Bound states and interactions of vortex solitons in the discrete Ginzburg-Landau equation

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(Received 25 October 2011; published 20 August 2012)

By using different continuation methods, we unveil a wide region in the parameter space of the discrete cubic-quintic complex Ginzburg-Landau equation, where several families of stable vortex solitons coexist. All these stationary solutions have a symmetric amplitude profile and two different topological charges. We also observe the dynamical formation of a variety of “bound-state” solutions composed of two or more of these vortex solitons. All of these stable composite structures persist in the conservative cubic limit for high values of their power content.

DOI: [10.1103/PhysRevA.86.023834](https://doi.org/10.1103/PhysRevA.86.023834)

PACS number(s): 42.65.Tg, 42.65.Wi, 05.45.Yv

I. INTRODUCTION

Optical beams whose phase circulates around a singular point, or central core, changing by $2\pi S$ times in each closed loop around it (with S being an integer number), are called optical vortices. The integer number S is known as the topological charge of the vortex, and its sign defines the direction of the phase circulation. Usually an optical vortex has a doughnutlike shape and diffracts when it propagates in free space. In quantum information they have an enormous potential for codifying information beyond two levels using their topological charge value [1]. In other fields, such as biophotonics, for example, they are useful due to their ability to affect the motion of particles (microorganisms) through angular momentum transfer [2]. Other scientific and technological applications for optical vortices are found in optical systems communication, spintronics, and optical tweezers [3–5]. These potential applications of optical vortices have sparked the interest of the scientific community regarding their basic properties and characteristics.

Diffraction is a fundamental phenomenon which leads to beam broadening upon propagation. In a nonlinear medium, self-focusing reduces diffraction whereas self-defocusing enhances the beam spreading. In a situation where the nonlinear self-focusing effects exactly balance diffraction, the beam can propagate as an optical spatial soliton, i.e., a self-trapped optical beam which preserves its shape upon propagation. Recently, spatial optical solitons have become attractive for several technological applications. They can be defined as self-localized solutions of nonlinear wave equations found in various physical systems [6]. Typical equations of this type in optics are the nonlinear Schrödinger equation (NLSE) for conservative systems, and the complex Ginzburg-Landau equation (CGLE) for dissipative equations [7,8]. The CGLE is a master model in which dissipative solitons [9] are probably its most interesting solutions. In conservative models, such as the ones described by the NLSE or its several variants, exchange of energy with the surroundings is not allowed. Self-localized solutions for the nonlinear Schrödinger equation originate from a balance between nonlinearity (e.g., the Kerr effect) and dispersion or diffraction. In contrast, for dissipative

systems the solutions also exchange energy with an external source, making the problem more complex and rich. In this case, an extra balance between gain and loss is required in order to obtain stationary solutions. In particular, dissipative vortex solitons in continuous media have been found to exist for several values of S , and they are stable in wide regions of the parameter space of the CGLE [10,11].

In this paper we concentrate on dissipative systems governed by the Ginzburg-Landau equation. This model appears in many branches of science such as nonlinear optics, Bose-Einstein condensates, chemical reactions, superconductivity, and many others [12,13]. Nonlinear periodic structures offer alternative ways to control light propagation by modification of diffraction properties through refractive index modulation. For instance, photonic crystals are structures of alternating refractive index that provide unprecedented control over light fields propagating through them. Recent works show that lasers with square-lattice photonic crystal cavities possess enhanced functionality and performance when compared to conventional lasers [14]. These systems can be analyzed within the framework of a set of coupled, linear equations which, in solid-state physics, is known as the *tight-binding* approximation, while in an optics context, it is known as the *coupled-mode approach*. Stationary solutions obtained in this framework are called discrete solitons. In particular, discrete vortex solitons in conservative systems have been reported on several theoretical and experimental works [15–18], while dissipative discrete solitons have been found, analytically and numerically, in one-dimensional waveguide arrays [19–21].

In this paper we report the finding of a wide region in the parameter space of the discrete cubic-quintic complex Ginzburg-Landau equation where different discrete vortex solitons coexist. All the individual solutions we examine in this paper possess simultaneously two topological charges, as those reported in some of our recent works [22,23]. We have studied their interactions and the formation of bound states.

The rest of the paper is organized as follows. In Sec. II we introduce the complex cubic-quintic Ginzburg-Landau model that we will use in this work. Section III describes the new families of solutions we have found, and in Sec. IV we show

the composite structures obtained when we let them interact as they evolve. Section V analyzes the discrete nonlinear Schrödinger equation case for all the solutions previously mentioned. Finally, Sec. VI summarizes our main results and conclusions.

