A Riemann solver for unsteady computation of 2D shallow flows with variable density

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Abstract

A novel 2D numerical model for vertically homogeneous shallow flows with variable horizontal density is presented. Density varies according to the volumetric concentration of different components or species that can represent suspended material or dissolved solutes. The system of equations is formed by the 2D equations for mass and momentum of the mixture, supplemented by equations for the mass or volume fraction of the mixture constituents. A new formulation of the Roe-type scheme including density variation is defined to solve the system on two-dimensional meshes. By using an augmented Riemann solver, the numerical scheme is defined properly including the presence of source terms involving reaction. The numerical scheme is validated using analytical steady-state solutions of variable-density flows and exact solutions for the particular case of initial value Riemann problems with variable bed level and reaction terms. Also, a 2D case that includes interaction with obstacles illustrates the stability and robustness of the numerical scheme in presence of non-uniform bed topography and wetting/drying fronts. The obtained results point out that the new method is able to predict faithfully the overall behavior of the solution and of any type of waves.

Key words: Multi-component transport; variable density; upwind method; solute; weak solutions; well-balanced approach; Roe solver; reactive terms; Riemann problem; source terms.

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1. Introduction

Multicomponent flows are widely present in various engineering, environmental and geophysical applications, involving complex flow configurations. When water in rivers and estuaries is well-mixed, the transported components are distributed uniformly over the water column, allowing the use of depth integrated models. When modeling natural waters, their description considers the transport of dissolved chemical with a degree of dilution low enough to assume that density is equal to the density of water. This is no longer valid in a wide range of applications that involve shallow flows. For instance, the degree of salinity in industrial effluents or the amount of transported sediments in debris flow makes more appropriate to describe these phenomena as a mixture of water and different components.

In the case of considering a variable horizontal density the mathematical model for multicomponent flow can be based on one set of equations for mass and momentum of the mixture, supplemented by equations for the mass or volume fraction of the mixture constituents.

The system of equations in realistic shallow water models includes non-conservative or source terms describing the presence of irregular geometries and friction. When incorporated in a given specific finite volume scheme the main focus has been traditionally put on keeping a discrete balance between flux and source terms in order to ensure preservation of motionless steady state, leading to the notion of well-balanced schemes or property C [1]. In [1] an upwind method for the treatment of the bed slope term was defined to ensure property C. In contrast, other authors [2, 3] have ensured property C by using a centered discretization of the bed slope source term. In [2] property C was ensured by using the water surface gradient as the basis for data reconstruction in combination with the HLL. In [3] the HLLC solver [4] was applied writing the set of equations in a deviatoric formulation.

On the other hand, the effect of the source terms in the weak solution can be included by using well-balanced augmented approximate Riemann solvers [5, 6, 7]. In [7] problems with m equations were presented using an augmented Roe's Riemann solver defined by $m + 1$ states, by means of an extra stationary wave linked to the source term. This approximate solver is based on a reinterpretation of the Roe's approach [8] and in the upwind treatment of the source term of Vázquez-Cendón [1]. This approximation provides accurate results, and in [9] it provided better results than those given by the centered discretization of the bed slope source term of [3].

When enlarging the shallow water system to incorporate the solute volume equations, it is convenient to solve all the equations in a coupled form [10, 11, 12, 13, 14] but when including the solute volume as a new conserved quantity, nonphysical solutions for the solute concentration may appear [15]. In [9], the understanding of the approximate augmented solver behind the numerical method allowed a detailed analysis of the complete set of approximate solutions, isolating those cases where unbounded solute concentrations appear. These analysis allowed an accurate and conservative solute fix ensuring a monotone description of the concentration values. Following a previous work [16], this additional effort is bypassed here still retaining a coupled formulation.

Godunov-type finite volume schemes can be implemented to solve a variable horizontal density mathematical model for multicomponent flow. In [17] a finite volume scheme in conjunction with a Roe’s approximate Riemann solver was used to solve the equations on a uniform collocated grid. Although the resulting numerical scheme provided plausible results, lack of conservation was reported. In [18] 2D cases with horizontally variable density flow were modeled using the HLLC approximate Riemann solver, extending the 1D formulation proposed by [17].

In this work, accurate, conservative and robust numerical tools are devised for the numerical modeling of multicomponent flows, able to handle complex situations. The outline of the paper is as follows: in Section 2, we present the 2D differential formulation of the mathematical model and in Section 3, the finite volume formulation is introduced. In Section 4 a 2D Roe’s solver for the homogeneous part of the multicomponent shallow equations is defined, based on the definition of an approximate Jacobian that follows strictly the conditions and properties of a Roe-type scheme. In Section 5, the augmented Riemann solver is used to formulate the first order numerical scheme in presence of bed slope and friction source terms, and in Section 6 reaction source terms are included. Section 7 presents numerical model validation for 1D cases using analytical steady-state solutions including uniform and variable-density flows, exact solutions for the particular case of initial value Riemann problems with variable bed level and reaction terms. Also a 2D academic case that illustrates the stability and robustness of the numerical scheme in presence of non-uniform bed topography and wetting/drying fronts is presented.

74 2. Mathematical model

75 The phenomena we are interested in regard transport of suspended ma-
 76 terial or dissolved species, leading to a multicomponent flow with variable
 77 density in time and space. The model involves the following assumptions: (i)
 78 shallow-water approach: the flow is confined to a layer which is thin compared
 79 to the horizontal scales of interest; (ii) multicomponent flow: the mixture of
 80 water, suspended material and dissolved species is described by using the
 81 continuum approach and assuming the same velocity for the liquid and for
 82 the solid phase; (iii) the different components are distributed uniformly in
 83 the mixture column and the pressure is hydrostatic.

84 Accordingly, ϕ_p represents the scalar depth-averaged volumetric concen-
 85 tration of component p , with $p = 1, \dots, N_p$ and N_p the number of different
 86 components transported. The mixture density is given by $\rho_w r$ where ρ_w is
 87 the density of the water and r represents the relative density of the bulk
 88 mixture to that of the clean water

$$r = 1 + \sum_{p=1}^{N_p} \Delta_p \phi_p \quad (1)$$

89 where $\Delta_p = (\rho_p - \rho_w)/\rho_w$ is the relative density of the solid phase p . We
 90 assume that dissolved species with low concentration do not change bulk
 91 density $\Delta_p = 0$.

92 The relevant formulation of the model derives, respectively, from the
 93 depth-averaged equation of bulk mass conservation, mixture momentum con-
 94 servation and conservation of the mass of the different constituents. That
 95 system of partial differential equations is formulated here in coupled form as
 96 follows [17, 18].

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} + \frac{\partial \mathbf{G}(\mathbf{U})}{\partial y} = \mathbf{S}(\mathbf{U}) + \mathbf{R}(\mathbf{U}) + \mathbf{D}(\mathbf{U}) \quad (2)$$

97 where

$$\mathbf{U} = \left(h r, \quad h u r, \quad h v r, \quad h \phi_1, \quad \dots, \quad h \phi_{N_p} \right)^T \quad (3)$$

98 are the conserved variables, with h representing the mixture depth, $q_x = hu$
 99 and $q_y = hv$ the unit discharges, with (u, v) the depth averaged components
 100 of the velocity vector $\mathbf{u}$ along the x and y coordinates respectively.

101 The fluxes are given by

$$\begin{aligned}\mathbf{F} &= \left(hur, \quad hu^2r + \frac{1}{2}gh^2r, \quad huvr, \quad hu\phi_1, \quad \dots \quad hu\phi_{N_p} \right)^T \\ \mathbf{G} &= \left(hvr, \quad huvr, \quad hv^2r + \frac{1}{2}gh^2r, \quad hv\phi_1, \quad \dots \quad hv\phi_{N_p} \right)^T\end{aligned}\quad (4)$$

where g is the acceleration of the gravity. The source terms of the system are split in three kind of terms. The term $\mathbf{S}$ is defined as

$$\mathbf{S} = \left(0, \frac{p_{b,x}}{\rho_w} - \frac{\tau_{b,x}}{\rho_w}, \frac{p_{b,y}}{\rho_w} - \frac{\tau_{b,y}}{\rho_w}, 0, \dots, 0 \right)^T \quad (5)$$

with $p_{b,x}, p_{b,y}$ and $\tau_{b,x}, \tau_{b,y}$ are the pressure force along the bottom and the bed shear stress in the x and y direction respectively, with ρ_w the density of water. The former can be formulated in terms of the bed slopes of the bottom level z

$$\frac{p_{b,x}}{\rho_w} = -ghr \frac{\partial z}{\partial x} = ghr S_{0x}, \quad \frac{p_{b,y}}{\rho_w} = -ghr \frac{\partial z}{\partial y} = ghr S_{0y} \quad (6)$$

and the friction losses are written in terms of the Manning's roughness coefficient n

$$\frac{\tau_{b,x}}{\rho_w} = gh S_{fx} \quad S_{fx} = \frac{n^2 u \sqrt{u^2 + v^2}}{h^{4/3}}, \quad \frac{\tau_{b,y}}{\rho_w} = gh S_{fy} \quad S_{fy} = \frac{n^2 v \sqrt{u^2 + v^2}}{h^{4/3}} \quad (7)$$

The reaction source terms $\mathbf{R}$, having a volumetric character, have to be defined by every particular application, and are written in general form as follows:

$$\mathbf{R} = \left(\sum_{p=1}^{N_p} \Delta_p R_p, 0, 0, R_1, \dots, R_{N_p} \right)^T \quad (8)$$

The diffusion term D_s for all the conserved variables U_s can be expressed by means of an associated primitive variable W_s , with $U_s = hW_s$, as

$$D_s = \vec{\nabla} \left(\mathbf{K} h \vec{\nabla} W \right)_s \quad (9)$$

where each $\mathbf{K}$ is an empirical dispersion matrix, of the form

$$\mathbf{K} = \begin{pmatrix} K_{xx} & K_{xy} \\ K_{yx} & K_{yy} \end{pmatrix} \quad (10)$$

116 In the present work, the effect of the diffusion terms has not been consid-
 117 ered. We are mostly interested in the numerical issues associated to advection
 118 and therefore these terms will be neglected from now on in the text.

