

Endogenous Growth and Wave-Like Business Fluctuations*

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Abstract

Schumpeter stated that “wave-like fluctuations in business...are the form economic development takes in the era of capitalism.” This paper argues that observed long lags in the implementation of innovations make modern economies to behave consistently with Schumpeter’s statement. In a simple endogenous growth model with implementation delays, the paper finds that: First, the equilibrium path admits a Hopf bifurcation where consumption, R&D and output permanently fluctuate. Innovations arrive en masse, moving the economy to a boom; the associated increase in purchasing power all over the business sphere induces research activities to flourish again; but, innovations will take a while to develop; when the new wave of innovations is eventually implemented, new products enter the market producing a second boom; a third will follow, then a fourth and so on and so forth. Second, this mechanism is quantitatively consistent with US aggregate data. Finally, a procyclical R&D subsidy rate moving around 10% and designed to half consumption fluctuations increases the growth rate from 2.4% to 3.4% with a 9.6% increase in welfare, 6.3% of the welfare gains due to consumption smoothing.

JEL Classification O3, E32

Keywords Endogenous growth; endogenous fluctuations; innovation implementation; time delays; medium term cycles; Hopf bifurcation

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1 Introduction

The conjecture that in the modern era business fluctuations and economic growth are two faces of the same coin comes back to Schumpeter [49], who pointed out that “wave-like fluctuations in business...are the form economic development takes in the era of capitalism.” Starting from this premise, Schumpeter raised the key question of “why is it that economic development does not proceed evenly..., but as it were jerkily; why does it display those characteristics ups and downs?” When searching for an answer, he drew attention to the critical fact that innovations “appear en masse at intervals”, “discontinuously in groups or swarms,” which “signifies a very substantial increase in purchasing power all over the business sphere.”

Following the seminal work by Aghion and Howitt [2], Grossman and Helpman [35] and Romer [47], important developments have been undertaken in the last twenty years addressed to improve our understanding on the main channels through which innovations promote development and growth. Endogenous growth theory is in a fundamental sense Schumpeterian, since it stresses the critical role played by innovations in the observed growth of total factor productivity. However, little has been written since then on the relation between innovation and business fluctuations.

A natural candidate for the study of Schumpeterian wave-like business fluctuations is the observed long delay elapsed between the realization of R&D activities and the implementation and adoption of the associated innovations.¹ Schumpeter [49]’s description of the periodicity of business fluctuations is, in this sense, very appealing: “the boom ends and the depression begins after the passage of the time which must elapse before the products of the new enterprise can appear on the market.” The argument in this paper is very close to Schumpeter’s description: waves of innovations arrive en masse, moving the economy to a boom; the associated increase in productivity raises purchasing power all over the business sphere, inducing research activities to flourish; but, the new products will take a while to develop; when the new wave of innovations is eventually implemented, the new products enter the market producing a second boom, which will generate a third, then a fourth and so on and so for.

It is important to notice that Schumpeterian wave-like business fluctuations as described in the previous paragraph substantially differ from the type of fluctuations studied in modern business cycle literature. Inspired on Kydland and Prescott [41], it has focused on the study of high frequency movements, those between 4 and 40 quarters. Schumpeter, indeed, was more interested in medium (Juglar) and low (Kondratieff) frequency movements lasting around 10 and 50 years, respectively. A description of economic fluctuations more in accordance with the Schumpeterian’s view was recently suggested by Comin and Gertler [20]. They estimate the medium term movements of US per capita GDP growth by analyzing frequencies between 40 and 200 quarters, and find that it permanently undulates with a periodicity of around 11 years and an amplitude of around 8 percentage points from pick to valley. This paper focuses on

¹Comin and Hobijn [22] study the pattern of technology diffusion around the globe and find that countries on average adopt technologies 47 years after their invention. Comin et al [23] find that, when compared to the US, lags in the use of technology are measured in decades for most countries. Adams [1] estimates that academic knowledge is a mayor contributor to productivity growth, but its effects lag roughly 20 years. Mansfield [42] estimates the mean adoption delay of twelve mayor 20th-century innovations in 8 years. Jovanovic and Lach [36] estimate at 8.1% the annual diffusion rate of new products.

Juglar cycles or, equivalently, on medium terms movements.

In this paper, Schumpeter’s wave-like fluctuations are modeled in a simple way by adding an implementation delay to an otherwise standard endogenous growth model with expanding product variety –see Romer [47]. The paper shows that the equilibrium path admits a Hopf bifurcation where consumption, research and output permanently fluctuate. The main mechanism relating growth to wave-like fluctuations is based on the assumption that innovations being fundamental for economic growth require long implementation and adoption lags. The mechanics is the following. Let say that the economy initially reacts by some concentration of research activities, which makes new ideas to appear en masse. This is the standard reaction of a dynamic general equilibrium model when the initial stock of (technological) capital is relatively low. However, the economic effects of this wave of research activity will be delayed in time. When a swarm of new businesses will become eventually operative, the associated increase in productivity will inject additional resources to the economy –“a substantial increase in purchasing power” in Schumpeter’s words. Consumption smoothing makes the rest, by allocating the additional resources to create a second wave of innovations. This process will repeat again and again as time passes. A simple quantitative exercise is undertaken by calibrating the model to some US aggregates. The paper finds that under this calibration, the model shows permanent cycles of the observed pattern. In this sense, the suggested mechanism relating the sources of growth and business fluctuations is not only theoretically possible but quantitatively relevant.

Additionally, the paper makes some welfare considerations. Firstly, it shows that detrended consumption is constant from the initial time in an optimal allocation, and both R&D and output converge by oscillations. Second, it proves that a procyclical subsidy/tax scheme would restore optimality. Finally, it quantitatively find that a procyclical 10% subsidy rate halving consumption fluctuations will increase the growth rate from 2.4% to 3.4% with a 9.6% increase in welfare, 6.3% due to consumption smoothing.

The model in this paper belongs to the literature on dynamic general equilibrium with time delays, including time-to-build and vintage capital theories. Firstly, fluctuations in the vintage capital literature are the result of machine replacement, as described in Benhabib and Rustichini [13], Boucekkine et al [16] and Caballero and Hammour [18].² Following the lumpy investment literature, initiated by Doms and Dunne [28], Cooper et al [24] find robust evidence on the existence of machine replacement, but little support for the contribution of machine replacement to the understanding of observed business fluctuations. Second, since the seminal paper by Kydland and Prescott [41], investment lags have been shown to make the business cycle highly persistent. Asea and Zak [3] and, more recently, Bambi [4] go further and prove that time-to-build may generate endogenous fluctuations. However, time-to-build delays are short relative to Juglar cycles, since they last some few quarters only. These observations make implementation delays a more appealing object to the understanding of Schumpeterian business fluctuations than vintage capital or time-to-build arguments.

There is an extensive literature on endogenous competitive equilibrium cycles in discrete time economies, along the seminal contributions of Benhabib and Nishimura [11] and Grandmont [34]. Benhabib and Nishimura [12] relate optimal cycles to the existence of a Hopf bifurcation in continuous time multisector growth models. Furthermore, our policy implications goes in the

²See also Boucekkine et al [15] and Boucekkine and de la Croix [14].

same direction of those found, even if in a different context, by Grandmont [34]; more precisely the policy designed to restore efficiency reduces the persistence of the business cycle and then make our economy to converge by damping fluctuations toward the balanced growth path.

This paper is also related to Matsuyama [43] and Francois and Lloyd-Ellis [32], among the few exceptions connecting endogenous growth with cycles. Firstly, Matsuyama [43] shows that, under some conditions regarding the saving rate, endogenous cycles arrive in a discrete time Rivera-Batiz and Romer [46] endogenous growth model, where monopoly rents last only one period and implementing an innovation entails fixed costs. Along the cycle, the economy moves periodically from a Neoclassical regime to an AK regime. Research activities come en masse as in Schumpeter’s theory, but, contrary to the empirical evidence, they are counter-cyclical. In our theory, indeed, R&D moves pro-cyclically.³ Second, Francois and Lloyd-Ellis [32] link growth and cycles combining animal spirits, such as in Schleifer [48], to a Schumpeterian endogenous growth model. In their framework, a cyclical equilibrium exists because firms are interested in delaying implementation to the boom in order to maximize the expected length of incumbency. In our model, cycles are also related to implementation delays too, but they are not the consequence of animal spirits but result from a Hopf bifurcation.

The idea that delayed gains in productivity may generate persistence has being deeply studied in the recent literature on “news shocks” –see Beaudry and Portier [8].⁴ However, the main source of fluctuations in this literature remains exogenous. In our theory, indeed, current research activities and the associated future innovations may be seen as perfectly forecasted, endogenous news shocks. Endogenous news are at the basis of the the cyclical behavior of our economy, since more resources are allocated to produce current news when past news realize.