II. MODEL

Optical beam propagation in a nonlinear, periodical two-dimensional waveguide array can be modeled by the following equation:

$$i\dot{\psi}_{m,n} + \hat{C}\psi_{m,n} + |\psi_{m,n}|^2\psi_{m,n} + \nu|\psi_{m,n}|^4\psi_{m,n} = i\delta\psi_{m,n} + i\varepsilon|\psi_{m,n}|^2\psi_{m,n} + i\mu|\psi_{m,n}|^4\psi_{m,n}. \quad (1)$$

Equation (1) is the discrete version of the complex cubic-quintic Ginzburg-Landau (CQGL) equation. Here, $\psi_{m,n}$ is the complex field amplitude at the (m,n) lattice site and $\dot{\psi}_{m,n}$ denotes its first derivative with respect to the propagation coordinate z . The set

$$\{m = -M, \dots, M\} \times \{n = -N, \dots, N\}$$

defines the array, with $2M + 1$ and $2N + 1$ being the number of sites in the horizontal and vertical directions, respectively. The tight-binding approximation establishes that the field propagating in each waveguide interacts linearly only with nearest-neighbor fields through their evanescent tails. This interaction is described by the discrete diffraction operator

$$\hat{C}\psi_{m,n} = C(\psi_{m+1,n} + \psi_{m-1,n} + \psi_{m,n+1} + \psi_{m,n-1}),$$

where C is a complex parameter. Its real part denotes the strength of the coupling between adjacent sites and its imaginary part denotes the gain or loss originated by this coupling. The nonlinear higher-order Kerr term is represented by ν , while $\varepsilon > 0$ and $\mu < 0$ are the coefficients for cubic gain and quintic losses, respectively. Linear losses are accounted for by a negative value of δ .

In contrast to the conservative discrete nonlinear Schrödinger (DNLS) equation, the optical power, defined as

$$Q(z) = \sum_{m,n=-M,-N}^{M,N} |\psi_{m,n}(z)|^2, \quad (2)$$

is not a conserved quantity in the present model. Nevertheless, for a self-localized solution, the power and its evolution will be the main quantity that we will monitor in order to identify different families of stationary solutions.

We look for stationary solutions of Eq. (1) of the form $\psi_{m,n}(z) = \phi_{m,n} \exp(i\lambda z)$, where λ is real and $\phi_{m,n}$ are complex amplitudes. We are interested in solutions localized in space whose phase changes azimuthally by an integer number (S) of 2π along a discrete closed circuit. In such cases, the self-localized solution is called a discrete vortex soliton [24] with vorticity S . By inserting the previous *ansatz* into Eq. (1) we obtain the following set of $(2M + 1) \times (2N + 1)$ algebraic coupled complex equations:

$$-\lambda\phi_{m,n} + \hat{C}\phi_{m,n} + |\phi_{m,n}|^2\phi_{m,n} + \nu|\phi_{m,n}|^4\phi_{m,n} = i\delta\phi_{m,n} + i\varepsilon|\phi_{m,n}|^2\phi_{m,n} + i\mu|\phi_{m,n}|^4\phi_{m,n}. \quad (3)$$

We solve Eq. (3) by using a multidimensional Newton-Raphson iterative algorithm. The method requires an initial guess, and it converges rapidly when using a highly localized profile (more details can be found in Ref. [22]).

A. Linear stability analysis

Small perturbations around the stationary solution can grow exponentially, leading to the destruction of the vortex soliton. A stability analysis provides us with the means for establishing which solutions are linearly stable. Let us introduce a small perturbation $\tilde{\phi}$ to the localized stationary solution:

$$\psi_{m,n} = [\phi_{m,n} + \tilde{\phi}_{m,n}(z)]e^{i\lambda z}, \quad \tilde{\phi}_{m,n} \in \mathbb{C}. \quad (4)$$

Then, after replacing Eq. (A3) into Eq. (1) and then linearizing with respect to $\tilde{\phi}$, we obtain the following:

$$i\dot{\tilde{\phi}}_{m,n} + \hat{C}\tilde{\phi}_{m,n} - i\delta\tilde{\phi}_{m,n} + [2(1 - i\varepsilon)|\phi_{m,n}|^2 + 3(\nu - i\mu)|\phi_{m,n}|^4 - \lambda]\tilde{\phi}_{m,n} + [(1 - i\varepsilon)\phi_{m,n}^2 + 2(\nu - i\mu)|\phi_{m,n}|^2\phi_{m,n}^*]\tilde{\phi}_{m,n}^* = 0. \quad (5)$$

The solutions of the above homogeneous linear system can be written as

$$\tilde{\phi}_{m,n}(z) = C_{m,n}^1 \exp[\gamma_{m,n}z] + C_{m,n}^2 \exp[\gamma_{m,n}^*z], \quad (6)$$

where $C^{1,2}$ are integration constants and $\gamma_{m,n}$ is the discrete spectrum of the associated *eigensystem* (see Appendix A2). The solutions are unstable if at least one eigenvalue has a positive real part, that is, if $\max\{\text{Re}(\gamma_{m,n})\} > 0$. By following the formalism developed in Appendix A2, we computed the eigenvalues spectrum and determined the linear stability of the solutions reported along this work. Hereafter, as a general convention, we plot stable and unstable solutions with solid and dashed lines, respectively.