119 System (2) is time dependent, non linear, and contains source terms. Un-
 120 der the hypothesis of dominant advection it can be classified and numerically
 121 dealt with as belonging to the family of hyperbolic systems. The mathemati-
 122 cal properties of (2) include the existence of a Jacobian matrix, $\mathbf{J}_{\mathbf{n}}$, of the flux
 123 normal to a direction given by the unit vector $\mathbf{n}$, $\mathbf{En} = \mathbf{F}n_x + \mathbf{G}n_y$, defined as

$$\mathbf{J}_{\mathbf{n}} = \frac{\partial \mathbf{En}}{\partial \mathbf{U}} = \frac{\partial \mathbf{F}}{\partial \mathbf{U}} n_x + \frac{\partial \mathbf{G}}{\partial \mathbf{U}} n_y \quad (11)$$

124 leading to

$$\mathbf{J}_{\mathbf{n}} = \begin{pmatrix} \mathcal{J}_1 & \mathcal{J}_2 \\ \mathcal{J}_3 & \mathcal{J}_4 \end{pmatrix}_{(3+N_p) \times (3+N_p)} \quad (12)$$

125 with

$$\begin{aligned} \mathcal{J}_1 &= \begin{pmatrix} 0 & n_x & n_y \\ \left(\frac{1}{2}gh(1+r) - u^2\right)n_x - uvn_y & 2un_x + vn_y & un_y \\ \left(\frac{1}{2}gh(1+r) - v^2\right)n_y - uvn_x & vn_x & un_x + 2vn_y \end{pmatrix} \\ \mathcal{J}_2 &= \frac{1}{2}ghr \begin{pmatrix} 0 & \cdots & 0 \\ \Delta_1 n_x & \cdots & \Delta_{N_p} n_x \\ \Delta_1 n_y & \cdots & \Delta_{N_p} n_y \end{pmatrix}_{3 \times N_p} \\ \mathcal{J}_3 &= \frac{1}{r} \begin{pmatrix} -\phi_1 \mathbf{un} & \phi_1 n_x & \phi_1 n_y \\ \vdots & \vdots & \vdots \\ -\phi_{N_p} \mathbf{un} & \phi_{N_p} n_x & \phi_{N_p} n_y \end{pmatrix}_{N_p \times 3} \quad \mathcal{J}_4 = (\mathbf{un}) \mathbf{I}_{N_p \times N_p} \end{aligned} \quad (13)$$

126 with $\mathbf{un} = un_x + vn_y$. The eigenvalues are calculated from $\det(\mathbf{J}_{\mathbf{n}} - \lambda \mathbf{I})$. The
 127 model provides a set of $3 + N_p$ real eigenvalues λ_k^m

$$\lambda^1 = (\mathbf{un} - c), \quad \lambda^2 = (\mathbf{un}), \quad \lambda^3 = (\mathbf{un} + c)_k \quad \lambda^{3+p} = \lambda^2 \quad (14)$$

128 and eigenvectors $\mathbf{e}_k^m$. Then, it is possible to define two approximate matrices
 129 $\mathbf{P} = (\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3, \dots, \mathbf{e}^{3+N_p})$, and $\mathbf{P}^{-1}$, where

$$\mathbf{P} = \begin{pmatrix} \mathcal{P}_1 & \mathcal{P}_2 \\ \mathcal{P}_3 & \mathcal{P}_4 \end{pmatrix}_{(3+N_p) \times (3+N_p)} \quad (15)$$

130 with

$$\mathcal{P}_1 = \begin{pmatrix} 1 & 0 & 1 \\ u - cn_x & -cn_y & u + cn_x \\ v - cn_y & cn_x & v + cn_y \end{pmatrix} \mathcal{P}_2 = \frac{1}{r} \begin{pmatrix} \Delta_1 & \cdots & \Delta_{N_p} \\ \Delta_1 u & \cdots & \Delta_{N_p} u \\ \Delta_1 v & \cdots & \Delta_{N_p} v \end{pmatrix}_{3 \times N_p} \quad (16)$$

131 and

$$\mathcal{P}_3 = \begin{pmatrix} \phi_1 & 0 & \phi_1 \\ \phi_2 & 0 & \phi_2 \\ \vdots & 0 & \vdots \\ \phi_{N_p-1} & 0 & \phi_{N_p-1} \\ \phi_{N_p} & 0 & \phi_{N_p} \end{pmatrix}_{N_p \times 3} \quad \mathcal{P}_4 = \frac{1}{2r} \begin{pmatrix} \Delta_1 \phi_1 + 2 & \cdots & \Delta_{N_p} \phi_1 \\ \Delta_1 \phi_2 & \cdots & \Delta_{N_p} \phi_2 \\ \vdots & \cdots & \vdots \\ \Delta_1 \phi_{N_p-1} & \cdots & \Delta_{N_p} \phi_{N_p-1} \\ \Delta_1 \phi_{N_p} & \cdots & \Delta_{N_p} \phi_{N_p} + 2 \end{pmatrix}_{N_p \times N_p} \quad (17)$$

132 that diagonalize $\mathbf{J}_n$

$$\mathbf{P} \mathbf{J}_n \mathbf{P}^{-1} = \mathbf{\Lambda} \quad (18)$$

133 with $\mathbf{\Lambda}$ is a diagonal matrix with eigenvalues λ^m in the main diagonal

$$\mathbf{\Lambda} = \begin{pmatrix} \lambda^1 & 0 & 0 & 0 & 0 \\ 0 & \lambda^2 & 0 & 0 & 0 \\ 0 & 0 & \lambda^3 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & \lambda^{3+N_p} \end{pmatrix} \quad (19)$$

134 3. Finite Volume Model

135 To introduce the finite volume scheme, (2) is integrated in a volume or
136 grid cell Ω using Gauss theorem:

$$\frac{\partial}{\partial t} \int_{\Omega} \mathbf{U} d\Omega + \oint_{\partial\Omega} \mathbf{E} n dl = \int_{\Omega} \mathbf{S} d\Omega + \int_{\Omega} \mathbf{R} d\Omega \quad (20)$$

137 where $\mathbf{n} = (n_x, n_y)$ is the outward unit normal vector to the volume Ω .

Figure 1: Cell parameters.

138 In order to obtain a numerical solution of system (2) we divide the domain
 139 in computational cells, Ω_i , using a mesh fixed in time, and (20) is applied to
 140 each cell

$$\frac{\partial}{\partial t} \int_{\Omega_i} \mathbf{U} d\Omega + \sum_{k=1}^{NE} \int_{e_k}^{e_{k+1}} (\mathbf{En})_k^\downarrow dl_k = \int_{\Omega_i} \mathbf{S} d\Omega + \int_{\Omega_i} \mathbf{R} d\Omega \quad (21)$$

141 with $(\mathbf{En})_k^\downarrow$ the value of the interface flux function through the edge k to be
 142 defined, $\mathbf{n}_k = (n_x, n_y)$ is the outward unit normal vector to the cell edge k ,
 143 and NE is the number of edges in cell i . Assuming a piecewise representation
 144 of the conserved variables

$$\mathbf{U}_i^n = \frac{1}{A_i} \int_{\Omega_i} \mathbf{U}(x, y, t^n) d\Omega \quad (22)$$

145 (21) is written as

$$\frac{\partial}{\partial t} \int_{\Omega_i} \mathbf{U} d\Omega + \sum_{k=1}^{NE} (\mathbf{En})_k^\downarrow l_k = \int_{\Omega_i} \mathbf{S} d\Omega + \int_{\Omega_i} \mathbf{R} d\Omega \quad (23)$$

146 where l_k is the corresponding edge length.

147 For the sake of clarity, the solution of (23) is analyzed separately consid-
 148 ering cases with and without source terms.

149

150 **4. Roe methods for the homogeneous case.**

151 In the first subproblem all the right hand side in (23) is dropped, and the
 152 problem reduces to

$$\frac{\partial}{\partial t} \int_{\Omega_i} \mathbf{U} d\Omega + \sum_{k=1}^{NE} (\mathbf{E}\mathbf{n})_k^\downarrow l_k = 0 \quad (24)$$

153 The upwind method for the advection part of (23) is derived as a special
 154 case of the reconstruction, evolving and averaging steps method proposed
 155 originally by Godunov [19], that provides new cell averages values for the
 156 conserved variables $\mathbf{U}_i$. The Godunov method starts from piecewise data
 157 reconstructions, so the theory of Riemann problems can be applied.

Figure 2: Riemann problem in 2D along the normal direction to a cell side.

158 Once the reconstruction step has been considered assuming a first order
 159 in space approach in (23), in order to solve a 2D problem, the next step
 160 consists of defining a local 1D RP at each k edge, as shown in Figure 2. This
 161 is done projecting the fluxes onto the normal direction $\mathbf{n}_k$ to each cell edge.
 162 Therefore in a k edge the following local RP along the x' direction is defined

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial (\mathbf{E}\mathbf{n})}{\partial x'} = 0 \quad (25)$$

163 with the following initial conditions

$$\mathbf{U}(x', 0) = \begin{cases} \mathbf{U}_i & \text{if } x' < 0 \\ \mathbf{U}_j & \text{if } x' > 0 \end{cases} \quad (26)$$

164 being x' the coordinate normal to the cell edge k . Its exact solution provides
 165 the variation of $\mathbf{U}$ in time and space, $\mathbf{U}(x', t)$, and also provides the exact
 166 value of $(\mathbf{E}\mathbf{n})_k^\downarrow$ in (24).