Our adoption delay are indeed very different from the delay elapsing between the arrival of a general purpose technology (GPT) and its implementation. In fact, GPT refers to a major technology breakthrough, as for example the discovery of the electric dynamo, whose implementation requires costly and very long restructuring. According to David [25], the implementation of a new GPT may generally take several decades: the electric dynamos takes for example three decades to attain a fifty percent diffusion level in the U.S.. Then the consequences of a discovery of a GPT may well reproduce the low (Kondratieff) frequency movements in the data but not the medium ones which are the objective of our analysis.

Finally, this paper shares with Comin and Gertler [20]’s the view that lags of technology adoption do generate medium-term movements in models of endogenous productivity growth. In Comin and Gertler’s view, medium-term movements “reflect a persistent response of economic activity to the high-frequency fluctuations normally associated with the cycle.” In our theory, indeed, medium-term movements are self-sustained.

The paper is organized as follows. Section 2 describes the decentralized economy and studies its main dynamic properties. In particular, it shows the existence of a Hopf bifurcation. Finally, it quantitatively studies its empirical relevance. Section 3 analyses optimal allocations and suggests a procyclical R&D subsidy as a Pareto improving policy. A counterfactual exercise is performed showing that a 10% R&D procyclical subsidy halving consumption fluctuations

³The empirical countercyclical behavior of R&D is reported in Geroski and Walters [33], Fatas [31] and Walde and Woitek [50], among others.

⁴More recently, Comin et al [21] stress the importance of endogenous adoption in the amplification of these shocks.

generates first order welfare gains.

2 The decentralized economy

The economy is populated by a continuum of infinitely lived, identical households of unit measure, holding a constant flow endowment of one unit of labor. There is a sole final good, used for consumption purposes only. Household *preferences* are represented by:

$$U = \int_0^\infty \log(c_t) e^{-\rho t} dt, \quad (1)$$

where c_t is per capita consumption and $\rho > 0$ represents the subjective discount rate.

In line with the literature on expanding product variety, see Romer [47], the final consumption good is produced by a CES technology defined on a continuum of intermediary inputs in the support $[0, n]$. As usual, the extend of product variety n represents also the aggregate state of knowledge. Knowledge positively affects the productivity of the consumption sector as an externality, meaning that n has a positive effect on the production of the consumption good. Differently from the existing literature, we assume that adopting new technologies requires a time delay $d > 0$, meaning that varieties discovered at time t become operative at time $t + d$. It can be interpreted as an *adoption delay* which elapses from the discovery of a new variety to its economic implementation. Then the consumption good technology is

$$c_t = n_{t-d}^{v+1-\frac{1}{\alpha}} \left(\int_0^{n_{t-d}} x_t(j)^\alpha dj \right)^{\frac{1}{\alpha}}, \quad 0 < \alpha < 1 \quad (2)$$

where n_{t-d} represents the extend of operative varieties at time t , and $x_t(j)$ is the amount of the intermediary input j used at time t in the production of c_t . This consumption good technology implies a constant (and equal) elasticity of substitution between every pair of varieties, $\theta = \frac{1}{1-\alpha} > 1$. The parameter v is the elasticity of the externality n , but also the return to specialization as explained extensively in Ethier [30] and Benassy [10]; from now on we assume $v = 1$ to simplify our analysis and at the same time to distinguish between the markup charged by the monopolistic firms producing $x(j)$ and the degree of returns to specialization.⁵ The assumption that the externality operates only through the measure of operative varieties n_{t-d} is consistent with the *love variety* argument as suggested by Dixit and Stiglitz [27].

Technology in the intermediary sector is assumed to be symmetric across varieties

$$x_t(j) = l_t(j), \quad (3)$$

where $l_t(j)$ is labor allocated to the production of variety j . Total labor L allocated to the production of the intermediary sector is given by

$$\int_0^{n_{t-d}} x_t(j) dj = L_t. \quad (4)$$

⁵The main qualitative properties of the Romer's model do not depend on the elasticity being unity –see Benassy [10]. However, when the adoption delay is strictly positive, a unit elasticity allows for a mathematical study of the main properties of the model, which would not be the case otherwise.

An efficient allocation of labor to the production of the consumption good, spreading through the intermediary sector, results from maximizing (2) subject to (4). It is easy to see that an efficient allocation is symmetric, meaning $x_t(j) = x_t$ for all j , which implies

$$c_t = n_{t-d}L_t \quad \text{and} \quad n_{t-d}x_t = L_t. \quad (5)$$

As stated above, labor allocated to the production of the consumption good benefits from a knowledge externality, n , which comes linearly in the reduced form of the consumption goods technology (5). In the following sections, we show that optimal and equilibrium allocations are both efficient in the sense defined above –see Koeninger and Licandro [38].

Finally, *R&D* activities are also assumed to be linear on labor and addressed to the creation of new intermediary inputs. The innovation technology creating these new varieties is assumed to be:

$$\dot{n}_t = An_{t-d}(1 - L_t), \quad (6)$$

where $1 - L_t$ is labor assigned to R&D production, its marginal productivity depending on parameter A , $A > 0$. It is also assumed that the R&D sector benefits from a positive externality depending linearly on the extend of operative varieties.

Note that consumption and R&D technologies, (5) and (6) respectively, collapse to

$$\dot{n}_t = A(n_{t-d} - c_t). \quad (7)$$

The AK structure of the model, see Rebelo [45], can be easily seen if the extend of product variety n_{t-d} is interpreted as (intangible) capital. In the following, we will refer to (7) as the feasibility constraint.⁶

2.1 Decentralized equilibrium

The economy is decentralized as in Romer [47]. The market for the final consumption good is supposed to be perfectly competitive, so that individuals and firms take the consumption price, normalized to unity, as given. Innovations are protected by an infinitely lived patent and the market for intermediary inputs is monopolistically competitive. The R&D sector is perfectly competitive, implying that research firms make zero profits. Finally, the labor market is also assumed to be perfectly competitive. In the following, the key equations are presented while their derivation can be found in the Appendix.

A representative firm produces the consumption good by the mean of technology (2). It takes intermediary prices as given and maximizes profits by choosing $x_t(j)$ for $j \in [0, n_{t-d}]$, which results on the inverse demands function

$$p_t(j) = n_{t-d}^{2\alpha-1} \left(\frac{c_t}{x_t(j)} \right)^{1-\alpha} \quad (8)$$

with $p(j)$ the relative price of the intermediate good j . Consequently, the intermediaries operating under monopolistic competition, and facing the inverse demand function (8), maximizes

⁶Equivalently, it can be assumed that labor is only used to the production of goods, and output is assigned to both consumption and R&D, with L representing the consumption to output ratio and A the rate at which the consumption good is transformed into innovations.

their profits by setting the following price rule

$$p_t(j) = \frac{1}{\alpha} w_t, \quad (9)$$

where w is the marginal cost of production (technology is linear in labor), and $\frac{1}{\alpha}$ represents the markup over marginal costs, which depends inversely on the elasticity of substitution across varieties. The equilibrium is then symmetric, meaning that (5) holds, and equation (8) becomes

$$p_t = n_{t-d}. \quad (10)$$

Recall that the consumption good is the numeraire, which implies that p_t is the price of the intermediary input relative to the price of consumption. An expansion in product variety improves productivity in the consumption sector, inducing an increase in the relative price of the intermediary input as reflected by (10).

From (5), (9) and (10), intermediary profits can be written as

$$\pi_t = (1 - \alpha) \frac{c_t}{n_{t-d}} > 0. \quad (11)$$

Profits are proportional to total sales per firm, the proportionally factor being directly related to the markup rate.

By assumption, the inventor of a new variety receives a patent of infinite life, which can be sold in the market for patents at the price v_t . Given the R&D technology (6), a new variety costs $\frac{w_t}{A n_{t-d}}$. From equations (10) and (9), the free entry condition implies

$$v_t = \frac{\alpha}{A}, \quad (12)$$

which is constant at equilibrium.

Finally, let us solve the representative household problem.

$$\max \int_0^{\infty} \log(c_t) e^{-\rho t} dt$$

subject to the instantaneous budget constraint

$$\dot{n}_t = \frac{1}{v} (\pi_t n_{t-d} + w_t - c_t)$$

and the initial condition $n_t = \bar{n}_t$, for $t \in [-d, 0]$, where \bar{n}_t is a known continuous function defined on the t domain. At equilibrium, patents are the only asset households may hold, paying dividends $\pi_t n_{t-d}$ at time t . Non consumed income is then saved in the form of new patents, priced v . In the following, it is assumed that the solution is interior, meaning $\dot{n}_t \geq 0$.