III. MULTIPLICITY OF STABLE VORTEX SOLITON FAMILIES

As stated above, the nonlinear gain in the system is mainly controlled by ε ; this parameter will be the only one that we will change in our simulations. Once we find a stationary solution, for a fixed set of parameters, we compute its linear stability and then change the parameters slightly and find the new solution using the previous one as an *ansatz*. In this manner we obtain all the solution families displayed in the Q vs ε diagram shown in Fig. 1.

The first of them (A family), was already reported in our previous work [22]. It was obtained by starting from the fundamental four-peaks discrete vortex soliton with $S = 1$, after passing throughout several saddle-node points. A striking property of all the solutions shown in Fig. 1 is that they have, simultaneously, two topological charges; i.e., for two different closed trajectories on the plane (m,n) , the measured topological charge is not the same. For a detailed explanation of how a vortex solution with two topological charges can be identified and measured, see Ref. [23]. Representative amplitude and phase profiles of these families are shown in Fig. 2. From the amplitude profiles we can see that there is a difference of four excited sites between one stable family and the next one. Family A has eight main excited sites and family E has 24

FIG. 1. Q versus ε diagram for several vortex interconnected soliton families. Solid lines correspond to stable families while dashed lines to unstable ones. (CQGL equation parameters: $C = 0.8$, $\delta = -0.9$, $\mu = -0.1$, $\nu = 0.1$.)

main peaks. The amplitude profile for case (A) shows a swirl spatial configuration. From its phase profile we can identify a topological charge $S = 1$ in the core, the most inner discrete contour, and a charge $S = -3$ away from the center. The phase profiles for families B, C, D, and E show a topological charge $S = -3$ in the core of these solutions. From B to D, the topological charge has the same value in the core and away from there, but the phase profile outside looks rotated with respect to the center. For the last family E, the topological charge has increased up to $S = -7$ away from the center.

Stable families B, C, D, and E, shown in Fig. 1, were unveiled after observing the dynamic evolution of solutions belonging to unstable branches (dashed gray lines). For most of their existence domains, stable and unstable families are very close, being almost indistinguishable in the scale of Fig. 1. The amplitude profiles of the unstable and the stable solutions have essentially the same spatial distribution and almost equal optical power. Despite this, the phase profiles and, most importantly, the topological charge of the stable and unstable solutions are completely different (see, for example, Fig. 3).

As stated above, the initial condition that we used to find a new family is the unstable solution from the nearest unstable branch, i.e., the stationary (unstable) solution belonging to the gray dashed line is used as an initial condition to solve Eq. (1), for exactly the same parameters. After propagating a finite

FIG. 2. (Color online) Color map plots of the amplitude (top) and the phase (bottom) profiles for stable vortex solutions belonging to the families marked in Fig. 1. $\varepsilon = 0.9$.

FIG. 3. (Color online) Color map plots of the amplitude (a) and phase profiles for unstable (b) and stable (c) vortex solutions close to the family A of Fig. 1.

distance, the profile decays into a new stable solution, which is used as a new seed to unveil the whole new family. For example, the unstable profile sketched in Figs. 3(a) and 3(b) was used as an initial condition in Eq. (1). After propagation, the amplitude profile did not change much, and the phase profile converged to the one shown in Fig. 3(c). The initial phase profile [Fig. 3(b)] shows a $S = 1$ structure that, after propagation, evolves into two topological charges, as shown in Fig. 3(c). The same procedure was followed to obtain all the stable solutions displayed in Figs. 2(b)–2(e). In all the cases, the spatial amplitude distribution hardly changes on propagation, while its phase structure does it significantly. Figure 4 illustrates another example for the D family. We may interpret this dynamical process as a stabilization mechanism by means of phase adjustments: stability is obtained when a two-charge structure is achieved.

IV. COMPOSITE STRUCTURES

Next, we study the formation of bound states composed of two vortex solitons belonging to the family E with $\varepsilon = 0.9$. We have chosen this family due to his high value of vorticity and square symmetry equivalent to that of the optical lattice. We study dynamically the evolution of an array of two of these solutions, horizontally shifted by a certain small distance. We tested two initial configurations differing in their initial separation, and for each one of them we try a broad range of initial phase differences, following a procedure similar to the one implemented in Ref. [25]. For this purpose, we multiply the second solution by a phase factor $e^{i\theta_\alpha}$, where $\theta_\alpha \equiv \frac{\pi}{20}\alpha$ with $\alpha = 1, 2, \dots, 40$. A bound state is reached when the relative phase ($\Delta\theta$, defined as the phase difference between two given sites in each solution) becomes a constant. In continuous and homogeneous systems, the separation distance also changes along the evolution and it becomes a constant when the bound state is formed. Here, in the dissipative discrete case, we do

FIG. 4. (Color online) Color map plots of the amplitude (a) and phase profiles for unstable (b) and stable (c) vortex solutions close to the family D of Fig. 1.