167 If the exact solution of the local RP in (25) $\mathbf{U}(x', t)$ cannot be derived
 168 or is very costly, the problem can be approximated by the following constant
 169 coefficient linear problem

$$\frac{\partial \hat{\mathbf{U}}}{\partial t} + \tilde{\mathbf{J}}_{\mathbf{n},k} \frac{\partial \hat{\mathbf{U}}}{\partial x'} = 0 \quad (27)$$

170 with the following initial conditions

$$\hat{\mathbf{U}}(x', 0) = \begin{cases} \mathbf{U}_i & \text{if } x' < 0 \\ \mathbf{U}_j & \text{if } x' > 0 \end{cases} \quad (28)$$

171 that will provide a linearized or weak solutions, that will be referred to as
 172 $\hat{\mathbf{U}}(x', t)$.

173 Even when ignoring the exact solution of the associated RP $\mathbf{U}(x', t)$, it
 174 is possible to provide its variation by integrating (25) over a suitable control
 175 volume such as the one shown in Figure 3. Figure 3 shows a local RP with
 176 initial values $\mathbf{U}_i, \mathbf{U}_j$, over the time interval $[0, \Delta t]$ and the space interval
 177 $[-X', X']$, where

$$-X' \leq \lambda_{min} \Delta t, \quad X' \geq \lambda_{max} \Delta t \quad (29)$$

178 being $\lambda_{min} \Delta t, \lambda_{max} \Delta t$ the positions of the slowest and the fastest wave given
 179 by the eigenvalues of (11). Integrating (25) over the control volume $[0, \Delta t] \times$
 180 $[-X', X']$

$$\int_{x'=-X'}^{x'=+X'} \int_{t=0}^{t=\Delta t} \left(\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{E}\mathbf{n}}{\partial x'} \right) dx' dt = 0 \quad (30)$$

181 the first term is

$$\int_{-X'}^{X'} dx' \int_0^{\Delta t} \left(\frac{\partial \mathbf{U}}{\partial t} \right) dt = \int_{-X'}^{X'} \mathbf{U}(x', t = \Delta t) dx' - X' (\mathbf{U}_j + \mathbf{U}_i) \quad (31)$$

Figure 3: Integration control volume defined by a time interval $[0, \Delta t]$ and a space interval $[-X', X']$

182 and the second term becomes

$$\int_0^{\Delta t} dt \int_{-X'}^{X'} \left(\frac{\partial \mathbf{En}}{\partial x'} \right) dx' = \delta(\mathbf{En})_k \Delta t \quad (32)$$

183 with $\delta(\mathbf{En})_k = (\mathbf{En})_j - (\mathbf{En})_i$. Reordering, the following expression of integral
184 volume for $\mathbf{U}(x', t)$ is obtained

$$\int_{-X'}^{+X'} \mathbf{U}(x', t = \Delta t) dx' = X' (\mathbf{U}_j + \mathbf{U}_i) - \delta(\mathbf{En})_k \Delta t \quad (33)$$

185 This result is of utmost importance, as when integrating the approximate
186 solution $\hat{\mathbf{U}}(x', t)$ over the same control volume, the Consistency Condition
187 [20]

$$\int_{-X'}^{+X'} \hat{\mathbf{U}}(x', t = \Delta t) dx' = \int_{-X'}^{+X'} \mathbf{U}(x', t = \Delta t) dx' \quad (34)$$

188 must be ensured.

189 In order to obtain the desirable conditions over the approximate Jacobian
190 matrix $\tilde{\mathbf{J}}_{\mathbf{n},k}$, (27) is integrated over the control volume pictured in Figure 3

$$\int_{x'=-X'}^{x'=+X'} \int_{t=0}^{t=\Delta t} \left(\frac{\partial \hat{\mathbf{U}}}{\partial t} + \tilde{\mathbf{J}}_{\mathbf{n},k} \frac{\partial \hat{\mathbf{U}}}{\partial x'} \right) dx' dt = 0 \quad (35)$$

191 the first term in (35) is

$$\int_{-X'}^{X'} dx' \int_0^{\Delta t} \left(\frac{\partial \hat{\mathbf{U}}}{\partial t} \right) dt = \int_{-X'}^{X'} \hat{\mathbf{U}}(x', t = \Delta t) dx' - X' (\mathbf{U}_j + \mathbf{U}_i) \quad (36)$$

192 and the second term in (35) becomes

$$\int_0^{\Delta t} dt \int_{-X'}^{X'} \left(\tilde{\mathbf{J}}_{\mathbf{n},k} \frac{\partial \hat{\mathbf{U}}}{\partial x'} \right) dx' = \tilde{\mathbf{J}}_{\mathbf{n},k} \delta \mathbf{U}_k \Delta t \quad (37)$$

193 with $\delta \mathbf{U}_k = \mathbf{U}_j - \mathbf{U}_i$. Reordering, the following expression is obtained in each
194 k edge

$$\int_{-X'}^{+X'} \hat{\mathbf{U}}(x', t = \Delta t) dx' = X' (\mathbf{U}_i + \mathbf{U}_j) - \tilde{\mathbf{J}}_{\mathbf{n},k} \delta \mathbf{U}_k \Delta t \quad (38)$$

195 and since we want to satisfy (34), the constraint that follows is

$$\delta(\mathbf{E}\mathbf{n})_k = \tilde{\mathbf{J}}_{\mathbf{n},k} \delta \mathbf{U}_k \quad (39)$$

196 where the approximate Jacobian is to be determined. In the present work,
197 the approximated Jacobian matrix, $\tilde{\mathbf{J}}_{\mathbf{n},k}$ is defined as

$$\tilde{\mathbf{J}}_{\mathbf{n}} = \begin{pmatrix} \tilde{\mathcal{J}}_1 & \tilde{\mathcal{J}}_2 \\ \tilde{\mathcal{J}}_3 & \tilde{\mathcal{J}}_4 \end{pmatrix}_{(3+N_p) \times (3+N_p)} \quad (40)$$

198 with

$$\begin{aligned}\tilde{\mathcal{J}}_1 &= \begin{pmatrix} 0 & n_x & n_y \\ \left(\frac{1}{2}g\tilde{h}(1+\tilde{r})-\tilde{u}^2\right)n_x - \tilde{u}\tilde{v}n_y & 2\tilde{u}n_x + \tilde{v}n_y & \tilde{u}n_y \\ \left(\frac{1}{2}g\tilde{h}(1+\tilde{r})-\tilde{v}^2\right)n_y - \tilde{u}\tilde{v}n_x & \tilde{v}n_x & \tilde{u}n_x + 2\tilde{v}n_y \end{pmatrix} \\ \tilde{\mathcal{J}}_2 &= \frac{1}{2}g\tilde{h}\tilde{r} \begin{pmatrix} 0 & \cdots & 0 \\ \Delta_1 n_x & \cdots & \Delta_{N_p} n_x \\ \Delta_1 n_y & \cdots & \Delta_{N_p} n_y \end{pmatrix}_{3 \times N_p}\end{aligned}\quad (41)$$

$$\tilde{\mathcal{J}}_3 = \frac{1}{\tilde{r}} \begin{pmatrix} -\tilde{\phi}_1 \tilde{\mathbf{u}}\mathbf{n} & \tilde{\phi}_1 n_x & \tilde{\phi}_1 n_y \\ \vdots & \vdots & \vdots \\ -\tilde{\phi}_{N_p} \tilde{\mathbf{u}}\mathbf{n} & \tilde{\phi}_{N_p} n_x & \tilde{\phi}_{N_p} n_y \end{pmatrix}_{N_p \times 3} \quad \tilde{\mathcal{J}}_4 = (\tilde{\mathbf{u}}\mathbf{n})\mathbf{I}_{N_p \times N_p}$$

199 with $\tilde{\mathbf{u}}\mathbf{n} = \tilde{u}n_x + \tilde{v}n_y$.

200

201 The definition of the average values in (41) is a difficult task due to the
202 non-linearity of the multi-phase problem. Following previous works [6, 22]
203 the average values are provided by setting an appropriate number of degrees
204 of freedom, and according to Roe [8] imposing that

$$\tilde{\mathbf{J}}_{\mathbf{n}}(\mathbf{U}_i, \mathbf{U}_i) = \tilde{\mathbf{J}}_{\mathbf{n}}(\mathbf{U}_i) \quad (42)$$

205 as well as (39). The Roe averages in the approximate Jacobian matrix $\tilde{\mathbf{J}}_{\mathbf{n},k}$
206 are given by

$$\begin{aligned}\tilde{r} &= \frac{h_i r_i + h_j r_j}{h_i + h_j}, & \tilde{h} &= \frac{h_i + h_j}{2} \\ \tilde{u} &= \frac{u_i \sqrt{h_i r_i} + u_j \sqrt{h_j r_j}}{\sqrt{h_i r_i} + \sqrt{h_j r_j}}, & \tilde{v} &= \frac{v_i \sqrt{h_i r_i} + v_j \sqrt{h_j r_j}}{\sqrt{h_i r_i} + \sqrt{h_j r_j}}\end{aligned}\quad (43)$$

$$\tilde{\phi}_p = \tilde{r} \frac{\phi_{i,p} h_i \sqrt{h_j r_j} + \phi_{j,p} h_j \sqrt{h_i r_i}}{h_i r_i \sqrt{h_j r_j} + h_j r_j \sqrt{h_i r_i}}$$

207 with $r_i = 1 + \sum_{p=1}^{N_p} \Delta_p \phi_{p,i}$ and $r_j = 1 + \sum_{p=1}^{N_p} \Delta_p \phi_{p,j}$.