The households problem is an optimal control problem with delays, which can be solved following the optimal control theory in Kolmanovskii and Myshkis [39]. The first order conditions are

$$\frac{v e^{-\rho t}}{c_t} = \mu_t$$

$$\frac{\dot{\mu}_t}{\mu_t} = -\frac{\pi_{t+d}}{v} \frac{\mu_{t+d}}{\mu_t},$$

and the transversality condition

$$\lim_{t \rightarrow \infty} n_t c_t^{-1} e^{-\rho t} = 0, \quad (13)$$

where c_t is a control, n_{t-d} is a delayed state and μ_t the associated costate. The representative household faces the following trade-off, consuming at time t or buying new patents which will become operative at time $t + d$. The return of a new patent $\frac{\pi_{t+d}}{v}$ has to be then discounted by the mean of the discount factor $\frac{\mu_{t+d}}{\mu_t}$.

After substituting equilibrium profits from (11), the two optimal conditions collapse into the following Euler-type equation

$$\frac{\dot{c}_t}{c_t} = \underbrace{\frac{1 - \alpha}{\alpha} \frac{c_{t+d}}{n_t} A}_{\text{private R\&D return}} \underbrace{e^{-\rho d} \left(\frac{c_t}{c_{t+d}} \right)}_{\text{discount factor}} - \rho = \frac{1 - \alpha}{\alpha} A e^{-\rho d} \frac{c_t}{n_t} - \rho. \quad (14)$$

The private return to R&D, π/v , arrives after a period of length d . For this reason, it has to be discounted using the appropriate ratio of marginal utilities. Moreover, the private return to R&D is different from the social return, which is equal to A . Under log utility, the term in c_{t+d} cancels and the Euler equation does not depend on it, but on the state n_t .

Equilibrium is then a path (c_t, n_t) , for $t \geq 0$, verifying the feasibility condition (7), the Euler equation (14), the initial condition $n_t = \bar{n}_t, \forall t \in [-d, 0]$, the transversality condition (13) and the irreversibility constraint $\dot{n}_t \geq 0$.

2.2 Balanced growth and transitional dynamics

At a balanced growth path, from (14), the consumption to knowledge ratio is

$$\frac{c_t}{n_t} = \frac{\alpha(g_e + \rho)e^{\rho d}}{(1 - \alpha)A}, \quad (15)$$

where g_e is the growth rate of both c and n . Substituting this expression into (7), we obtain

$$Ae^{-g_e d} - g_e = \frac{\alpha(g_e + \rho)e^{\rho d}}{1 - \alpha}. \quad (16)$$

It is easy to show that a strictly positive growth rate g_e exists and is unique under the following parametric conditions:

$$A > \frac{\alpha \rho e^{\rho d}}{1 - \alpha} \equiv A_{\min}^e. \quad (17)$$

A straightforward application of the implicit function theorem on (16) shows that $\frac{\partial g_e}{\partial A} > 0$ and $\frac{\partial g_e}{\partial \alpha} < 0$, implying that both more productive economies and economies with larger markups grow faster.

In order to proceed with the stability analysis, let us define $\tilde{x}_t = x_t e^{-g_e t}$, $x_t = \{c_t, n_t\}$, with \tilde{c}_t, \tilde{n}_t representing detrended consumption and detrended knowledge stock, respectively. Equations (14) and (7) then become

$$\frac{\dot{\tilde{c}}_t}{\tilde{c}_t} = \frac{1 - \alpha}{\alpha} A e^{-\rho d} \frac{\tilde{c}_t}{\tilde{n}_t} - (\rho + g_e) \quad (18)$$

$$\dot{\tilde{n}}_t = A(\tilde{n}_{t-d} e^{-g_e d} - \tilde{c}_t) - g_e \tilde{n}_t. \quad (19)$$

By linearizing the Euler equation (18) around the steady state and using (15), we get

$$\dot{\tilde{c}}_t = (g_e + \rho)\tilde{c}_t - \frac{(g_e + \rho)^2 \alpha e^{\rho d}}{A(1 - \alpha)} \tilde{n}_t. \quad (20)$$

Existence and uniqueness of a continuous solution for the system of delay differential equations (19)-(20) is guaranteed by Theorem 6.1 page 167 and Theorem 6.2 page 171 in Bellman and Cooke [9]. It is worth noting that our detrending generates a spurious zero root (eigenvalue) which does not, consequently, play any role in the asymptotic behaviour of the detrended system. The linearized system (20)-(19) is a good approximation of the original one (18)-(19), provided that all the other roots of its characteristic equation –(23) below– have no zero real part (Bellman and Cooke [9], pages 337-392, or more recently Diekmann et al [26]).⁷ The Laplace transform solution and its series expansion is in Proposition 1.

Proposition 1 *The series expansion of the Laplace transform solution of (19)-(20) is*

$$\tilde{n}_t = \sum_{r=0}^{+\infty} p_r e^{\lambda_r t} \quad (21)$$

$$\tilde{c}_t = \frac{1}{A} \sum_{r=0}^{+\infty} (Ae^{-(g_e + \lambda_r)d} - g_e - \lambda_r) p_r e^{\lambda_r t} \quad (22)$$

where $\{\lambda_r\}_{r=0}^{+\infty}$ are the roots of the characteristic equation:

$$h(\lambda) = \lambda^2 - \rho\lambda - \lambda Ae^{-(g_e + \lambda)d} + A(g_e + \rho)e^{-(g_e + \lambda)d} - A(g_e + \rho)e^{-g_e d} \quad (23)$$

and $\{p_r\}_{r=0}^{+\infty}$ the residues:

$$p_r = \frac{\dot{\tilde{n}}_0 + \tilde{n}_0 (\lambda_r - \rho - Ae^{-(g_e + \lambda_r)d}) + Ae^{-(g_e + \lambda_r)d} \int_{-d}^0 [\dot{\tilde{n}}_t - (g_e + \rho)\tilde{n}_t] e^{-\lambda_r t} dt}{h'(\lambda_r)} \quad (24)$$

with $\dot{\tilde{n}}_t = \frac{d}{dt}(\tilde{n}_t e^{-g_e t})$ for $t \in [-d, 0)$, and $\dot{\tilde{n}}_0 = A(\tilde{n}_{-d} e^{-g_e d} - \tilde{c}_0) - g_e \tilde{n}_0$.

Proof. See Appendix. ■

In order to study the stability properties of the solution, we need information about the spectrum of roots of the characteristic equation (23). For a given delay d and A sufficiently close to A_{\min}^e by its right, let us define the D-Subdivision D_i as a set in the space (ρ, α) , $\rho > 0$ and $\alpha \in (0, 1)$, such that the characteristic equation (23) has i and only i roots with strictly positive real part. The assumption “ A sufficiently close to A_{\min}^e by its right” corresponds to situations where the growth rate is positive but small. Remind that from (17), A_{\min}^e is a function of α, ρ and d . Figure 1 divides the space (ρ, α) in D-Subdivisions. The curve separating the D-Subdivision D_1 from the D-Subdivision D_3 corresponds to a parameters configuration where the spectrum has a pair of purely imaginary roots. Then, for continuous variation of the two parameters (ρ, α) crossing this curve the number of roots with positive real part changes from one to three since a couple of conjugate roots passes through the imaginary axis. This feature will be critical for the rising of permanent cycles. Figure 2 shows how this curve separating D_1 and D_3 , moves when the delay d increases. As it can be seen, it moves to the left making permanent cycles more plausible for smaller values of ρ and α .

⁷A similar local stability analysis of a functional differential equation around the balanced growth path can be found in Boucekkine and Pintus [17].

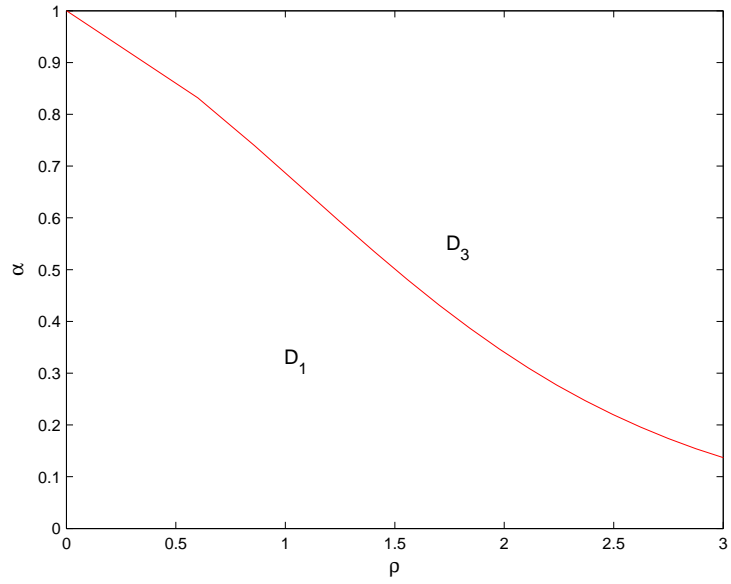


Figure 1: D-Subdivision of $h(\lambda)$ in the parameters space (ρ, α) when $A \simeq A_e$.