FIG. 5. (Color online) Color map plots showing the amplitude (left) and phase (right) profiles for the stable solutions corresponding with the basins of attraction shown in Fig. 6. For basins $\mathbf{b1}^{\leftrightarrow}$ the stable vortex soliton is similar to the profiles shown in (a) and (b). Profiles for the $\mathbf{b2}$ basin look slightly different and are shown in (c) and (d).

not observe any soliton mobility and, therefore, the separation distance remains invariant.

We have measured the phase difference, in both configurations, for the sites enclosed by the white circles shown in Fig. 5(a). Figures 6(a) and 6(b) show $\Delta\theta$ and Q versus z , respectively, for the first configuration shown in Fig. 5(a). We clearly identify two attraction basins, labeled $\mathbf{b1}^{\leftarrow}$ and $\mathbf{b1}^{\rightarrow}$. Both ($\mathbf{b1}^{\leftrightarrow}$) correspond to the lower power value shown in Fig. 6(b). This implies that both basins are symmetrically equivalent solutions. The unstable configuration is labeled as $\mathbf{b2}$ and it corresponds to the upper power value in Fig. 6(b). [Figures 5(c) and 5(d) show the amplitude and phase profiles of this unstable solution.]

All these solutions preserve the central core structure, keeping the same topological charge as the initial input condition. Figures 5(a) and 5(c) show the amplitude profile for both of them; although they are very similar, the first one

FIG. 6. (Color online) (a) Dynamic evolution of the relative phase between the two sites enclosed by the white circles in the vortex solutions shown in Fig. 5. (b) Optical power evolution for the same vortex solutions. The inset in (b) shows a magnification of the initial stage of evolution.

FIG. 7. (Color online) $\sin(\theta_l)$ versus l for contours (a) Γ_1 and (b) Γ_2 .

has an extra central core (marked by a red circle) located at the center of the structure. For the second structure we can note that the column in the middle ($m = 0$) is filled by small tails, without a central core. By taking a close look at the rectangular contour sketched in Fig. 5(b), we find that the phase varies continuously. The charge increases in the direction indicated by the arrows in this contour, with an accumulated charge of $S = 11$. On the other hand, if we look at how the phase changes along the rectangle sketched in Fig. 5(d), we see that the topological charge is not well defined on this contour. Indeed, the topological charge is truncated (see gray filled circles), meaning that this structure is not a composed vortex beam. Nevertheless, this profile can be thought as two noninteracting vortex solitons with a π radians rotation between them. As we can see from Fig. 6(a), any small variation in the phase leads this bound solution to evolve towards the basin of attraction $\mathbf{b1}^{\leftrightarrow}$. No other initial condition goes to $\mathbf{b2}$, meaning that this is not properly a basin of attraction. So, we can say that vorticity is a necessary condition, achieved during the propagation, for the stability of this kind of structure.

For the sake of clarity, we plot $\sin(\theta_l)$ vs l , where l corresponds to the site on the inner (Γ_1) and outer (Γ_2) discrete contours sketched in Fig. 5(a). Figure 7(a) shows a good correspondence between the data (green points) and a sinusoidal function (gray line) with seven periods ($S = 7$) along the 21 sites of the Γ_1 contour. For the Γ_2 contour, which contains 29 sites, we count 11 periods ($S = 11$), as shown in Fig. 7(b). Figure 7 explicitly shows the different topological charges contained simultaneously in this solution. This supports the right identification of discrete vortex solitons, which is not an easy task for discrete systems.

We consider now a second initial configuration where the two initially independent $\mathbf{E}$ -family vortices are placed closer to each other. We find a similar evolution than before, but now there are three different stable attraction basins for the relative phase evolution [see Fig. 8(a)]. Two of them, the lowest ($\mathbf{b2}^{\leftarrow}$) and the highest ($\mathbf{b1}^{\rightarrow}$), correspond to the larger Q -value basin [$\mathbf{b2}^{\leftrightarrow}$ in Fig. 8(b)]. The amplitude and phase profiles

FIG. 8. (Color online) (a) Dynamic evolution of the relative phase between two sites enclosed by the white circles, on the bound states showed in Fig. 9. (b) Optical power evolution for the same bound state. The inset in (b) shows a magnification of the initial stage of evolution.

for these two vortex solutions are shown in Figs. 9(a), 9(b), 9(c), and 9(d), respectively. We can see from the amplitude profiles that these solutions lost one of the two original central cores (both solutions are equivalent if we perform an inversion symmetry through the $\hat{n}$ axis). The global vorticity is lost, and we can see from Figs. 9(b) and 9(d) how the phase circulation is truncated when we move to the region without a

FIG. 9. (Color online) Color map plots showing the amplitude (left) and phase (right) profiles for the stable solutions corresponding with the basins of attraction shown in Fig. 8. For basin $b2^{\leftarrow}$ the stable structure is similar to the profiles shown in (a) and (b), while (c) and (d) correspond with $b2^{\rightarrow}$. For the remaining $b1$ basin, the vortex soliton is similar to the profiles shown in (e) and (f).