208

209 With these average quantities the approximate Jacobian provides a set of
210 $3 + N_p$ real eigenvalues $\tilde{\lambda}^m$

$$\tilde{\lambda}_k^1 = (\tilde{\mathbf{u}}\mathbf{n} - \tilde{c})_k, \quad \tilde{\lambda}_k^2 = (\tilde{\mathbf{u}}\mathbf{n})_k, \quad \tilde{\lambda}_k^3 = (\tilde{\mathbf{u}}\mathbf{n} + \tilde{c})_k \quad \tilde{\lambda}_k^{3+p} = \tilde{\lambda}_k^2 \quad (44)$$

211 where

$$\tilde{c} = \sqrt{\frac{g}{2}\tilde{h}\left(\tilde{r} + 1 - \sum_{p=1}^{N_p} \tilde{\phi}_p \Delta_p\right)} \quad (45)$$

212 so that

$$\tilde{c}^2 = \frac{g}{4} \left(h_i + h_j + \frac{(h_i r_i + h_j r_j) (h_i \sqrt{h_j r_j} + h_j \sqrt{h_i r_i})}{h_j r_j \sqrt{h_i r_i} + h_i r_i \sqrt{h_j r_j}} \right) \quad (46)$$

213 is always positive, and a set of $3 + p$ eigenvectors $\mathbf{e}_k^m$. At this point, it is
 214 worth remarking that in the case of setting all Δ_p relative density values
 215 zero, all average values become equal to the solute transport equations for
 216 low concentration in [21].

217 The set of eigenvectors generates matrix $\tilde{\mathbf{P}} = (\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3, \dots, \mathbf{e}^{3+N_p})$,

$$\tilde{\mathbf{P}} = \begin{pmatrix} \tilde{\mathcal{P}}_1 & \tilde{\mathcal{P}}_2 \\ \tilde{\mathcal{P}}_3 & \tilde{\mathcal{P}}_4 \end{pmatrix}_{(3+N_p) \times (3+N_p)} \quad (47)$$

218 with

$$\tilde{\mathcal{P}}_1 = \begin{pmatrix} 1 & 0 & 1 \\ \tilde{u} - \tilde{c}n_x & -\tilde{c}n_y & \tilde{u} + \tilde{c}n_x \\ \tilde{v} - \tilde{c}n_y & \tilde{c}n_x & \tilde{v} + \tilde{c}n_y \end{pmatrix} \quad \tilde{\mathcal{P}}_2 = \frac{1}{r} \begin{pmatrix} \Delta_1 & \dots & \Delta_{N_p} \\ \Delta_1 \tilde{u} & \dots & \Delta_{N_p} \\ \Delta_1 \tilde{v} & \dots & \Delta_{N_p} \end{pmatrix}_{3 \times N_p} \quad (48)$$

219 and

$$\tilde{\mathcal{P}}_3 = \begin{pmatrix} \tilde{\phi}_1 & 0 & \tilde{\phi}_1 \\ \tilde{\phi}_2 & 0 & \tilde{\phi}_2 \\ \vdots & 0 & \vdots \\ \tilde{\phi}_{N_p-1} & 0 & \tilde{\phi}_{N_p-1} \\ \tilde{\phi}_{N_p} & 0 & \tilde{\phi}_{N_p} \end{pmatrix}_{N_p \times 3} \quad \tilde{\mathcal{P}}_4 = \frac{1}{2\tilde{r}} \begin{pmatrix} \tilde{m} + \Delta_1 \tilde{\phi}_1 & \dots & \Delta_{N_p} \tilde{\phi}_1 \\ \Delta_1 \tilde{\phi}_2 & \dots & \Delta_{N_p} \tilde{\phi}_2 \\ \vdots & \dots & \vdots \\ \Delta_1 \tilde{\phi}_{N_p-1} & \dots & \Delta_{N_p} \tilde{\phi}_{N_p-1} \\ \Delta_1 \tilde{\phi}_{N_p} & \dots & \tilde{m} + \Delta_{N_p} \tilde{\phi}_{N_p} \end{pmatrix}_{N_p \times N_p} \quad (49)$$

220 with $\tilde{m} = \tilde{r} + 1 + -\sum_{p=1}^{N_p} \tilde{\phi}_p \Delta_p$. Matrices $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{P}}^{-1}$ diagonalize $\tilde{\mathbf{J}}_k$

$$\tilde{\mathbf{P}}_k \tilde{\mathbf{J}}_{\mathbf{n},k} \tilde{\mathbf{P}}_k^{-1} = \tilde{\mathbf{\Lambda}}_k \quad (50)$$

221 with $\tilde{\mathbf{\Lambda}}_k$ is a diagonal matrix with eigenvalues $\tilde{\lambda}_k^m$ in the main diagonal

$$\tilde{\mathbf{\Lambda}}_k = \begin{pmatrix} \tilde{\lambda}^1 & 0 & 0 & 0 & 0 \\ 0 & \tilde{\lambda}^2 & 0 & 0 & 0 \\ 0 & 0 & \tilde{\lambda}^3 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & \tilde{\lambda}^{3+N_p} \end{pmatrix}_k \quad (51)$$

222 The difference in vector $\mathbf{U}$ cross the grid edge is projected onto the matrix
223 eigenvectors basis

$$\delta \mathbf{U}_k = \tilde{\mathbf{P}}_k \mathbf{A}_k \quad (52)$$

224 where $\mathbf{A}_k = (\alpha^1, \alpha^2, \alpha^3, \dots, \alpha^{3+N_p})_k^T$ is the matrix of the wave strengths,
225 given by

$$\begin{aligned} \alpha^1 &= \frac{\delta U_1((\tilde{r}+1)\tilde{c} + \tilde{m}\mathbf{un}) - (\delta U_2 n_x + \delta U_3 n_y)\tilde{m} - \tilde{r}\tilde{c}\sum_{p=1}^{N_p} \tilde{\phi}_p \Delta_p}{2\tilde{m}\tilde{c}} \\ \alpha^2 &= \frac{\delta U_1(\tilde{u}n_y - \tilde{v}n_x) + \delta U_3 n_x - \delta U_2 n_y}{\tilde{c}} \\ \alpha^3 &= \frac{\delta U_1((\tilde{r}+1)\tilde{c} - \tilde{m}\mathbf{un}) + (\delta U_2 n_x + \delta U_3 n_y)\tilde{m} - \tilde{r}\tilde{c}\sum_{p=1}^{N_p} \tilde{\phi}_p \Delta_p}{2\tilde{m}\tilde{c}} \\ \alpha^{3+p} &= -2 \frac{(\tilde{\phi}_p \delta U_1 - \delta U_{3+p} \tilde{r})}{\tilde{m}} \end{aligned} \quad (53)$$

226 Then, considering (39) and (52) the flux difference $\delta \mathbf{E}_k$ can be written as

$$\delta \mathbf{E}_k = \sum_{m=1}^{3+N_p} (\tilde{\lambda} \alpha \tilde{\mathbf{e}})_k^m \quad (54)$$

227 that in matricial form allows to express the approximate linear problem (27)
228 as

$$\frac{\partial \hat{\mathbf{U}}}{\partial t} + [\tilde{\mathbf{P}} \tilde{\mathbf{\Lambda}} \tilde{\mathbf{P}}^{-1}]_k \frac{\partial \hat{\mathbf{U}}}{\partial x'} = 0 \quad (55)$$

Figure 4: Values of the solution $\mathbf{U}$ in each wedge of the (x', t) plane for the subcritical case and $\tilde{\mathbf{u}}\mathbf{n} > 0$.

229 This expression is particularly useful as it provides the solution of the
 230 RP, that consists of a 3 wave structure. The weak solution is given by

$$\hat{\mathbf{U}}(x, t) = \begin{cases} \mathbf{U}_i^n & \text{if } x' - \tilde{\lambda}^1 t < 0 \\ \mathbf{U}^* & \text{if } x' - \tilde{\lambda}^1 t > 0 \text{ and } x' - \tilde{\lambda}^2 t < 0 \\ \mathbf{U}^{**} & \text{if } x' - \tilde{\lambda}^2 t > 0 \text{ and } x' - \tilde{\lambda}^3 t < 0 \\ \mathbf{U}_j^n & \text{if } x' - \tilde{\lambda}^3 t > 0 \end{cases} \quad (56)$$

231 as depicted in Figure 4. It is possible to define a continuous value of the
 232 approximate solution at $x' = 0$, that will be referred to as $\mathbf{U}^\downarrow$, with $\mathbf{U}^\downarrow =$
 233 $\hat{\mathbf{U}}(x' = 0, t)$,

$$\mathbf{U}^\downarrow = \mathbf{U}_i^n + \sum_{m=1, \tilde{\lambda} < 0}^{3+N_p} (\alpha \tilde{\mathbf{e}})_k^m = \mathbf{U}_j^n - \sum_{m=1, \tilde{\lambda} > 0}^{3+N_p} (\alpha \tilde{\mathbf{e}})_k^m \quad (57)$$

234 and an approximate value for $\mathbf{En}_k^\downarrow$ in (24):

$$\mathbf{En}^\downarrow = \mathbf{En}_i^n + \sum_{m=1, \tilde{\lambda} < 0}^{3+N_p} (\tilde{\lambda} \alpha \tilde{\mathbf{e}})_k^m \quad \mathbf{En}^\downarrow = \mathbf{En}_j^n - \sum_{m=1, \tilde{\lambda} > 0}^{3+N_p} (\tilde{\lambda} \alpha \tilde{\mathbf{e}})_k^m \quad (58)$$

235 4.1. The Godunov first order method

236 Averaging the previous definitions in (58) the approximate numerical flux
 237 $\mathbf{En}_k^\downarrow$ is commonly expressed as

$$\mathbf{En}_k^\downarrow = \frac{1}{2}(\mathbf{E}_i \mathbf{n}_k + \mathbf{E}_j \mathbf{n}_k) - \frac{1}{2} \sum_{m=1}^{3+N_p} (|\tilde{\lambda}| \alpha \tilde{\mathbf{e}})_k^m \quad (59)$$

238 The flux version of the Godunov first order method is

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \sum_{k=1}^{NE} \mathbf{En}_k^\downarrow \frac{\Delta t l_k}{A_i} \quad (60)$$

239 or in one dimension

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{\Delta x} (\mathbf{F}_{i+1/2}^\downarrow - \mathbf{F}_{i-1/2}^\downarrow) \quad (61)$$

240 Also, the Godunov first order method can be written splitting the flux
241 difference $\delta \mathbf{En}_k$ in right-going and left-going wave propagations,

$$\delta \mathbf{En}_k = \delta \mathbf{En}_{i,k}^+ + \delta \mathbf{En}_{j,k}^- \quad (62)$$