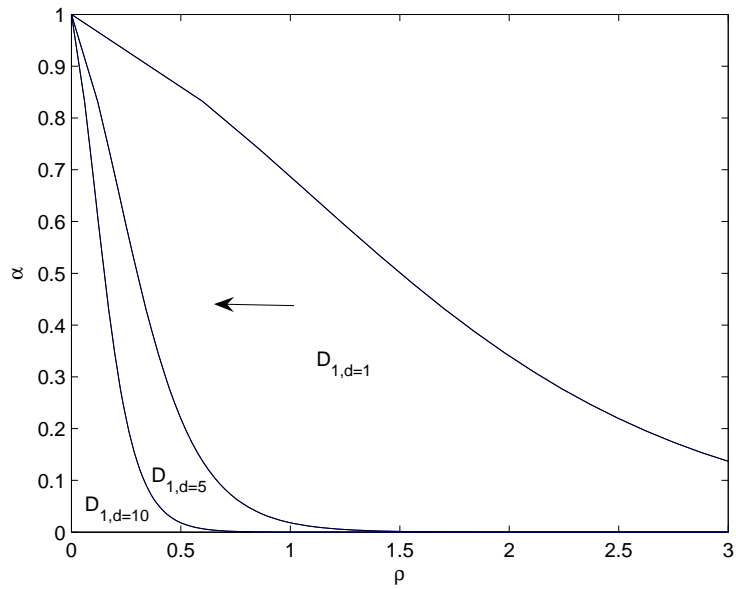


Figure 2: D_1 regions for different values of the parameter d , with $d_1 < d_2 < d_3$.

Proposition 2 For any admissible choice of parameters, the characteristic equation (23)

- has a spurious zero root, λ_1 , and a positive real root, λ_0 ;
- when A is sufficiently close to $(A_{r_{min}}^e)^+$, subdivisions D_1, D_3 are non empty.

Proof. See Appendix. ■

Proposition 2 shows the two fundamental properties of the model. Firstly, as usual in endogenous growth models with one state variable, when parameters belong to the D-Subdivision D_1 the spectrum has one and only one strictly positive real root. Local stability is proved in the proposition below by using the transversality condition to rule out this root. Second, Proposition 2 shows that permanent cycles may arise in endogenous growth models with adoption delays through a Hopf bifurcation. It is the case when parameters belong to the frontier between regions D_1 and D_3 –see Figure 1– where two complex roots cross the imaginary axes. In this case, the solution has two pure imaginary roots showing a permanent cycle (see Diekmann et al [26]).⁸ It is in this last sense that our results are in line with Schumpeter’s statement that “wave-like fluctuations in business are the form economic development takes in the era of capitalism.”

Proposition 3 Let us assume parameters belong to the D-Subdivision D_1 , then the equilibrium paths n_t, c_t follow

$$n_t = p_1 e^{g_e t} + \sum_{r=2}^{+\infty} p_r e^{(g_e + \lambda_r) t} \quad (25)$$

$$c_t = \frac{1}{A} \left[\left(A e^{-g_e d} - g_e \right) p_1 e^{g_e t} + \sum_{r=2}^{+\infty} \left(A e^{-(g_e + \lambda_r) d} - g_e - \lambda_r \right) p_r e^{(g_e + \lambda_r) t} \right] \quad (26)$$

with

$$c_0 = \tilde{n}_{-d} e^{-g_e d} + \frac{\tilde{n}_0}{A} \left(-g_e + \lambda_0 - \rho - A e^{-(g_e + \lambda_0) d} \right) + e^{-(g_e + \lambda_0) d} \int_{-d}^0 [\dot{\tilde{n}}_t - (g_e + \rho) \tilde{n}_t] e^{-\lambda_0 t} dt. \quad (27)$$

Proof. See Appendix. ■

Under log utility, consumption is expected to depend linearly on wealth. This is implicit in equation (27), where the left hand side implicitly defines initial wealth as an equilibrium valuation of the flow of past innovation activities. When the economy is in the D-Subdivision D_1 , the equilibrium path is unique and both n_t and c_t converge to the balanced growth path by damping oscillations.

2.3 Quantitative analysis and medium-term movements

In this section, we undertake a quantitative exercise to show that the conditions required for our economy to be on a permanent cycle equilibrium are quantitatively sensible. For this purpose, we calibrate the model to the US economy by setting the following parameters values:

$$d = 8.2, \quad \rho = 0.03, \quad \alpha = 0.9 \quad \text{and} \quad A = 0.786.$$

⁸It has been assumed that $\dot{n} \geq 0$, otherwise negative labor should be allocated to R&D, which is not feasible. Since we were not able to exclude in general that oscillations require negative innovation activities, we systematically check for this condition in our numerical exercises.

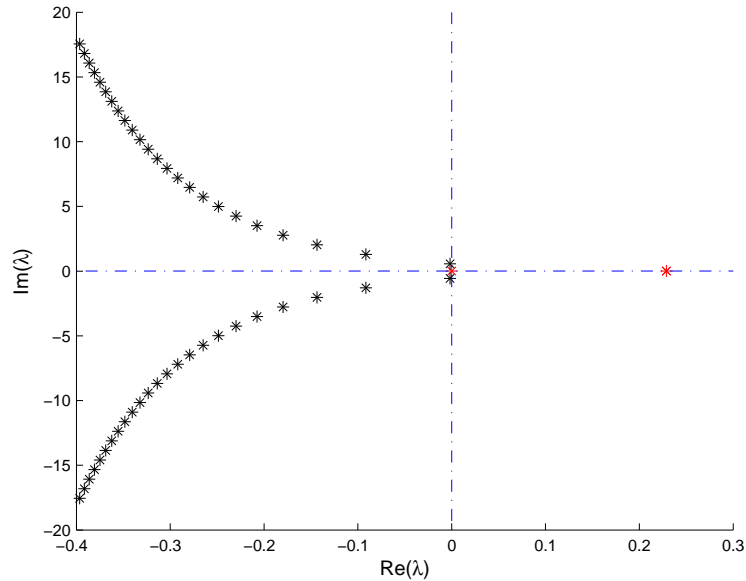


Figure 3: Spectrum of Roots.

The adopted value of d is consistent with Mansfield's estimations, and $\alpha = .9$ is in line with estimated markups in Basu and Fernald [7], implying a markup rate of 11%. Parameters A and ρ were chosen for the growth rate $g_e = 2.4\%$ as in Comin and Gertler [20] and the economy be in D-Subdivision D_1 , but close to its admissible border.

We use the software DDE-BIFTOOL developed by Engelborghs and Roose [29] to compute the subset of the rightmost roots of the characteristic equations (23) corresponding to the equilibrium allocation. The spectrum of roots is represented in Figure 3. As stated in Proposition 2, the detrended system has a spurious zero root and a strictly positive real root, the latter being ruled out by the transversality condition. Given our calibration strategy, the spectrum shows two conjugate complex roots very close to the imaginary axes, all the other conjugate roots having strictly negative real part.

To calibrate the initial conditions, we assume that during the years 1948 to 1959 the US economy faced a wave-like movement of 11 years and an amplitude of around 8% of per capita GDP when adjusting to the new economic environment emerging after World War II.⁹ The corresponding initial conditions are represented by

$$\bar{n}_t = a \cos(bt/\pi) + 1$$

where the amplitude of oscillations is given by parameter a , set equal to .375 for the amplitude be close to 8%, and the period by parameter b , set equal to $20/11$ for the period be equal to 11 years.¹⁰

⁹A similar figure emerges from the medium-term movements estimated by Comin and Gertler [20], for example.

¹⁰The particular choice $n_0 = 1$ comes without any loss of generality, since the profile of the solution does not depend on the level of the state variable, as usual in endogenous growth models, but on the profile of the initial conditions.