FIG. 10. (Color online) $\sin(\theta_l)$ vs l (site number) diagram for the first (a) and second (b) discrete contours shown in Fig. 8(a) for the bound state vortex soliton.

core. Here, we claim that this mixed bound state is composed of an **E**-family vortex soliton and a staggered bright soliton (with a π -phase shift between nearest neighbors). The third basin (**b1**), which corresponds to the lower Q -value basin in Fig. 8(b), has the amplitude and phase profiles displayed in Figs. 9(e) and 9(f). We clearly see that it preserves the initial two central cores, keeping the same topological charge as the initial condition. Unlike the previous two basins, the global topological charge of this solution is well defined. As for the first configuration, there are also two different topological charges for this composite vortex soliton. Again, to corroborate this, we plot $\sin(\theta_l)$ vs l in order to show in detail the topological charge along two different contours. The first one, Γ_3 , corresponds to the inner rectangular contour sketched in Fig. 9(e), while Γ_4 corresponds with the outer rectangular contour sketched in the same figure. Figure 10(a) shows how the inner charge is $S = 7$, while Fig. 10(b) indicates a charge $S = 11$ for contour Γ_4 .

We now show two more examples of composite structures built from an initial superposition of two and four solutions taken from the **D** and **E** families shown in Fig. 1. In both cases, the values of the parameters used are $C = 0.8$, $\delta = -0.9$, $\varepsilon = 0.9$, $\mu = -0.1$, and $\nu = 0.1$. The typical propagation distance was $z \approx 300$, enough for the power content to become constant.

Figure 11 shows three stable solutions obtained by superposing two vortex solitons belonging to the **D** family in Fig. 1. The first one is constructed by overlapping two of these vortices spaced by one site between their central cores. Figures 11(a) and 11(b) show the amplitude and phase profiles for this stable solution. We note that this state has only one central core, located halfway between the initial ones. On the other hand, the phase profile shows a charge $S = 5$ at the inner contour and rotated with respect to the next discrete contours.

The next configuration is constructed in the same manner as the previous one, but now the center-to-center distance between the cores has been increased to two sites. Figure 11(c)

FIG. 11. (Color online) Color map plots showing the amplitude (left) and phase (right) profiles for three stable vortex solutions obtained by superposing two vortex solitons belonging to the *D* family, at different initial distances.

shows a dynamically stable solution with two central cores located at the same positions as the initial condition. The phase profile [see Fig. 11(d)] shows a value of $S = 5$ for the topological charge, as in the previous case. A third stable, composed structure is obtained by superposing again two vortex solitons with an initial center-to-center distance of three sites. The amplitude profile for the new dynamically stable solution has one horizontally elongated core, along two lattice sites, as shown in Fig. 11(e), and its topological charge has two different values as shown in Fig. 11(f). Indeed, the innermost discrete contour exhibits a charge $S = 6$, while the remaining contours have a charge $S = 10$.

Finally, we show another example of a composed structure that was obtained by combining four solutions belonging to the *E* family. We locate each *E* vortex with their central cores forming the vertices of a 8×8 square. We use this configuration as the initial condition for model (1) and find a dynamically stable stationary solution. Figures 12(a) and 12(b) show the amplitude and phase profiles for this composite solution. We observe a spatial amplitude distribution similar to the initial condition, where the four initial cores preserve their initial position and vorticity. In addition, an extra phase core appears at the lattice center ($n = m = 0$), around which an $S = -1$ topological charge is observed. If we consider a new contour enclosing all the sites with large amplitude,

FIG. 12. (Color online) Color map plots of the amplitude (a) and phase (b) profiles for a dynamically stable solution obtained by combining four solutions belonging to the *E* family.

the measured topological charge will be $S = 15$. This last superposition could be interpreted as four solutions evolving independently, and not as a bound state. If $\psi_{n,m}$ is a solution of Eq. (1), $\psi_{n,m} \exp(i\theta z)$ is a solution too. Accordingly, we will show that the superposition of four individual solutions, each one multiplied by a different phase, will always evolve to the same bound solution. In order to define this composed solution as a bound state, a phase-locking mechanism is necessary. Thus, to study this issue we use the following initial conditions:

$$\psi_{m,n}(0) = \tilde{\psi}_1 e^{i\theta_1} + \tilde{\psi}_2 e^{i\theta_2} + \tilde{\psi}_3 e^{i\theta_3} + \tilde{\psi}_4 e^{i\theta_4}. \quad (7)$$

Here, $\tilde{\psi}_i$ is a solution of the *E* family centered in a given position, and θ_i corresponds to the phase applied to each *E* solution, chosen from the interval $[0, 2\pi]$.

If we measure the phase difference between two equivalent points of the array it will converge to a constant value while the beam is propagating. Figures 13(a)–13(d) show how for different initial conditions the phase locks, leading to the formation of the bound solution depicted in Fig. 12. All the different initial conditions converge to a same value of Q and to a phase difference equal to zero. In order to test this, we used the following initial values of the phases in Eq. (7): $\theta_1 = 0$, $\theta_2 \equiv \pi\alpha/20$ (for $\alpha = 1, 2, \dots, 40$), $\theta_3 = -\pi/8$, and $\theta_4 = -\pi/4$. We define $\Delta\theta_{1,i}(z) \equiv \theta_i(z) - \theta_1(z)$, which is computed between four points of the array (points enclosed by gray circles in Fig. 12). Figure 13 shows the phase differences $\Delta\theta_{1,2}$ (b), $\Delta\theta_{1,3}$ (c), and $\Delta\theta_{1,4}$ (d) versus propagation.