242 with

$$\delta \mathbf{En}_{i,k}^+ = \sum_{m=1}^{3+N_p} (\tilde{\lambda}^+ \alpha \tilde{\mathbf{e}})_k^m \quad \delta \mathbf{En}_{j,k}^- = \sum_{m=1}^{3+N_p} (\tilde{\lambda}^- \alpha \tilde{\mathbf{e}})_k^m \quad (63)$$

243 and $\tilde{\lambda}^\pm = \frac{1}{2}(\tilde{\lambda} \pm |\tilde{\lambda}|)$. The flux splitting version formulation is derived if,
244 following Godunov's method, the approximate solutions of all k RP's are
245 evolved for a time equal to the time step and the resulting solution is cell-
246 averaged [7]. The flux splitting version of the Godunov first order method
247 is

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \sum_{k=1}^{NE} \delta \mathbf{En}_{i,k}^- \frac{\Delta t l_k}{A_i} \quad (64)$$

248 or in one dimension

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{\Delta x} (\delta \mathbf{F}_{i-1/2}^- + \delta \mathbf{F}_{i+1/2}^+) \quad (65)$$

249 The flux formulation can be directly recovered using the following geo-
250 metrical property

$$\sum_{k=1}^{NE} \mathbf{E}_i \mathbf{n}_k = 0 \quad (66)$$

251 Including (66) in (64)

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \sum_{k=1}^{NE} (\mathbf{E}_i \mathbf{n}_k + \delta \mathbf{E} \mathbf{n}_{i,k}^-) \frac{\Delta t l_k}{A_i} = \mathbf{U}_i^n - \sum_{k=1}^{NE} \mathbf{E} \mathbf{n}_k^+ \frac{\Delta t l_k}{A_i} \quad (67)$$

252 the flux formulation is obtained following definitions in (58). Although both
 253 formulations are completely equivalent, it must be stressed that in practical
 254 computation differences appear when updating boundary cells. In boundary
 255 cells, the flux splitting formulation updates the conserved variables averaging
 256 the incoming information of the different RP's and no special treatment is
 257 required. But if using the flux formulation in a boundary cell, a ghost cell
 258 must be used to complete the definition in (66) over the entire cell, otherwise
 259 wrong results are obtained. Although it can be argued that the cost of
 260 computing the fluxes arising from ghost cells is not traumatic, the problem
 261 arises in deciding which are the correct values of the conserved variables that
 262 have to be imposed in those cells in complex situations.

263 4.2. Reduced Godunov first order method

264 The Roe's approximate solution provides the variation of the conserved
 265 variables on both sides of the initial discontinuity. Considering that one of
 266 the conserved variables is the mass, in some situations unphysical results
 267 in the concentration variable may appear [22]. In the case of the solute
 268 transport equations, a solute fix was derived in order to avoid such problems,
 269 ensuring both a monotone description of the concentration values and mass
 270 conservation. The procedure to provide the solute fix relies on the derivation
 271 of the conditions under which unphysical results appear, by comparing the
 272 integrals of the evolved solutions of the RP for the solute mass and water
 273 depth. This solution has proved accurate and robust and may be extended
 274 to the case of a multicomponent flow analyzed in this work, but the solute
 275 fix derivation is not simple [9].

276 Following [16], a simpler algorithm is proposed, that will be referred to
 277 as Reduced Godunov and presented for the first order method. Instead of
 278 updating all the conserved variables using the same procedure, the reduced
 279 set of conserved variables referred to the mixture $\mathbf{U}^r$

$$\mathbf{U}^r = \begin{pmatrix} hr \\ hur \\ hvr \end{pmatrix} \quad (68)$$

280 is computed using the procedure in (60) or (64) and an alternative procedure
 281 is used to compute the transported variables $\mathbf{U}^\phi$

$$\mathbf{U}^\phi = \begin{pmatrix} h\phi_1 \\ \vdots \\ h\phi_{N_p} \end{pmatrix} \quad (69)$$

282 The mixture mass discharge evaluated at $x' = 0$, is used and it will be
 283 referred to as $q_m^\downarrow = (h\mathbf{un}r)^\downarrow$,

$$q_m^\downarrow = q_{m,i} + \sum_{m=1}^{3+N_p} (\tilde{\lambda}^+ \alpha \tilde{e}_1)_k^m = q_{m,j} - \sum_{m=1}^{3+N_p} (\tilde{\lambda}^- \alpha \tilde{e}_1)_k^m \quad (70)$$

284 For the sake of clarity the Reduced Godunov first order method is presented
 285 in two steps. These are:

- 286 • Step (I). Compute the updating value of the reduced set of conserved
 287 variables $\mathbf{U}^r$

$$\mathbf{U}_i^{r,n+1} = \mathbf{U}_i^{r,n} - \sum_{k=1}^{NE} \mathbf{E} \mathbf{n}_k^{r,\downarrow} \frac{\Delta t l_k}{A_i} = \mathbf{U}_i^{r,n} - \sum_{k=1}^{NE} \delta \mathbf{E} \mathbf{n}_{i,k}^{r,-} \frac{\Delta t l_k}{A_i} \quad (71)$$

288 as in (60) or (64), using the complete set of eigenvalues and eigenvec-
 289 tors.

- 290 • Step (II). Compute the updating value of the set of transported vari-
 291 ables $\mathbf{U}_i^\phi$ as

$$\mathbf{U}_i^{\phi,n+1} = \mathbf{U}_i^{\phi,n} - \sum_{k=1}^{NE} (q\mathbf{W})_{i,k}^\downarrow \frac{\Delta t l_k}{A_i} \quad (72)$$

292 where

$$(q\mathbf{W})_{i,k}^\downarrow = \begin{cases} \frac{q_m^\downarrow}{r_i} \mathbf{W}_i & \text{if } q_m^\downarrow > 0 \\ \frac{q_m^\downarrow}{r_j} \mathbf{W}_j & \text{if } q_m^\downarrow < 0 \end{cases} \quad (73)$$

293 with $\mathbf{W}$ the set of concentrations $\mathbf{W} = (\phi_1, \dots, \phi_{N_p})^T$, or in one dimen-
 294 sion

$$\mathbf{U}_i^{\phi,n+1} = \mathbf{U}_i^{\phi,n+1} - \frac{\Delta t}{\Delta x} [(q\mathbf{W})_{i+1/2}^\downarrow - (q\mathbf{W})_{i-1/2}^\downarrow] \quad (74)$$

295 Using this procedure, the set of mass conservation equations for the dif-
 296 ferent components is solved in coupled way that ensures the correct bounds
 297 on the volumetric concentrations.

298 As stated previously, the flux formulation of the advected components
 299 requires the introduction of ghost cells in boundary cells. If using the flux
 300 splitting version for the set of reduced components $\mathbf{U}^r$, their definition is only
 301 necessary in the outflow region and remains quite simple, as only involves
 302 the set of reduced components $\mathbf{U}^\phi$.

303 4.3. Stability region and entropy fix

304 The updated value $\mathbf{U}_i^{n+1}$ is defined cell averaging the contributions of the
 305 local RPs, and in consequence the time step Δt is taken small enough so
 306 that there is no interaction of waves from the k neighboring RPs. According
 307 to [20], it is only necessary that the edge values of the approximate solution
 308 remain constant in time over the entire time step. In the 2D framework,
 309 considering unstructured meshes, the relevant distance, that will be referred
 310 to as χ_i in each cell i must consider the volume of the cell and the length of
 311 the shared k edges

$$\chi_i = \frac{A_i}{\max_{k=1,NE} l_k} \quad (75)$$

312 Considering that each k RP is used to deliver information to a pair of
 313 neighboring cells of different size, the distance $\min(A_i, A_j)/l_k$ is relevant, so
 314 in case that the water depth is greater than zero in all the regions of the RP
 315 solution the time step is limited by

$$\Delta t \leq CFL \Delta t^{\tilde{\lambda}} \quad \Delta t^{\tilde{\lambda}} = \frac{\min(\chi_i, \chi_j)}{\max |\tilde{\lambda}^m|} \quad (76)$$

316 with $CFL=1/2$ in the case of rectangular or structured triangular cells and,
 317 according to computational experience, CFL close to 1 for triangular un-
 318 structured grids as the construction of finite volume schemes from direct
 319 application of one-dimensional fluxes leads to reduced stability ranges [23].

320 The Roe method presented in this work provides linearised Riemann solu-
 321 tions that consist of discontinuous jumps only, and in practical computation
 322 transonic rarefactions encounter difficulties. As a result unphysical, discon-
 323 tinuous waves appear in such cases. Attending to the approximate solution
 324 given to the conserved variables, in situations where data that is near the

325 “vacuum state” or situations where there is a strong expansion, Roe average
 326 gives a nonphysical solution such as negative depth. To avoid unphysical re-
 327 sults derived from the linearization, the version of the Harten-Hyman entropy
 328 fix [24] is applied.

329 5. Augmented Roe’s Riemann solver.