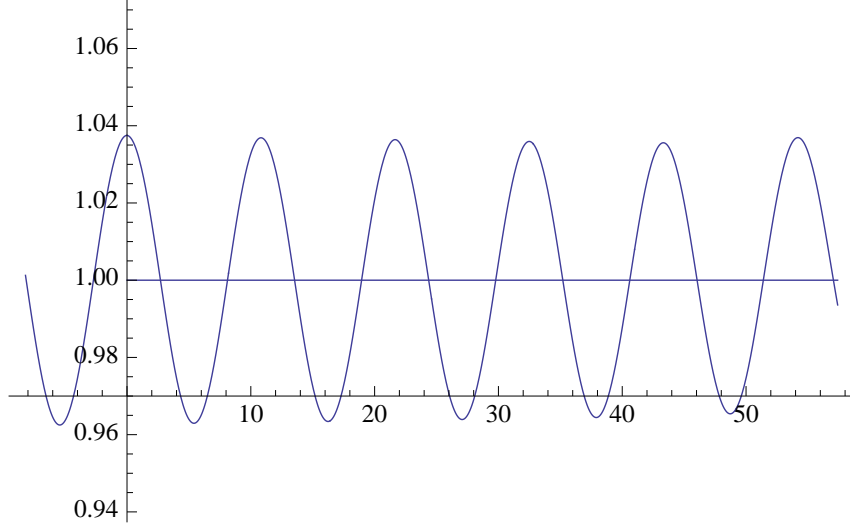


Figure 4: Equilibrium path for n_t .

To compute a numerical solution, we use the strategy proposed by Collard et al [19], which combines the method of steps suggested by Bellman and Cooke [9] with a shooting algorithm –see Judd [37]. We apply this strategy to the nonlinear system (18)-(19) and use the solution (27) of the linearized system to initialize c_0 when applying the shooting algorithm. The solution for n_t is represented in Figure 4. As expected from Propositions 2 and 3, the decentralized equilibrium converges to a Juglar cycle with periodicity close to 11 years and an amplitude of around 8 percentage points. The amplitude of the cycle depends crucially on the amplitude of the initial conditions as previously defined in the time interval $[-d, 0]$. Given that initial conditions are periodic with a periodicity close to the permanent cycle period, the economy converges to its permanent cycle very fast.

As can be observed in Figure 4, in a permanent cycle equilibrium the period of the solution is larger than the adoption delay. Remember that the behavior of n is governed by the feasibility condition (7), whose detrended version is in (19). Let first show that the solution cannot be periodic of period d . We can prove it by contradiction. Suppose the solution is periodic of period d , then $\tilde{n}_t = \tilde{n}_{t-d}$, implying that (19) becomes

$$\frac{\dot{\tilde{n}}_t}{\tilde{n}_t} = Ae^{-g_e d} - g_e - \frac{\tilde{c}_t}{\tilde{n}_t}.$$

Firstly, when detrended \tilde{n}_t is at its maximum value, because of consumption smoothing the ratio \tilde{c}_t/\tilde{n}_t is at its minimum value, implying that the growth rate is maximal at this point. Second, since the solution is periodic, it has to be that the growth rate $\dot{\tilde{n}}_t/\tilde{n}_t = 0$ at a maximum, but positive before. This contradicts the result that the growth rate is maximal at the maximum.

Let us now show that if a periodic solution exists, it has to be that the period is larger than d . Since the solution is periodic, \tilde{n}_t has to be bounded, meaning that $\tilde{n}_t \in [n^{\min}, n^{\max}]$. Since the period of the solution is different from d , $\tilde{n}_{t-d} \neq n^{\max}$. Let us call t_m at a time t at which $\tilde{n}_t = n^{\max}$. From (19), at any t larger than but close to t_m

$$Ae^{-g_e d} \Delta \tilde{n}_{t-d} - \Delta \tilde{c}_t = \Delta \dot{\tilde{n}}_t + g_e \Delta \tilde{n}_t,$$

where Δx_t refers to the discrete change in variable x with respect to t_m . The right-hand-side is strictly negative, since \tilde{n}_t is decreasing and concave at the right of the maximum, meaning that $\Delta \tilde{n}_t < 0$ and $\Delta \dot{\tilde{n}}_t < 0$. From consumption smoothing, we know that detrended output $Ae^{-ge^d \tilde{n}_{t-d}}$ reacts more than consumption, meaning that the left-hand-side has the same sign as $\Delta \tilde{n}_{t-d}$, which has to be negative then. Consequently, when n_t is at n^{\max} , n_{t-d} has to be close, but at the right of the previous spike, which proves that the period of the solution is larger than d .

How do cycles work? When the economy is on a recession, i.e. on a neighborhood of n^{\min} , purchasing power is relatively low allowing few innovators to invest on R&D. This period of low innovation activity will eventually generate a recession in the near future with negative effects on future innovation. For a similar argument, when the economy is on a boom, purchasing power is relatively high allowing many innovators to undertake R&D activities, creating the bases of a new boom when all these innovation will eventually become operative.¹¹

3 R&D Subsidies

An optimal allocation solves the following social planner problem¹²

$$\max \int_0^{\infty} \log(c_t) e^{-\rho t} dt$$

subject to the feasibility constraint

$$\dot{n}_t = A(n_{t-d} - c_t), \quad (7)$$

the irreversibility constraint $\dot{n} \geq 0$ and the initial condition $n_t = \bar{n}_t, \forall t \in [-d, 0]$, the same \bar{n}_t as in the decentralized equilibrium. Notice that for $d = 0$ the variable change $\hat{c} = Ac$ renders this problem formally identical to the AK model as in Rebelo [45].

Following Kolmanovskii and Myshkis [39] and operating as in the decentralized economy, optimality requires the Euler-type equation

$$\frac{\dot{c}_t}{c_t} = Ae^{-\rho d} \frac{c_t}{c_{t+d}} - \rho, \quad (28)$$

and the transversality condition

$$\lim_{t \rightarrow \infty} n_t c_t^{-1} e^{-\rho t} = 0, \quad (13)$$

where λ_t is the costate associated to the state n_{t-d} . The social planner faces a trade-off between consuming at time t or saving and consuming at $t + d$. For this reason, in (28) the R&D productivity, A , is weighted by the ratio of marginal utilities of consuming at $t + d$ and t , which multiplied by $e^{-\rho d}$ represents the discount factor on a period of length d . It is useful to observe that, as in the AK model, the Euler-type mixed functional differential equation (28) does not depend on the state variable n . Consequently, since the social return to R&D is constant, the planer may allocate consumption over time without caring about the path of knowledge n . As shown in Proposition 4, optimal consumption is in its balanced growth path from time zero.

¹¹This property is referred as *echo effects* in the vintage capital literature. See Boucekkine et al (1997).

¹²We implicitly assume that the solution is interior, meaning that $L_t \in (0, 1)$. Bambi et al [5] in a similar framework explicitly states the needed parameter restriction.

However, since initial conditions affect production from zero to time d , R&D has to adjust to fulfill the feasibility condition. This mechanism will repeat again and again making the optimal allocations to fluctuate, as shown in Proposition 4, converging by damping oscillations.

An optimal allocation is then a path (c_t, n_t) , for $t \geq 0$, verifying the mixed functional differential equations system (7) and (28), the transversality condition (13), the initial condition $n_t = \bar{n}_t, \forall t \in [-d, 0]$ and the irreversibility constraint $\dot{n} \geq 0$.

At a balanced growth path, from (28), consumption grows at the constant rate g holding

$$g + \rho = A e^{-(g+\rho)d}. \quad (29)$$

The following parameter condition

$$A > \rho e^{\rho d} \equiv A_{min}^* \quad (30)$$

is necessary and sufficient for g to be strictly positive. When $d = 0$, this condition collapses to the standard assumption in the AK model that $A > \rho$. Even if the transcendental equation (29) has an infinity of complex solutions, under assumption (30), existence and uniqueness of a real solution are trivial, since for $g > 0$ the right-hand-side of (29) is decreasing from $A e^{-\rho d}$ to zero and the left-hand-side is increasing from ρ to infinity.

The main properties of the transitional dynamics are stated in the proposition below. The proof follows from the maximum principle approach developed by Bambi [4] and the dynamic programming approach as in Bambi et al [5].

