalism in this limit. Then the parameters of soliton pairs, if they exist in this limit, can be found analytically. The tails of the solitons play a minor role in their interaction dynamics in the strongly dissipative regime, and their effect can be ignored. We believe the same is true when dissipative terms are small. However, the existence of stable bound states in the latter case is still an open question.

7. MULTISOLITON SOLUTIONS

As a consequence of the existence of two-soliton solutions, three-soliton and other multisoliton solutions also exist. An example of the three-soliton solution is given in Fig. 9 and a multisoliton solution is shown in Fig. 10. As a result of the above-mentioned asymmetry, multisoliton solutions are also asymmetric and move with the same constant velocity along the τ axis.

Periodic solutions of CGLE can clearly be constructed in this way. Then the whole train will move with a constant velocity, as is also the case for an NLSE train.²¹ Schöpf and Karmner were the first to discuss periodic solutions of the CGLE, and their numerical results³⁸ in fact indicate a small transverse velocity of the periodic train (see Fig. 2a of Ref. 38). Some examples of periodic solutions in our numerical simulations evolve from a single pulse (because δ is positive) (Fig. 11.) The pulses in such a train have small nonzero velocities. This means that the periodicity of this train is also due to the above mechanism. On the other hand, doubly periodic solutions with zero velocity also exist.³⁹ Hence there can be other reasons for the existence of stable periodic solutions.

We studied other types of bound states to find the basic building blocks for more complicated structures. As we

found above, there is no stable two-soliton solution with zero velocity. However, it is possible to have stable three-soliton solutions with zero velocity, if the outer solitons have a common phase that is different from that of the central one. The structure is therefore symmetric in phase profile and must have zero velocity. This structure is similar to a particular three-soliton solution of the NLSE.⁴⁰

An example of a three-soliton solution with zero velocity is given in Fig. 12. The phase profile shows that the phase difference between the adjacent solitons in this

Fig. 9. Evolution of a three-soliton bound state.

Fig. 10. Stable propagation of a four-soliton bound state.

Fig. 11. Creation of a soliton train from a single pulse. The parameters of the simulation are shown on the plot.

Fig. 12. (a) Amplitude and (b) phase profiles of a three-soliton bound state with zero velocity. Each curve is labeled with its value of ϵ .

case is not exactly $\pi/2$ but is closer to 2.0 (i.e., 115°). This solution exists and is stable for a wide range of parameters. In Fig. 12, a three-soliton solution is shown for three values, $\epsilon = 0.49, 0.58$, and 0.67 , with fixed values of the other parameters of the equation. This solution can be, for example, a basic building block for the doubly periodic solution found in Ref. 39.

8. CONCLUSIONS

In conclusion, we have found novel stable soliton pairs and trains of the one-dimensional complex Ginzburg–Landau equations and analyzed them. We have suggested that the distances between the pulses and the phase differences between them are defined by energy and momentum balance equations, rather than by equations of standard perturbation theory. We have presented a two-dimensional phase plane (interaction plane) for analyzing the stability properties and general dynamics of two-soliton solutions of the CGLE.

APPENDIX A: DARK-SOLITON SOLUTION OF THE CUBIC EQUATION

Here we consider the dark-soliton solutions of Eq. (1) with $\mu = \nu = 0$. The analytic solution has been found in Ref. 35 and refined in Ref. 32. Here we give a simpler form to allow convenient numerical simulations. We are interested in the traveling solution of Eq. (1) with transverse velocity V :

$$\psi(\tau, \xi) = \left[\kappa u \tanh(\kappa \zeta) + (d + 2i) \frac{w}{d} \right] \times \exp[i\phi(\zeta)] \exp(iK\zeta - i\Omega\xi), \quad (\text{A1})$$

where u , w , d , κ , K , and Ω are real constants and ϕ is a real function of $\zeta = \tau - V\xi$:

$$\phi(\zeta) = d \ln[\cosh(\kappa \zeta)]. \quad (\text{A2})$$

d is the phase modulation parameter, which is expressed in terms of ϵ and β only:

$$d = \frac{3(2\epsilon\beta + 1) + \sqrt{9(2\epsilon\beta + 1)^2 + 8(2\beta - \epsilon)^2}}{2(2\beta - \epsilon)}. \quad (\text{A3})$$

The parameter u^2 also depends on ϵ and β only (because d depends on ϵ and β):

$$u^2 = \frac{3d(1 + 4\beta^2)}{2(2\beta - \epsilon)} = \frac{(1 + 4\beta^2)(d^2 - 2)}{2(1 + 2\epsilon\beta)}. \quad (\text{A4})$$

Equation (A1) represents a one-parameter family of solutions. It is most convenient to take κ as the independent parameter. Then the other parameters of solution (A1) will depend on κ :

$$\left(\frac{w}{d}\right)^2 = \beta \frac{\kappa^2(3d + 4\beta) - 2\delta}{3\epsilon(d + 4\beta)}, \quad (\text{A5})$$

$$\Omega = \frac{\delta}{2\beta} + \frac{2\beta - \epsilon}{2\epsilon} \left(K^2 - \kappa^2 d^2 + \frac{\delta}{\beta} \right), \quad (\text{A6})$$

$$K = \frac{\epsilon u w}{\beta d},$$

$$K^2 = \frac{d\epsilon(1 + 4\beta^2)}{2\beta(2\beta - \epsilon)} \frac{[\kappa^2(3d + 4\beta) - 2\delta]}{(d + 4\beta)}, \quad (\text{A7})$$

$$V = \frac{K}{\epsilon} (\epsilon - 2\beta). \quad (\text{A8})$$

Fig. 13. Parameters in the solution (A1) versus κ . Note that $w = V = K = 0$ when $\kappa = 0.02579$. This point corresponds to Eq. (A9).

A plot of all these parameters versus κ is given in Fig. 13. Note that the rest of the parameters, apart from Ω , are zero at finite κ . This is one of the reasons for taking κ as the arbitrary independent parameter. This set leads to the following special case.

This particular case, the black soliton, which has been found in Ref. 41, is especially simple. Setting $K = w = V = 0$ in the above equations, we then have fixed values of κ^2 ,

$$\kappa^2 = \frac{2\delta}{3d + 4\beta}, \quad (\text{A9})$$

and Ω ,

$$\Omega = \kappa^2(1 - 3\beta d) = \frac{2\delta(1 - 3\beta d)}{3d + 4\beta}. \quad (\text{A10})$$

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