330 If the problem includes bed slope and friction terms, the resulting system
 331 becomes

$$\frac{\partial}{\partial t} \int_{\Omega_i} \mathbf{U} d\Omega + \sum_{k=1}^{NE} (\mathbf{En})_k^\downarrow l_k = \int_{\Omega_i} \mathbf{S} d\Omega \quad (77)$$

332 where now, the approximate solutions derived to approximate $(\mathbf{En})_k^\downarrow$ must
 333 include the presence of the source term $\mathbf{S}$. In consequence, the local RP
 334 along the x' direction in (25) is reformulated as follows

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial (\mathbf{En})}{\partial x'} - \mathbf{S} = 0 \quad (78)$$

335 with the same initial conditions in (26). Integrating (78) over the control
 336 volume $[0, \Delta t] \times [-X', X']$

$$\int_{x'=-X'}^{x'=+X'} \int_{t=0}^{t=\Delta t} \left(\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{En}}{\partial x'} - \mathbf{S} \right) dx' dt = 0 \quad (79)$$

337 and reordering the following expression of integral volume for $\mathbf{U}(x', t)$ is
 338 obtained

$$\int_{-X'}^{+X'} \mathbf{U}(x', t = \Delta t) dx' = X' (\mathbf{U}_j + \mathbf{U}_i) - \delta(\mathbf{En})_k \Delta t + \int_{-X'}^{X'} \int_0^{\Delta t} \mathbf{S} dx' dt \quad (80)$$

339 Following the linearization concept, the source term is involved in the
 340 Riemann solver as a singular source at the discontinuity point $x' = 0$. Con-
 341 sidering that source terms are not necessarily constant in time, the following
 342 time linearization of the nonconservative term is applied [1, 7]

$$\int_{-X'}^{X'} \int_0^{\Delta t} \mathbf{S} dx' dt \approx \Delta t \int_{-X'}^{+X'} \mathbf{S}(x', 0) dx' = \Delta t \bar{\mathbf{S}}_k^n \quad (81)$$

343 where $\bar{\mathbf{S}}$ is a suitable numerical source vector, that can be projected in the
 344 normal direction writing it as

$$\bar{\mathbf{S}}_k = \begin{pmatrix} 0, & \bar{S}n_x, & \bar{S}n_y, & 0, & \dots, & 0 \end{pmatrix}_k^T \quad (82)$$

345 The integration of the source term is not a trivial task, and depending on
 346 the integration procedure significant differences appear in the results. The
 347 integration of the source term $\mathbf{S}$ is detailed in Appendix A.

348 For the approximate solution $\hat{\mathbf{U}}(x', t)$, the following formulation for the
 349 Consistency Condition is proposed [9, 7]

$$\int_{-X'}^{+X'} \hat{\mathbf{U}}(x', t = \Delta t) dx' = X (\mathbf{U}_i + \mathbf{U}_j) - \delta \mathbf{M}_{1,k} \Delta t \quad (83)$$

350 with $\delta \mathbf{M}_1 = (\delta \mathbf{E} \mathbf{n} - \bar{\mathbf{S}})$.

Figure 5: Integration control volume in interval $[0, \Delta t] \times [-X', X']$ and interpretation of the source term.

351 The local RP in (78) is approximated by using another constant coefficient
 352 linear problem

$$\frac{\partial \hat{\mathbf{U}}}{\partial t} + \mathbf{L}_{n,k} \frac{\partial \hat{\mathbf{U}}}{\partial x'} = 0 \quad (84)$$

353 with the same initial conditions. Integrating (84) over the control volume
 354 pictured in Figure 5, and comparing with (83) the constraint that follows is

$$\delta \mathbf{M}_{1,k} = \mathbf{L}_{\mathbf{n},k} \delta \mathbf{U}_k \quad (85)$$

355 The source vector $\bar{\mathbf{S}}$ defined at the grid edge is projected onto the matrix of
 356 eigenvectors basis

$$\bar{\mathbf{S}}_k = \tilde{\mathbf{P}}_k \mathbf{B}_{\bar{S},k} \quad \mathbf{B}_{\bar{S},k} = (\beta_{\bar{S}}^1 \quad \beta_{\bar{S}}^2 \quad \beta_{\bar{S}}^3 \dots \beta_{\bar{S}}^{3+N_p})_k^T \quad (86)$$

357 so that matrix $\mathbf{B}_{\bar{S},k}$ contains the source strengths associated to each wave,
 358 given by

$$\beta_{\bar{S}}^1 = -\frac{1}{2\bar{c}}\bar{S}, \quad \beta_{\bar{S}}^2 = 0, \quad \beta_{\bar{S}}^3 = -\beta_{\bar{S}}^1, \quad \beta_{\bar{S}}^{3+p} = 0 \quad (87)$$

359 Then, (85) is expressed as

$$\delta \mathbf{M}_{1,k} = \sum_{m=1}^{3+N_p} (\tilde{\lambda} \theta_1 \alpha \tilde{\mathbf{e}})_k^m \quad \theta_{1,k}^m = \left(1 - \frac{\beta_{\bar{S}}}{\tilde{\lambda} \alpha} \right)_k^m \quad (88)$$

360 or in matricial form [7]

$$\delta \mathbf{M}_{1,k} = [\tilde{\mathbf{P}} \tilde{\boldsymbol{\Lambda}} \tilde{\boldsymbol{\Theta}} \tilde{\mathbf{P}}^{-1}]_k \delta \mathbf{U}_k \quad (89)$$

361 where $\boldsymbol{\Theta}$ is a diagonal matrix

$$\boldsymbol{\Theta}_k = \begin{pmatrix} \theta^1 & 0 & 0 & 0 & 0 \\ 0 & \theta^2 & 0 & 0 & 0 \\ 0 & 0 & \theta^3 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & \theta^{3+N_p} \end{pmatrix}_k \quad (90)$$

362 transforming (78) in the following approximate linear problem

$$\frac{\partial \hat{\mathbf{U}}}{\partial t} + [\tilde{\mathbf{P}} \tilde{\boldsymbol{\Lambda}} \tilde{\boldsymbol{\Theta}} \tilde{\mathbf{P}}^{-1}]_k \frac{\partial \hat{\mathbf{U}}}{\partial x'} = 0 \quad (91)$$

363 Following [9] a four wave approximate solution can be built from (78).
 364 These solutions for $\hat{\mathbf{U}}(x', t)$ are governed by the celerities in $\tilde{\boldsymbol{\Lambda}}_k$ and each
 365 consists of five regions connected by four waves, one of them steady, located
 366 at $x' = 0$. This jump divides the solutions in a left and a right plane solution,
 367 associated to cells i and j respectively.

Figure 6: Values of the solution $\mathbf{U}$ in each wedge of the (x', t) plane for the subcritical case, $\tilde{\mathbf{u}}\mathbf{n} > 0$, where $\mathbf{U}_i^* = \mathbf{U}^{-\downarrow}$ and $\mathbf{U}_j^{**} = \mathbf{U}^{+\downarrow}$. A single value of $\mathbf{U}^\downarrow$ can not be defined.

368 Now it is not possible to define a continuous value of the approximate
 369 solution at $x = 0$. Two different solutions appear, one moving from the left
 370 region $x^- \rightarrow 0$

$$\mathbf{U}^{-\downarrow} = \mathbf{U}_i^n + \sum_{m=1, \tilde{\lambda} > 0}^{3+N_p} (\theta_1 \alpha \tilde{\mathbf{e}})_k^m \quad (92)$$

371 and the other moving from the right region $x^+ \leftarrow 0$.

$$\mathbf{U}^{+\downarrow} = \mathbf{U}_j^n - \sum_{m=1, \tilde{\lambda} < 0}^{3+N_p} (\theta_1 \alpha \tilde{\mathbf{e}})_k^m \quad (93)$$

372 Figure 6 shows the structure of the solution for the subcritical case with
 373 $\tilde{\mathbf{u}}\mathbf{n} > 0$, where $\mathbf{U}_i^* = \mathbf{U}^{-\downarrow}$ and $\mathbf{U}_j^{**} = \mathbf{U}^{+\downarrow}$. The complete description of the four
 374 wave approximate solutions is detailed in [9]. The definitions in (92) and (93)
 375 provide the variation of the conserved variables in presence of source terms,
 376 allowing to detail if a particular integration approach of (81) ensures a well
 377 balanced formulation (Appendix B).
 378 Also a discontinuous flux function for $\mathbf{M}_1$ can be defined at $x' = 0$, one
 379 associated to the left region $x^- \rightarrow 0$

$$\mathbf{M}_1^{-\downarrow} = \mathbf{E}\mathbf{n}_i^n + \sum_{m=1, \tilde{\lambda} > 0}^{3+N_p} (\tilde{\lambda} \theta_1 \alpha \tilde{\mathbf{e}})_k^m \quad (94)$$

380 and the other associated to the right region $x^+ \leftarrow 0$.

$$\mathbf{M}_1^{+\downarrow} = \mathbf{E}\mathbf{n}_j^n - \sum_{m=1, \tilde{\lambda} < 0}^{3+N_p} (\tilde{\lambda}\theta_1\alpha\tilde{\mathbf{e}})_k^m \quad (95)$$

381 with

$$\mathbf{M}_{1,k}^{+\downarrow} - \mathbf{M}_{1,k}^{-\downarrow} = \delta\mathbf{E}\mathbf{n}_k - \sum_{m=1}^{3+N_p} (\tilde{\lambda}\theta_1\alpha\tilde{\mathbf{e}})_k^m = \bar{\mathbf{S}}_k \quad (96)$$

382 that can be interpreted as the standard Rankine-Hugoniot relations (here-
383 after, RH) for an steady shock, of shock speed S

$$\mathbf{M}^{+\downarrow} - \mathbf{M}^{-\downarrow} - \bar{\mathbf{S}} = S(\mathbf{U}^{+\downarrow} - \mathbf{U}^{-\downarrow}) \quad (97)$$

384 with $S=0$.

385 When dealing with simulation problems that involve bed variations and
386 transient flow over a dry bed, a heavier restriction than the classical Courant-
387 Friedrichs-Lewy (CFL) condition [25] on the time step size that may lead
388 to inefficient computations. By analyzing the augmented Riemann solver
389 behind the numerical scheme it is possible to avoid the necessity of reducing
390 the time step and, at the same time, preventing instability and ensuring
391 conservation at all times by means of an efficient reduction of the source
392 wave strengths. These concepts are detailed in Appendix C.