Proposition 4 *Let's assume that $A > A_{min}^*$, then the optimal equilibrium paths for n_t and c_t are*

$$n_t^* = a_L e^{gt} + \sum_{j=1}^{+\infty} a_j e^{z_j t} \quad (31)$$

$$c_t^* = c_0 e^{gt} \quad (32)$$

where g is the unique real solution of (29), a_L and $\{a_j\}_{j=1}^{+\infty}$ are the residues associated to the roots $\{z_j\}_{j=0}^{+\infty}$ of the characteristic equation $h(z) \equiv z - A e^{-z d} = 0$,

$$a_L = A \sum_{j=0}^{+\infty} \frac{c_0^*}{(z_j - g)h'(z_j)} \quad a_j = \frac{\bar{n}_0 + z_j \int_{-d}^0 \bar{n}_s e^{-z_j s} ds}{h'(z_j)} - \frac{A c_0^*}{(z_j - g)h'(z_j)} \quad (33)$$

with $z_0 = g + \rho$ and the initial value of consumption, c_0 , equals to

$$c_0^* = \frac{\rho}{A} \left(\bar{n}_{-d} + \int_{-d}^0 \dot{\bar{n}}_s e^{(g+\rho)s} ds \right). \quad (34)$$

Proof. See Appendix. ■

From the transversality condition, as usual, the proposition above states that detrended consumption is constant all along the transition path. Optimal detrended n , however, converges by damping oscillations to a positive constant.¹³

¹³See Bambi [4] and Bambi et al [5] for details; in particular, the discussion about the conditions for the solution to be interior.

Under log utility, consumption equals the return on wealth, the latter being represented by the term within brackets at the right hand side of (34) divided the relative productivity A –see (7). Notice that initial wealth is the sum at time zero of the value of operative varieties n_{-d} plus the value of produced but still non operative varieties, i.e., those produced between $-d$ and zero. The factor $e^{(g+\rho)s}$, multiplying the mass of varieties $\dot{\tilde{n}}_s$ created at time s , $s \in [-d, 0]$, discounts the varieties' value for the period still remaining until those varieties will become operative.

3.1 Comparing centralized and decentralized balanced growth path allocations

Optimal and equilibrium allocations differ in at least two dimensions. First, consumption is perfectly smoothed in the optimal allocation, but fluctuates at equilibrium. Second, the growth rates are different at the balanced growth path. We develop these two arguments below, before suggesting an optimal R&D policy.

The fact that consumption does not fluctuate in the optimal allocation comes from the Euler-type equation (28), which does not depend on the state of knowledge n due to linearity in the aggregate technology. This is not the case at the equilibrium Euler-type equation (14), since the private return to R&D depends on future profits, which are a negative function of the market share. For example, when the economy is expected to be on a boom at the time the innovation will be implemented, market shares are expected to be relatively low, making current R&D investments less attractive and the growth rate of current consumption low too. Individuals would like to smooth consumption at equilibrium, but since they expect returns to R&D be fluctuating, they adjust consumption consistently.

At the equilibrium and optimal balanced growth paths

$$g_e \left(e^{-\rho d} + \frac{\alpha}{A(1-\alpha)} \right) + \frac{\alpha \rho}{A(1-\alpha)} = A e^{-(g_e+\rho)d}, \quad (16)$$

$$g + \rho = A e^{-(g+\rho)d} \quad (29)$$

where g_e and g represent the equilibrium and optimal growth rates, respectively. The following proposition studies the relation between them.

Proposition 5 For $\alpha \in (0, 1)$, $g_e = g$ iff $\alpha = \underline{\alpha}$ and $g_e < g$ iff $\underline{\alpha} < \alpha < 1$, where

$$\underline{\alpha} \equiv \frac{g + \rho - g e^{-\rho d}}{2(g + \rho) - g e^{-\rho d}} < 1/2. \quad (35)$$

Proof. See Appendix. ■

This proposition is consistent with Benassy [10], who shows for $d = 0$ that the equilibrium growth rate is smaller than the optimal rate if and only if the knowledge externality, v in equation (2), is small enough or, equivalently, the elasticity of substitution α is large enough. Since in our framework v is assumed to be unity, let argue in terms of the elasticity of substitution for a given knowledge externality. For $d = 0$, $\underline{\alpha} = \left(1 + \frac{A}{\rho}\right)^{-1}$, meaning that there is a range of parameters for which the optimal growth rate is smaller than the equilibrium growth rate at the balanced growth path. Increasing α makes goods more substitutable, reducing markups, the return to R&D and the growth rate. Consequently, there is a degree of substitutability beyond which the optimal growth rate is larger than the equilibrium rate.

Since private R&D returns are different from public returns, optimality may be restored by the mean of a time dependent subsidy/tax scheme imposed on current R&D investments or, equivalently, on the return to R&D. By comparing the Euler equation associated to the optimal allocation (28) to the one associated to the equilibrium allocation (14), after using (5), it is easy to see that private and public returns equalize when the subsidy rate is

$$1 + s_t = \frac{\alpha}{1 - \alpha} \frac{n_t}{c_{t+d}}.$$

An optimal policy has two components. Firstly, as in the Romer model, it has to equalize the average private return to the social return. Second, it has to compensate for fluctuations in the private return. The social return to R&D is constant and equal to A , but the private return fluctuates following the consumption to knowledge ratio c_t/n_t , which moves countercyclically due to consumption smoothing. To render the equilibrium allocation optimal, the subsidy has to be procyclical to counterbalance fluctuations in this ratio.

3.2 A quantitative comparison

This section suggests a R&D policy designed to partially remedy the distortions underlined in the previous section, with the purpose of undertaking some counterfactual exercise around the equilibrium computed in section 2.3 and evaluate the corresponding welfare gains. The model is then extended to study a time varying R&D subsidy addressed to increase the average return to R&D and reduce the volatility of consumption. Let assume the R&D policy follows

$$1 + s_t = (1 + s) \left(\frac{c_t}{n_t} \right)^{\sigma-1},$$

where s is a constant rate and $\sigma < 1$ represents the additional smoothing introduced by the R&D policy. The Euler equation (14) becomes

$$\frac{\dot{c}_t}{c_t} = \frac{1 - \alpha}{\alpha} (1 + s) A e^{-\rho d} \left(\frac{c_t}{n_t} \right)^{\sigma} - \rho.$$

Notice that an equilibrium without R&D policy requires $s = 0$ and $\sigma = 1$.

In order to make welfare comparisons, we compute a consumption equivalent measure defined as the constant rate at which consumption in the decentralized equilibrium should increase all over the equilibrium path to make equilibrium welfare equal to the corresponding welfare of the equilibrium path with subsidies. Since utility is logarithmic, our welfare measure collapses to

$$\omega = e^{\rho(W_{R\&D} - W_e)} - 1,$$

where $W_{R\&D}$ and W_e measure welfare, as defined by the utility function (1), evaluated at equilibrium with and without subsidies, respectively.

When the R&D policy pays a 10% average subsidy, $s = .10$, and the subsidy rate moves procyclically in order to smooth consumption, with a smoothing parameter $\sigma = 1/2$, the growth rate increases from 2.4% to 3.4%. In Figure 5, detrended consumption paths, relative to initial consumption, are represented for the economies with and without subsidies. The smoother corresponds to the economy with procyclical subsidies. As can be observed, the subsidy halves consumption fluctuations. Moreover, consistent with Proposition 4, the economy slowly converges

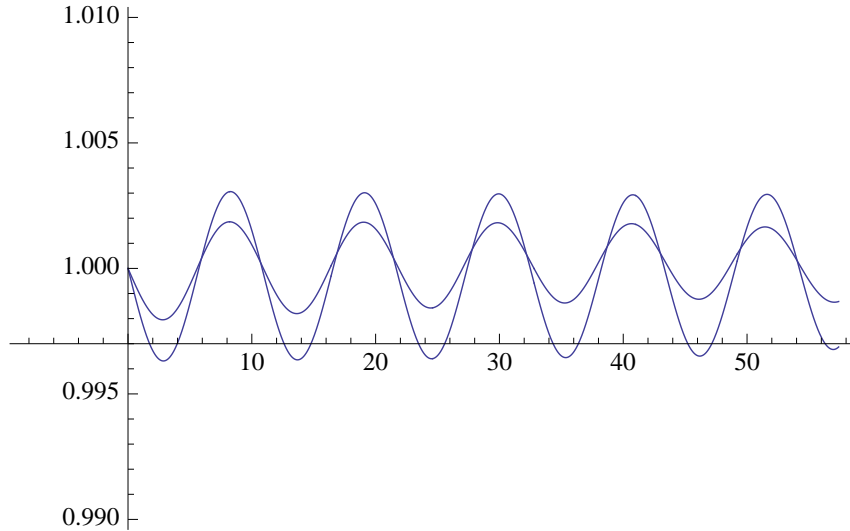


Figure 5: Consumption paths with and without subsidy.

by oscillations instead of permanently cycling. There are welfare gains of 9.6% as measured by ω . The order of magnitude is consistent with the findings in Barlevy [6]. If the 10% subsidy were constant, the growth rate would be 2.8% and the welfare gains 3.3%. Consequently, a 6.3% welfare gain may be attributed to consumption smoothing alone.