Is worth mentioning here that we studied the interaction between solitons from different families but did not observe the formation of any bound state. Each soliton propagates with its own propagation constant, and their phases are never locked.

V. SCHRÖDINGER LIMIT

Most of the present experiments with optical beams are performed under conditions that are close to the cubic conservative limit, so we are interested in knowing if all these previous dissipative structures can be observed also here. In this scenario, all the parameters of the CGLE are zero, i.e., $\nu = \mu = \varepsilon = \delta = 0$, and model (3) reduces to the discrete NLSE equation

$$-\lambda \psi_{m,n} + \hat{C} \psi_{m,n} + |\psi_{m,n}|^2 \psi_{m,n} = 0. \quad (8)$$

Then, as a first approximation, we assume that the previously found solutions (dissipative ones) also exist in the

FIG. 13. (Color online) Evolution of the power Q (a) and the phase differences $\Delta\theta_{1,i}$ (b)–(d).

Schrödinger limit. To solve Eq. (7), we took these dissipative bound states as new seeds for the iterative algorithm. After a few iterations, we found stationary solutions with the same distribution fields and constructed the corresponding families and their stability. We note that all the previously found solutions also exist in the Schrödinger limit, being stable only for high values of their optical power content. For example, from Figs. 6(b) and 8(b) we note that the attractors are located at around $Q \approx 380$. In the conservative cubic case, equivalent localized structures are stable only for $Q \gtrsim 800$ (see blue and black lines in Fig. 14).

In Fig. 14 we show five families of conservative stationary solutions of the discrete NLSE that are related to the reported dissipative solutions: the blue, black, red, green, and gray lines are related to the composed solutions displayed in Figs. 5(a) and 5(b), Figs. 9(d) and 9(e), Figs. 11(a) and 11(b), Figs. 11(c) and 11(d), and Figs. 11(e) and 11(f), respectively. The upper-

FIG. 14. (Color online) Q versus λ diagram for several solutions in the Schrödinger limit. The gray region represents the linear band.

left inset in Fig. 14 shows a magnified view of the region close to the linear band (gray zone). As usual for two-dimensional systems [22], each family tends to increase its power steeply after passing through the point of minimum optical power. On the other hand, the lower-right inset shows a zoom of the black and blue curves, which are almost indistinguishable due to the similarity of their spatial profiles.

VI. SUMMARY AND CONCLUSIONS

We have reported several families of discrete vortex solitons, characterized for having two topological charges simultaneously, and coexisting for the same set of parameters. By superposing two or more of these vortices, we have been able to produce, dynamically stable composite vortex solitons that are also endowed with multiple vorticity charges. Additionally, we have shown that these composite structures persist in the conservative limit, where they are stable for high values of their power content.

ACKNOWLEDGMENTS

C.M.C. and J.M.S.C. acknowledge support from the Ministerio de Ciencia e Innovación under contracts FIS2006-03376 and FIS2009-09895. R.A.V. and M.I.M. acknowledge support from Fondecyt Grants No. 1110142 and No. 1080374, Programa ICM P10-030-F, and Programa de Financiamiento Basal de Conicyt (FB0824/2008).

APPENDIX: STABILITY OF DISCRETE SOLUTIONS

We now focus on the stability analysis of the two-dimensional (2D) discrete CQGL equation (1). As we can see, this set of equations relates the wave function $\psi_{m,n}$ with its nearest neighbors ($\psi_{m\pm 1,n}$ and $\psi_{m,n\pm 1}$). Although this set of ordinary differential equations describes a system with a bidimensional geometry, it is possible to rewrite these equations in such a way that we use only one index to enumerate the sites of the optical array. The idea is to map the 2D problem into a one-dimensional one, while keeping the corresponding interactions.

1. Euclidean dimensionality reduction

Without loss of generality, we can consider that our optical array has a square symmetry, i.e., the indices n and m have the same domains,

$$\{(n,m) | n \text{ and } m \in (1,2,\dots,K)\},$$

where K is the horizontal and vertical size of the square lattice. With this in mind, we can reorganize the 2D optical array as only one linear chain (see Fig. 15). From this reorganization it is natural to think that the physical description of the original problem must be reformulated. In fact, all the couplings have been redistributed along the same dimension such that the ψ_k field is coupled two times to the left and two more to the right. Two of them come from the original left and right nearest-neighbor fields; the other two couplings correspond to the upper and lower neighbors, which have been relocated at K sites to the left and to the right from k th field. Having in

FIG. 15. (Color online) Transformation of a two-dimensional lattice into a linear chain, where all the original couplings are preserved.

mind the above description, Eq. (1) can now be rewritten as

$$i\dot{\psi}_k + C(\psi_{k+1} + \psi_{k-1} + \psi_{k+K} + \psi_{k-K}) + |\psi_k|^2 \psi_k + v|\psi_k|^4 \psi_k = i\delta\psi_k + i\varepsilon|\psi_k|^2 \psi_k + i\mu|\psi_k|^4 \psi_k, \quad (\text{A1})$$

where $k \in (1, 2, \dots, K^2)$.