393 5.1. The Godunov first order method with source terms

394 Now discrete fluxes and source terms are combined in the approximate solu-
395 tion, and a single value of the equivalent flux $\mathbf{M}_1$ can not be defined in the
396 discontinuity so an equivalent value to the flux function for the homogeneous
397 case as in (59) can not be provided. In that case, the flux version of the
398 Godunov first order method is

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \sum_{k=1}^{NE} \mathbf{M}_{1,k}^{-\downarrow} \frac{\Delta t l_k}{A_i} \quad (98)$$

399 or in one dimension

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{\Delta x} (\mathbf{M}_{1,i+1/2}^{-\downarrow} - \mathbf{M}_{1,i-1/2}^{+\downarrow}) \quad (99)$$

400 The augmented approach has allowed to write the flux version in (98) in a
 401 straightforward manner. This formulation is an important advance, as it is of
 402 general application to systems of conservation laws. This result is equivalent
 403 to the numerical flux-based volume scheme in [26], where searching for a
 404 balancing between source terms and flux gradients scheme (99) was written
 405 as

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - (\mathbf{F}_{i+1/2}^\downarrow - \mathbf{F}_{i-1/2}^\downarrow) \frac{\Delta t}{\Delta x} + (\bar{\mathbf{S}}_{i+1/2}^- + \bar{\mathbf{S}}_{i-1/2}^+) \frac{\Delta t}{\Delta x} \quad (100)$$

406 in which

$$\bar{\mathbf{S}}_{i+1/2}^\pm = (\mathbf{P} \mathbf{I}^\pm \mathbf{P}^{-1} \bar{\mathbf{S}})_{i+1/2} \quad (101)$$

407 and $\mathbf{I}^\pm = \mathbf{\Lambda}^{-1} \frac{1}{2} (\mathbf{\Lambda} \pm |\mathbf{\Lambda}|)$. The formulation in (100), although correct, does
 408 not explain directly the internal structure of the RP solution, which turns
 409 on an issue when analysing the solution in complex cases.

410 If the Godunov first order method is written splitting the flux $\delta \mathbf{M}_k$ in
 411 right-going and left-going wave propagations,

$$\delta \mathbf{M}_k = \delta \mathbf{M}_{1,i,k}^+ + \delta \mathbf{M}_{1,j,k}^- \quad (102)$$

412 with

$$\delta \mathbf{M}_{1,i,k}^+ = \sum_{m=1}^{3+N_p} (\tilde{\lambda}^+ \theta_1 \alpha \tilde{\mathbf{e}})_k^m \quad \delta \mathbf{M}_{1,j,k}^- = \sum_{m=1}^{3+N_p} (\tilde{\lambda}^- \theta_1 \alpha \tilde{\mathbf{e}})_k^m \quad (103)$$

413 the flux splitting version of the Godunov first order method is

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \sum_{k=1}^{NE} \delta \mathbf{M}_{1,i,k}^- \frac{\Delta t l_k}{A_i} \quad (104)$$

414 or in one dimension

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{\Delta x} (\delta \mathbf{M}_{1,i-1/2}^- + \delta \mathbf{M}_{1,i+1/2}^+) \quad (105)$$

415 5.2. Reduced Godunov first order method with source terms.

416 The proposed algorithm is an extension of the homogeneous case. The
 417 source terms are included to force changes in momentum, not affecting mass
 418 conservation. In this way, although flux $\mathbf{M}_1$ varies at $x' = 0$, the first compo-
 419 nent, the mixture mass discharge, $q_m^\downarrow$ remains constant and equal to

$$q_m^\downarrow = q_{m,i} + \sum_{m=1}^{3+N_p} (\tilde{\lambda}^+ \theta_1 \alpha \tilde{e}_1)_k^m = q_{m,j} - \sum_{m=1}^{3+N_p} (\tilde{\lambda}^- \theta_1 \alpha \tilde{e}_1)_k^m \quad (106)$$

420 The basic scheme is similar to the one presented for the homogeneous case,
421 and is defined in two steps. These are:

- 422 • Step (I). Compute the updating value of the reduced set of conserved
423 variables of the mixture $\mathbf{U}^r$

$$\mathbf{U}_i^{r,n+1} = \mathbf{U}_i^{r,n} - \sum_{k=1}^{NE} \mathbf{M}_{1,i,k}^{r,-} \frac{\Delta t l_k}{A_i} = \mathbf{U}_i^{r,n} - \sum_{k=1}^{NE} \delta \mathbf{M}_{1,i,k}^{r,-} \frac{\Delta t l_k}{A_i} \quad (107)$$

- 424 • Step (II). Compute the updating value of the set of transported vari-
425 ables $\mathbf{U}_i^\phi$ as

$$\mathbf{U}_i^{\phi,n+1} = \mathbf{U}_i^{\phi,n} - \sum_{k=1}^{NE} (q\mathbf{W})_{i,k}^\downarrow \frac{\Delta t l_k}{A_i} \quad (108)$$

426 with $(q\mathbf{W})^\downarrow$ as in (73), or in case that the k edge acts as a solid wall,
427 imposing $(q\mathbf{W})^\downarrow = 0$.

428 6. Augmented Roe methods including reaction terms.

429 When extending the problem to cases with reaction, the resulting system
430 becomes

$$\frac{\partial}{\partial t} \int_{\Omega_i} \mathbf{U} d\Omega + \sum_{k=1}^{NE} (\mathbf{En})_k^\downarrow l_k = \int_{\Omega_i} \mathbf{S} d\Omega + \int_{\Omega_i} \mathbf{R} d\Omega \quad (109)$$

431 and the local RP along the x' direction in (25) is now defined

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial (\mathbf{En})}{\partial x'} - \mathbf{S} - \mathbf{R} = 0 \quad (110)$$

432 The following time linearization of the nonconservative term [9]

$$\int_{-X'}^{X'} \int_0^{\Delta t} \mathbf{R} dx' dt \approx \Delta t \int_{-X'}^{+X'} \mathbf{R}(x', 0) dx' = \Delta t \bar{\mathbf{R}}_k^n \quad (111)$$

433 with

$$\bar{\mathbf{R}} = (\bar{R}_0, 0, 0, \bar{R}_1, \dots, \bar{R}_{N_p})^T \quad \bar{R}_0 = \sum_{p=1}^{N_p} \Delta_p \bar{R}_p \quad (112)$$

434 can be applied and projected onto the matrix eigenvectors basis

$$\bar{\mathbf{R}}_k^n = \tilde{\mathbf{P}}_k \mathbf{B}_{\bar{R},k} \quad (113)$$

435 where matrix $\mathbf{B}_{\bar{R}} = (\beta_{\bar{R}}^1 \ \beta_{\bar{R}}^2 \ \beta_{\bar{R}}^3 \dots \beta_{\bar{R}}^{3+N_p})_k^T$ contains the reaction source
436 wave strengths associated to each wave, given by

$$\begin{aligned} \beta_{\bar{R}}^1 &= \bar{R}_0 \frac{\tilde{m}(\tilde{c} + \tilde{\mathbf{u}}\tilde{\mathbf{n}}) + \tilde{c}(\sum_{p=1}^{N_p} \tilde{\phi}_p \Delta_p - \tilde{r})}{2\tilde{m}\tilde{c}} \\ \beta_{\bar{R}}^2 &= \bar{R}_0 \frac{\tilde{u}n_y - \tilde{v}n_x}{\tilde{c}} \\ \beta_{\bar{R}}^3 &= \bar{R}_0 \frac{\tilde{m}(\tilde{c} - \tilde{\mathbf{u}}\tilde{\mathbf{n}}) + \tilde{c}(\sum_{p=1}^{N_p} \tilde{\phi}_p \Delta_p - \tilde{r})}{2\tilde{m}\tilde{c}} \\ \beta_{\bar{R}}^{3+p} &= 2 \frac{\tilde{r}\bar{R}_p - \tilde{\phi}_p \bar{R}_0}{\tilde{m}} \end{aligned} \quad (114)$$

437 and transforming (110) in the following approximate linear problem

$$\frac{\partial \hat{\mathbf{U}}}{\partial t} + [\tilde{\mathbf{P}}\tilde{\mathbf{\Lambda}}\mathbf{\Theta}\tilde{\mathbf{P}}^{-1}]_k \frac{\partial \hat{\mathbf{U}}}{\partial x'} = 0 \quad (115)$$

438 where $\mathbf{\Theta}$ is a diagonal matrix with

$$\theta_{2,k}^m = \left(1 - \frac{\beta_{\bar{S}} + \beta_{\bar{R}}}{\tilde{\lambda}\alpha}\right)_k^m \quad (116)$$

439 and a global flux function can be expressed

$$\delta \mathbf{M}_{2,k} = (\delta \mathbf{E}\mathbf{n} - \bar{\mathbf{S}} - \bar{\mathbf{R}})_k = \sum_{m=1}^{3+N_p} (\tilde{\lambda} \theta_2 \alpha \tilde{\mathbf{e}})_k^m \quad (117)$$

440 with the following RH relations

$$\mathbf{M}_2^{+\downarrow} - \mathbf{M}_2^{-\downarrow} = \bar{\mathbf{S}} + \bar{\mathbf{R}} \quad (118)$$

441 *6.1. Reduced Godunov first order method*

442 When defining the Godunov first order method, the definitions in (92-95) and
 443 the numerical schemes in (98) and (104) can be directly applied by simply
 444 replacing in $\mathbf{M}$ the definitions of subindex 1 by the definitions of subindex 2.