4 Conclusions

This paper studies the relation between Schumpeterian wave-like business fluctuations and economic development in an endogenous growth framework with implementation delays. The paper shows that the equilibrium path admits a Hopf bifurcation where consumption, research and output permanently fluctuate around a positive trend. The main mechanism relating growth to wave-like fluctuations is based on the assumption that innovations being fundamental for economic growth require long implementation and adoption lags. A simple quantitative exercise shows that such an endogenous mechanism relating the sources of growth and business fluctuations is not only theoretically possible but quantitatively relevant.

Additionally, the paper makes some welfare considerations. Firstly, it shows that detrended consumption is constant from the initial time in an optimal allocation, and both R&D and output converge by oscillations. Second, it proves that a procyclical subsidy/tax scheme would restore optimality. Finally, it quantitatively find that a procyclical 10% subsidy rate halving consumption fluctuations will increase the growth rate from 2.4% to 3.4% with a 9.6% increase in welfare, 6.3% due to consumption smoothing.

Appendix

More details on the three sectors

We start with the *consumption good sector*. The profit maximization problem which leads to the inverse demand function for the intermediate good j , equation (8), is

$$\max_{x_t(j)} p_c c_t - \int_0^{n_{t-d}} p_t(j) x_t(j) dj \quad (36)$$

subject to the consumption good technology (2), and assuming $p_c = 1$.

Each firm j in the *intermediary good sector* sets the monopolistic prices of $x_t(j)$ by solving the following maximization problem

$$\max_{p_t(j)} p_t(j) x_t(j) - w_t l_t(j)$$

subject to the technology constraint (3), and the inverse demand function (8) of its intermediate good j , coming from the consumption good sector. It is straightforward to show that once all the constraints are substituted into the objective function the problem is equivalent to:

$$\max_{p_t(j)} p_t(j)^{\frac{1}{\alpha-1}} (p_t(j) - w_t)$$

which implies the monopolistic price equation (9).

Firms may enter freely into *R&D*. Each new patent has a value of v_t and cost $w_t(1 - L_t)$ to be produced. Then the value to be maximized is

$$\max_{1-L_t} v_t \dot{n}_t - w_t(1 - L_t)$$

subject to the *R&D* technology (6). This implies

$$\max_{1-L_t} (1 - L_t)[v_t A n_{t-d} - w_t]$$

and then

- $L_t = 0$ if $v_t > \frac{w_t}{A n_{t-d}}$ not possible (why?)
- $L_t = 1$ if $v_t < \frac{w_t}{A n_{t-d}}$ which implies $\dot{n}_t = 0$;
- $L_t \in (0, 1)$ and $v_t = \frac{w_t}{A n_{t-d}}$ if $\dot{n}_t > 0$

In the paper we focus on this interior solution and we will show that the inequality $\dot{n}_t > 0$ will be always respected both in the market and the central planner economy. Observe also that at the symmetric equilibrium this condition implies the free entry condition (12).

Proof of Proposition 1. We first rewrite the system (19), (20) as a second order delay differential equation

$$\ddot{\tilde{n}}_t - \rho \dot{\tilde{n}}_t - A e^{-g_e d} \dot{\tilde{n}}_{t-d} - \left(g_e (g_e + \rho) + \frac{\alpha (g_e + \rho)^2 e^{\rho d}}{1 - \alpha} \right) \tilde{n}_t + A (g_e + \rho) e^{-g_e d} \tilde{n}_{t-d} = 0$$

Taking the Laplace transformation $\mathbf{L}(\tilde{n}_t)(\lambda) = \int_0^\infty \tilde{n}_t e^{-\lambda t} dt$ of this equation and taking into account that

$$\begin{aligned}\mathbf{L}(\ddot{\tilde{n}}_t)(\lambda) &= -\dot{\tilde{n}}_0 - \lambda \tilde{n}_0 + \lambda^2 \mathbf{L}(\tilde{n}_t)(\lambda) \\ \mathbf{L}(\dot{\tilde{n}}_t)(\lambda) &= -\tilde{n}_0 + \lambda \mathbf{L}(\tilde{n}_t)(\lambda) \\ \mathbf{L}(\dot{\tilde{n}}_{t-d})(\lambda) &= e^{-\lambda d} \left[-\tilde{n}_0 + \int_{-d}^0 \dot{\tilde{n}}_t e^{-\lambda t} dt + \lambda \mathbf{L}(\tilde{n}_t)(\lambda) \right] \\ \mathbf{L}(\tilde{n}_{t-d})(\lambda) &= e^{-\lambda d} \left[\int_{-d}^0 \tilde{n}_t e^{-\lambda t} dt + \mathbf{L}(\tilde{n}_t)(\lambda) \right]\end{aligned}$$

we have that

$$\mathbf{L}(\tilde{n}_t)(\lambda) \cdot h(\lambda) = \phi(\lambda)$$

where

$$\phi(\lambda) = \dot{\tilde{n}}_0 + \tilde{n}_0 \left(\lambda - \rho - A e^{-(g_e + \lambda)d} \right) + A e^{-(g_e + \lambda)d} \int_{-d}^0 [\dot{\tilde{n}}_t - (g_e + \rho) \tilde{n}_t] e^{-\lambda t} dt$$

and $h(\lambda)$ is the characteristic equation (23) associated to the second order delay differential equation. Since \tilde{n}_t is a continuous differentiable function in $[0, +\infty)$,¹⁴ and therefore certainly continuous and of bounded variation on any finite interval, then we can use the inversion formula for the Laplace transformation on the set of circle contours C_ℓ with $\ell = 1, 2, \dots$, center in the origin of the complex plane, and radius y_ℓ , to obtain its solution:

$$\tilde{n}_t = \oint \frac{\dot{\tilde{n}}_0 + \tilde{n}_0 (\lambda - \rho - A e^{-(g_e + \lambda)d}) + A e^{-(g_e + \lambda)d} \int_{-d}^0 [\dot{\tilde{n}}_t - (g_e + \rho) \tilde{n}_t] e^{-\lambda t} dt}{h(\lambda)} e^{\lambda t} d\lambda \quad (37)$$

Then we can obtain the series expansion (21) of this solution by using the residue theorem. Since the argument of the contour integral in (37) is not complex differentiable in all of its domain due to the singularities represented by the roots of $h(\lambda)$, then we may use the Residue theorem (see for example Bellman and Cooke [9], chapter 4.6 page 121-126) to rewrite the solution of \tilde{n}_t as:

$$\tilde{n}_t = \lim_{\ell \rightarrow \infty} \sum_{\lambda_r \in C_\ell} \text{Res} \left(\frac{\phi(\lambda)}{h(\lambda)}, e^{\lambda t} \right) = \lim_{\ell \rightarrow \infty} \sum_{\lambda_r \in C_\ell} p_{\lambda_r} e^{\lambda_r t} = \sum_{r=0}^{\infty} p_r e^{\lambda_r t} \quad (38)$$

where the residues $p_r = \frac{\phi(\lambda_r)}{h'(\lambda_r)}$ are defined in the complex field \mathbb{C} . Finally the solution of \tilde{c}_t can be derived from (21) and (19). ■

Proof of Proposition 2. First of all, the system in the normalized variables $\tilde{c}(t)$ and $\tilde{n}(t)$ implies that $h(0) = 0$, and then $\lambda_1 = 0$ is a spurious root of (23) coming from the detrending. Moreover, a positive real root, λ_0 , always exists since $\lim_{\lambda \rightarrow \pm\infty} h(\lambda) = +\infty$ and $h'(0) = -\rho - A e^{-g_e d} - dA(g_e + \rho)e^{-g_e d} < 0$ for any admissible choice of the parameters. No other positive real root exists since

$$h'(\lambda) = 2\lambda - \rho - A[(1 + d(-\lambda + g_e + \rho))]e^{-\lambda d} e^{-g_e d}$$

has only one critical point. This comes directly by looking at $f(\lambda) = A[1 + d(-\lambda + g_e + \rho)]e^{-\lambda d} e^{-g_e d}$ and noticing that the following relations always hold: $f(0) > 0$, $f'(0) = A\{-d[2 + d(g_e + \rho)]\}e^{-g_e d} < 0$, and $\lim_{\lambda \rightarrow +\infty} f(\lambda) = 0$.