2. Linear stability analysis

As it was mentioned in Sec. II, there exist stationary solutions for the discrete CQCGLE equation. Usually they can be written as

$$\psi_k(z) = \phi_k \exp(i\lambda z), \quad \text{where } \lambda \in \mathbb{R} \text{ and } \phi_k \in \mathbb{C}. \quad (\text{A2})$$

When the beam is propagating, small perturbations around the stationary solutions can grow exponentially, leading to the destruction of the vortex soliton. A stability analysis provides us with the means for establishing which solutions are linearly stable. Let us introduce a small perturbation $\tilde{\phi}$ to the localized stationary solution (A2):

$$\tilde{\psi}_k = [\phi_k + \tilde{\phi}_k(z)]e^{i\lambda z}, \quad \tilde{\phi}_k \in \mathbb{C}. \quad (\text{A3})$$

Considering that $\tilde{\phi}_k^p \rightarrow 0$ for $p \geq 2$, the nonlinear terms become

$$|\tilde{\psi}_k|^2 \tilde{\psi}_k = [\phi_k + \tilde{\phi}_k(z)][\phi_k^* + \tilde{\phi}_k^*(z)]^2 e^{i\lambda z} \approx (\phi_k^2 \tilde{\phi}_k^* + 2|\phi_k|^2 \tilde{\phi}_k + |\phi_k|^2 \phi_k) e^{i\lambda z} \quad (\text{A4})$$

and

$$|\tilde{\psi}_k|^4 \tilde{\psi}_k = [\phi_k + \tilde{\phi}_k(z)]^2 [\phi_k^* + \tilde{\phi}_k^*(z)]^3 e^{i\lambda z} \approx (2|\phi_k|^2 \phi_k^2 \tilde{\phi}_k^* + 3|\phi_k|^4 \tilde{\phi}_k + |\phi_k|^4 \phi_k) e^{i\lambda z}. \quad (\text{A5})$$

Then, after replacing (4), (A4), and (A5) into Eq. (A1) and then linearizing with respect to $\tilde{\phi}$, we obtain

$$i\dot{\tilde{\phi}}_k - i\delta\tilde{\phi}_k + C(\tilde{\phi}_{k+1} + \tilde{\phi}_{k-1} + \tilde{\phi}_{k+K} + \tilde{\phi}_{k-K}) - \lambda\tilde{\phi}_k - 2i\varepsilon|\phi_k|^2 \tilde{\phi}_k - 3i\mu|\phi_k|^4 \tilde{\phi}_k + 3v|\phi_k|^4 \tilde{\phi}_k + 2|\phi_k|^2 \tilde{\phi}_k - 2i\mu\phi_k^2 |\phi_k|^2 \tilde{\phi}_k^* + 2v\phi_k^2 |\phi_k|^2 \tilde{\phi}_k^* - i\varepsilon\phi_k^2 \tilde{\phi}_k^* + \phi_k^2 \tilde{\phi}_k^* = 0, \quad (\text{A6})$$

and factorizing the perturbation function in (A6), we have

$$i\dot{\tilde{\phi}}_k + \hat{C}\tilde{\phi}_k - i\delta\tilde{\phi}_k + [2(1-i\varepsilon)|\phi_k|^2 + 3(v-i\mu)|\phi_k|^4 - \lambda]\tilde{\phi}_k + [(1-i\varepsilon)\phi_k^2 + 2(v-i\mu)|\phi_k|^2 \phi_k^2]\tilde{\phi}_k^* = 0. \quad (\text{A7})$$

The previous equation describes how the perturbation evolves in the presence of a stationary solution. In general terms, this kind of differential equation has solutions that can be written as linear combinations of exponential functions, where their arguments determine when the perturbation grows or remains bounded during its evolution. To solve the equations (5) we first separate the real and imaginary part of the solution and its perturbation:

$$\phi_k = u_k + iv_k, \quad \tilde{\phi}_k(z) = x_k(z) + iy_k(z), \quad (\text{A8})$$

where u, v, x , and $y \in \mathbb{R}$. By replacing (A8) in (A7) and separating real and imaginary parts, we obtain two ordinary differential equations, namely,

$$\begin{aligned} \dot{x}_k + \hat{C}y_k + h_1y_k + h_2x_k &= 0, \\ \dot{y}_k - \hat{C}x_k + h_3x_k + h_4y_k &= 0, \end{aligned} \quad (\text{A9})$$