¹⁴See the previously mentioned theorem of existence and uniqueness of solution in Bellman and Cooke [9]

Following Kolmanovskii and Nosov [40], we use the D-Subdivision method to determine the regions D_i (separated each other by what we call D curves) having i roots with strictly positive real part (from here on p -roots). Moreover we focus our analysis on the quasi-polynomial $h(\lambda)$ when $A \rightarrow A_e^+$, and then $g_e \rightarrow 0^+$; in this case, the characteristic equation becomes a continuous function of only two parameters, (ρ, α) , and a visual representation of the results can be provided. Moreover the stability results obtained for this restriction still hold for any sufficiently small and continuous variation of A . Under this assumption on A we have that

$$h(\lambda) = \lambda^2 - \rho\lambda - \lambda e^{-\lambda d} \hat{\alpha} \rho e^{\rho d} + e^{-\lambda d} \hat{\alpha} \rho^2 e^{\rho d} - \hat{\alpha} \rho^2 e^{\rho d} = 0 \quad (39)$$

where $\hat{\alpha} = \frac{\alpha}{1-\alpha} \in [0, +\infty)$ since $\alpha \in [0, 1)$. We also extend the domain of ρ to the interval $(-\varepsilon, 1 + \varepsilon)$, with ε positive and infinitely small, in order to pin down more easily the different D_i .

Let's start with the analysis of the two *extreme cases*: $\hat{\alpha} = 0$ and $\rho = 0$. When $\hat{\alpha} = 0$ then the parameters space is partitioned in two regions, D_1 if $\rho \in (0, 1 + \varepsilon)$ and D_0 if $\rho \in (-\varepsilon, 0]$; in fact when $\hat{\alpha} = 0$ then $h(\lambda) = \lambda(\lambda - \rho) = 0$ and there are only two real roots $\lambda_0 = \rho$, and $\lambda_1 = 0$. On the other hand $\rho = 0$ implies $h(\lambda) = \lambda^2 = 0$.

Let's now focus on the purely imaginary roots $\lambda = iv$ when $\hat{\alpha}$ and ρ can take any values in their respective domains in order to identify the D -curves, $\hat{\alpha} = \hat{\alpha}(v)$ and $\rho = \rho(v)$, separating different D_i regions. The characteristic equation writes

$$h(iv) = -v^2 - \rho iv - \hat{\alpha} \rho e^{\rho d} v [i \cos(vd) + \sin(vd)] + \hat{\alpha} \rho^2 e^{\rho d} [\cos(vd) - i \sin(vd)] - \hat{\alpha} \rho^2 e^{\rho d} = 0 \quad (40)$$

Observe also that $h(iv) = U(v) + iW(v) = 0$ with:

$$U(v) = 0 \Leftrightarrow v^2 + \hat{\alpha} \rho^2 e^{\rho d} + \hat{\alpha} \rho e^{\rho d} [v \sin(vd) - \rho \cos(vd)] = 0 \quad (41)$$

$$W(v) = 0 \Leftrightarrow v + \hat{\alpha} e^{\rho d} [v \cos(vd) + \rho \sin(vd)] = 0 \quad (42)$$

From $W(v) = 0$ follows immediately that

$$\hat{\alpha} = \frac{-v}{e^{\rho d} [v \cos(vd) + \rho \sin(vd)]} \quad (43)$$

when $v \cos(vd) + \rho \sin(vd) \neq 0$; then substituting (43) into (41) leads to

$$\rho = \rho(\omega) = \pm \frac{1}{d} \sqrt{\frac{\omega^2 \cos(\omega)}{1 - \cos(\omega)}} \quad \text{with } \omega = vd \quad (44)$$

Then substituting back (44) into (43) leads to

$$\hat{\alpha}(\omega) = \frac{-\omega}{e^{\sqrt{\frac{\omega^2 \cos(\omega)}{1 - \cos(\omega)}}} \left[\omega \cos(\omega) + \sqrt{\frac{\omega^2 \cos(\omega)}{1 - \cos(\omega)}} \cdot \sin(\omega) \right]} \quad (45)$$

Relations (44) and (45) determine the point $(\rho_1, \hat{\alpha}_1) = (\rho(\omega_1), \hat{\alpha}(\omega_1))$ of a D curve for $\omega = \omega_1$. If ω varies in its domain $\omega \in \left(\frac{(2k-1)\pi}{2}, \frac{(2k+1)\pi}{2} \right) \setminus 2k\pi$ with $k = \dots, -2, -1, 0, 1, 2, \dots$, we obtain all the D -curves.¹⁵ Besides these curves, the D-subdivision may contain some straight lines for

¹⁵The domain of ω excludes the points $2k\pi$ and $\frac{(2k+1)\pi}{2}$ which are discontinuity for $1 - \cos(\omega)$ and $\frac{\omega}{d} \cos(\omega) + \rho(\omega) \sin(\omega) = 0$ respectively.

the values of ω which imply an indeterminate form of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ to $\rho(\omega)$ or $\hat{\alpha}(\omega)$. However in our specific case, the only indeterminate form emerges at $\omega = 0$ which implies $h(0) = 0$ confirming the presence of a zero root in all the parameters space. Then the properties of the parametric D -curves can be analytical derived; among them, we show in the following why the region $[-\varepsilon, 0] \times [0, +\infty]$ in the parameters space $(\rho, \hat{\alpha})$ is a subset of D_0 .

To show this fact we will prove that if $\rho \rightarrow 0^-$ then $\hat{\alpha} \rightarrow \pm\infty$ and then no D -curve can be in the region under analysis. From (44) it is clear that $\rho \rightarrow 0^-$ if and only if $\omega \rightarrow \frac{(2k+1)\pi}{2}$, and then we have to study the following limit:

$$\begin{aligned} \lim_{\omega \rightarrow \frac{(2k+1)\pi}{2}} \hat{\alpha}(\omega) &= \lim_{\omega \rightarrow \frac{(2k+1)\pi}{2}} \frac{-\omega}{e^{\rho(\omega)d} \left[\omega \cos(\omega) \pm |\omega| \sqrt{\frac{\cos(\omega)}{1-\cos(\omega)}} \cdot \sin(\omega) \right]} \\ &= \lim_{\omega \rightarrow \frac{(2k+1)\pi}{2}} \frac{-\omega}{e^{\rho(\omega)d} \left[\omega \cos(\omega) \pm \frac{|\omega|}{|\sin(\omega)|} \sqrt{\cos(\omega) \cdot (1 + \cos(\omega))} \cdot \sin(\omega) \right]} \end{aligned}$$

if k is even then \pm otherwise \mp ; let's assume k even, then

$$\begin{aligned} \lim_{\omega \rightarrow \frac{(2k+1)\pi}{2}} \hat{\alpha}(\omega) &= \lim_{\omega \rightarrow \frac{(2k+1)\pi}{2}} \frac{-1}{\cos(\omega) \pm \sqrt{\cos(\omega)(1 + \cos(\omega))}} \\ &= \lim_{\omega \rightarrow \frac{(2k+1)\pi}{2}} \frac{1}{\sqrt{\cos(\omega)}} \cdot \lim_{\omega \rightarrow \frac{(2k+1)\pi}{2}} \frac{1}{\sqrt{\cos(\omega) \pm \sqrt{1 + \cos(\omega)}}} = \mp\infty \end{aligned}$$

On the other hand if k is odd then $\pm\infty$.

Each curve separating two regions is obtained by studying the values that the two parameters can have in each of the intervals of v . It is also clear that the D_1 region changes as shown in Figure 2 because $\frac{\partial \rho(\omega)}{\partial d} < 0$, while $\frac{\partial \hat{\alpha}(\omega)}{\partial d} = 0$. ■

Proof of Proposition 3. Given our assumptions, the only positive root to be ruled out in order to have convergence to the balanced growth path is λ_0 . To do that we have to specify \tilde{c}_0 as in (27) so that $p_0 = 0$. Uniqueness of the equilibrium path is a direct consequence of the fact that (27) is the only choice of the initial condition of consumption which rules out λ_0 . Oscillatory convergence follows from the properties of the spectrum of roots as discussed in the previous proposition. ■

Proof of Proposition 4. We refer the interested reader to the proof of Theorem 4 in Bambi et al [5]. ■

Proof of Proposition 5. Let's assume $g = g_e$. Combining (16) and (29) to solve for α gives $\underline{\alpha}$ as defined above. Notice that from (29), g does not depend on α , meaning that $\underline{\alpha}$ in (35) only depends on the other three parameters A, d, ρ . It is straightforward to observe that $\underline{\alpha}$ is always smaller than $1/2$. Finally $g_e < g$ iff $\alpha > \underline{\alpha}$, since from (16)

$$\frac{dg_e}{d\alpha} = - \frac{\frac{(g_e + \rho)e^{\rho d}}{(1-\alpha)^2}}{1 + dAe^{-g_e d} + \frac{\alpha}{1-\alpha}e^{\rho d}} < 0,$$

and g in (29) does not depend on α . ■

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