where the h factors are given by

$$\begin{aligned} h_1 &= -2uv - \delta - u^2\varepsilon - 3v^2\varepsilon - u^4\mu - 6u^2v^2\mu \\ &\quad - 5v^4\mu - 4u^3v\nu - 4uv^3\nu, \\ h_2 &= -3u^2 - v^2 - 2uv\varepsilon + \lambda - 4u^3v\mu - 4uv^3\mu \\ &\quad - 5u^4\nu - 6u^2v^2\nu - v^4\nu, \\ h_3 &= 2uv - \delta - 3u^2\varepsilon - v^2\varepsilon - 5u^4\mu - 6u^2v^2\mu \\ &\quad - v^4\mu + 4u^3v\nu + 4uv^3\nu, \\ h_4 &= u^2 + 3v^2 - 2uv\varepsilon - \lambda - 4u^3v\mu - 4uv^3\mu \\ &\quad + u^4\nu + 6u^2v^2\nu + 5v^4\nu. \end{aligned} \quad (\text{A10})$$

We define the following four matrices:

$$\begin{aligned} \mathbf{A} &= C(\delta_{k+1,l} + \delta_{k-1,l}) + h_1\delta_{k,l}, \quad \mathbf{B} = h_2\delta_{k,l}, \\ \mathbf{C} &= -C(\delta_{k+1,l} + \delta_{k-1,l}) + h_3\delta_{k,l}, \quad \mathbf{D} = h_4\delta_{k,l}, \end{aligned}$$

where $\delta_{k,l}$ is the Kronecker symbol. The system (A9) can be expressed then as

$$\dot{\vec{x}} + \mathbf{A}\vec{y} + \mathbf{B}\vec{x} = 0, \quad \dot{\vec{y}} + \mathbf{C}\vec{x} + \mathbf{D}\vec{y} = 0,$$

or in matricial form,

$$\dot{\mathbf{R}} = -\begin{pmatrix} \mathbf{B} & \mathbf{A} \\ \mathbf{D} & \mathbf{C} \end{pmatrix} \mathbf{R}, \quad \text{where } \mathbf{R} = \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}. \quad (\text{A11})$$

The matrix equation (A11) corresponds to a linear and homogeneous system of ordinary differential equations, for which its solutions can be written as

$$\mathbf{R} = \begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_N \end{pmatrix} e^{\gamma z} = \mathbf{L}e^{\gamma z}, \quad (\text{A12})$$

with $N = 2K^2$. If we replace the ansatz (A12) in (A11) the system transforms into

$$\mathbf{L}\gamma e^{\gamma z} = \mathbf{H}\mathbf{L}e^{\gamma z}, \quad \text{where } \mathbf{H} = -\begin{pmatrix} \mathbf{B} & \mathbf{A} \\ \mathbf{D} & \mathbf{C} \end{pmatrix}. \quad (\text{A13})$$

We can rewrite this equation as

$$(\mathbf{H} - \gamma \mathbf{I})\mathbf{L} = \mathbf{0}, \quad (\text{A14})$$

which is equivalent to the algebraic system

$$\begin{aligned} (h_{11} - \gamma)l_1 + h_{12}l_2 + \cdots + h_{1N}l_N &= 0 \\ h_{21}l_1 + (h_{22} - \gamma)l_2 + \cdots + h_{2N}l_N &= 0 \\ h_{N1}l_1 + h_{N2}l_2 + \cdots + (h_{NN} - \gamma)h_{NN}l_N &= 0. \end{aligned} \quad (\text{A15})$$

Thus, to determine a nontrivial solution $\mathbf{L}$ of the system (A11), it is necessary that

$$\det(\mathbf{H} - \gamma \mathbf{I}) = 0. \quad (\text{A16})$$

This is the characteristic equation of the $\mathbf{H}$ matrix; in other words, $\mathbf{R} = \mathbf{L}e^{\gamma z}$ will be a solution of the system (A11) if and only if γ is an eigenvalue of $\mathbf{H}$, and $\mathbf{L}$ is an eigenvector

corresponding to γ . The general solution of this system is

$$\mathbf{R} = c_1 \mathbf{L}_1 e^{\gamma_1 z} + c_2 \mathbf{L}_2 e^{\gamma_2 z} \cdots + c_N \mathbf{L}_N e^{\gamma_N z}, \quad (\text{A17})$$

$c_1, c_2, \dots$ being integration constants. The set $\{\gamma_1, \gamma_2, \dots, \gamma_N\}$ is the spectrum of the *eigensystem* associated with (A11). If at least one of the eigenvalues is a complex number, then $\mathbf{R}^*$ is also a solution of system (A11).

Here, we have to remember that the vector $\mathbf{R}$ contains all the components of the perturbation function $\tilde{\phi}_k$. The stability of the localized structure ϕ_k is determined by the discrete spectrum of the eigenvalues of (A16) with a nonzero real part. More precisely, a localized structure is unstable if $\Re(\gamma) > 0$, where the maximum is chosen among all the roots of (A16). If at least one of the eigenvalues has a real part greater than zero, the perturbation $\tilde{\phi}_k$ grows exponentially, leading to an unstable evolution of the stationary solution profile.

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