445 The Reduced version, must consider that in this case the volumetric re-
 446 action term $\mathbf{R}$ affects to mass conservation, and two values of mixture mass
 447 discharge at $x = 0$ appear, $q_m^{-\downarrow}$ and $q_m^{+\downarrow}$ that can be expressed as

$$\begin{aligned} q_m^{-\downarrow} &= q_{m,i} + \sum_{m=1}^{3+N_p} (\tilde{\lambda}^+ \theta_2 \alpha \tilde{e}_1)_k^m = q_m^\downarrow + \sum_{m=1, \tilde{\lambda} < 0}^{3+N_p} (\beta_{\bar{\mathbf{R}}} \tilde{e}_1)_k^m \\ q_m^{+\downarrow} &= q_{m,j} - \sum_{m=1}^{3+N_p} (\tilde{\lambda}^- \theta_2 \alpha \tilde{e}_1)_k^m = q_m^\downarrow - \sum_{m=1, \tilde{\lambda} > 0}^{3+N_p} (\beta_{\bar{\mathbf{R}}} \tilde{e}_1)_k^m \end{aligned} \quad (119)$$

448 The basic scheme is presented in two steps. These are:

- 449 • Step (I). Compute the updating value of the reduced set of conserved
 450 variables $\mathbf{U}^r$

$$\mathbf{U}_i^{r,n+1} = \mathbf{U}_i^{r,n} - \sum_{k=1}^{NE} \mathbf{M}_{2,i,k}^{-\downarrow} \frac{\Delta t l_k}{A_i} = \mathbf{U}_i^{r,n} - \sum_{k=1}^{NE} \delta \mathbf{M}_{2,i,k}^{r,-} \frac{\Delta t l_k}{A_i} \quad (120)$$

- 451 • Step (II). Compute the updating value of the set of transported vari-
 452 ables $\mathbf{U}_i^\phi$ as

$$\mathbf{U}_i^{\phi,n+1} = \mathbf{U}_i^{\phi,n} - \sum_{k=1}^{NE} (q\mathbf{W})_{i,k}^\downarrow \frac{\Delta t l_k}{A_i} \quad (121)$$

453 where

$$(q\mathbf{W})_{i,k}^\downarrow = \begin{cases} \frac{q_m^{-\downarrow}}{r_i} \mathbf{W}_i & \text{if } q_m^\downarrow > 0 \\ (\frac{q_m^{+\downarrow}}{r_j} \mathbf{W}_j - \bar{\mathbf{R}}_k) & \text{if } q_m^\downarrow < 0 \end{cases} \quad (122)$$

454 It is worth noting that the time linearization of the reaction term in (111)
 455 can lead to negative values for the volumetric concentration in case that
 456 $\bar{R}_p < 0$. This can be avoided by using an implicit cell centered approximation
 457 of the reaction term [9]. In one dimensional cases of pure transport with
 458 null relative density, upwind approach of the reaction terms leads to more
 459 accurate results than the cell centered approximation. However, practical

460 computation shows that this is no longer true on 2D unstructured triangular
 461 meshes where both techniques provide similar accuracy [9]. In multicompo-
 462 nent problems with relevant relative density, the reaction terms affect the
 463 density and therefore the solution of the Riemann problem. In these cases a
 464 cell centered approximation is grossly incorrect.

465 As the RP solution is linearized and explicit, unphysical values for the
 466 volumetric concentration can be avoided by decreasing numerically the value
 467 of the reaction terms $\bar{R} < 0$ ensuring correct values of transported concentra-
 468 tions in (122) with almost no computational cost.

469 7. Applications

470 7.1. Quiescent equilibrium with four mixture constituents

471 The first test is the simulation of an initially steady configuration to prove
 472 the preservation of motionless steady state, where the mixture has a uniform
 473 density. A rectangular tank of length $L=100\text{m}$, with a sinusoidal frictionless
 474 bed containing a liquid-species mixture is considered. The bed topography
 475 is defined by

$$z(x) = 0.2 \left\{ 1 - \cos \left(\frac{2\pi x}{L} \right) \right\} \quad (123)$$

476 The mixture is composed by four dissolved constituents in water. The solute
 477 distribution of each constituent is given by

$$\begin{aligned} \phi_1(x) &= 0.1 \cos^2 \left(\frac{2\pi x}{L} \right) & \phi_2(x) &= 0.1 \sin^2 \left(\frac{2\pi x}{L} \right) \\ \phi_3(x) &= 0.2 \tanh \left(\frac{2x}{L} \right) & \phi_4(x) &= 0.2 \left(1 - \tanh \left(\frac{2x}{L} \right) \right) \end{aligned} \quad (124)$$

478 and the relative density of each phase is $\Delta_1 = \Delta_2 = 1.0$, $\Delta_3 = \Delta_4 = 0.2$. The
 479 water level surface is constant and given by $h + z = 1\text{m}$.

480 Figure 7 (left) shows the bed elevation and the plot of the water level
 481 surface after 50 seconds obtained with approaches (137) ($-\circ-$) and in (133)
 482 ($-\bullet-$), dividing the domain in 1000 uniform cells. The volumetric concentra-
 483 tion of each phase after 50 seconds in the same mesh is displayed in Figure
 484 7, (right), using approaches (137) and (133), represented by symbols ($-\circ-$)
 485 and ($-\bullet-$) for ϕ_1 , ($-\triangle-$) and in (133) ($-\blacktriangle-$) for ϕ_2 , ($-\diamond-$) and ($-\times-$)
 486 for ϕ_3 and ($-\square-$) and i ($-\blacksquare-$) for ϕ_4 . Figures 8 and 9 show the results for
 487 the same test case reducing the number of cells to 100 and 10 respectively.

Figure 7: Numerical solutions for the water level surface (left) and for the solute distribution of each constituent (right) after 50 seconds setting $\Delta x=0.01$ m.

Figure 8: Numerical solutions for the water level surface (left) and for the solute distribution of each constituent (right) after 50 seconds setting $\Delta x=0.1$ m.

488 In all cases initial equilibrium is kept exact in time with independence of the
 489 cell size, as in this particular case both integral procedures are equivalent.
 490 Regarding the computational cost, negligible differences appear when using
 491 approaches (133) or (137), due to the reduced number of operations needed
 492 to formulate the latter in comparison with the overall number of operations
 493 required, in special in 2D cases.

494 7.2. Quiescent equilibrium with four mixture constituents and wetting/drying

495 The second test is the simulation of an initially steady configuration
 496 to proof the preservation of motionless steady state in presence of wet-
 497 ting/drying fronts. In this 1d academic test, the numerical experiment con-
 498 sists of flat level still surface, $d = h + z = 3$ m, with the following function for
 499 the bed level

Figure 9: Numerical solutions for the water level surface (left) and for the solute distribution of each constituent (right) after 50 seconds setting $\Delta x=1.0\text{m}$.

$$z(x) = 2.5 + \sin(2\phi|x - 500|/100) + \cos(2\phi x/1000) \quad (125)$$

500 The mixture, with constant density, presents a solute distribution for each
501 constituent given by

$$\begin{aligned} \phi_1(x) &= 0.1 \cos^2\left(\frac{2\pi x}{1000}\right) & \phi_2(x) &= 0.1 \sin^2\left(\frac{2\pi x}{1000}\right) \\ \phi_3(x) &= 0.2 \tanh\left(\frac{2x}{500}\right) & \phi_4(x) &= 0.2 \left(1 - \tanh\left(\frac{2x}{500}\right)\right) \end{aligned} \quad (126)$$

502 and the relative density of each phase is $\Delta_1 = \Delta_2 = 1.0$, $\Delta_3 = \Delta_4 = 0.2$.
503 The domain, 1000 meters long, is totally closed. As no external influence is
504 present, no change in free surface or velocity field (nil in this case) must be
505 observed. Figure 10 shows the initial plot of the water level surface and bed
506 elevation.

507 In cases of initial equilibrium and still water it is possible to integrate
508 exactly the source term (81) forces using integral procedure in (137). Another
509 possibility is to use the integral procedure in (133), based on the differential
510 form of the bed slope term in (6). Figure 11 (a) and (b) show a detail of
511 the numerical solutions for the water level surface $h+z$ and for the discharge
512 hu respectively, using the integral procedure in (137) ($- \circ -$) and in (133)
513 ($- \bullet -$) after 50 seconds using a mesh with $\Delta x=1\text{m}$ and neglecting the flux fix
514 in (143,144), that avoids negative values of water depth. Figure 11 (c) and
515 (d) show a detail of the numerical solutions for the constituent distributions
516 using the integral procedure in (133) and in (137) respectively, under the

Figure 10: Initial condition for the water level surface and bed elevation.

517 same conditions. From the results it is obvious that steady state is only
 518 maintained when using the integration form in (137). Otherwise spurious
 519 velocities and changes in free surface appear with no physical meaning. If
 520 the integral procedure in (137) is combined with the flux fix, motionless
 521 steady state is preserved. The same conclusions are derived if the cell size is
 522 incremented to 10 and 50 meters as shown in Figures 12 and 13 respectively.

523 7.3. Riemann problems with frictionless discontinuous bed

524 In this section we present comparisons among exact solutions of the Rie-
 525 mann problems and numerical solutions. They can be constructed adapting
 526 solution procedures [27, 28] for the shallow water equations with constant
 527 density. The results are presented in the form of plots of the water level
 528 surface, Froude numbers Fr_x and Fr_y and volumetric concentration. The
 529 examples are chosen to represent different combinations of wave patterns.
 530 The acceleration due to gravity is $g = 9.8m^2/s$. Only one component ϕ_1 is
 531 considered with a relative density value of $\Delta_1 = 1.0$. In all cases the bottom
 532 step is positioned at $x = 0$ and has a variable height. In all cases $\Delta x = 1$ and
 533 $CFL = 1$.

534 Test case 1 is a dam-break type problem, with a combination of rarefac-
 535 tion and shock waves. The initial condition consists of two columns of water
 536 of different height, different density and zero velocity in the x direction. The
 537 solution, presented in Figure 14, contains a left moving rarefaction wave, a
 538 stationary shock at the step, a contact wave and a right-moving shock wave.
 539 The presence of the step leads to a reduction of the total water height run-
 540 ning to the right as compared to the flat bottom case. This reduction is
 541 due to the stationary shock, which dissipates part of the energy of the shock

Figure 11: Exact solutions (—) and numerical solutions for (a) $h + z$ and for (b) hu using the integral procedure in (133) (— • —) and in (137) (— ○ —). Exact solutions (—) and numerical solutions for constituent distributions using the integrals in (133) and (137) in (c) and (d) respectively. $\Delta x = 1\text{m}$ and $t = 50\text{s}$.

Figure 12: Exact solutions (—) and numerical solutions for (a) $h+z$ and for (b) hu using the integral procedure in (133) ($-\bullet-$) and in (137) ($-\circ-$). Exact solutions (—) and numerical solutions for constituent distributions using the integrals in (133) and (137) in (c) and (d) respectively. $\Delta x=10\text{m}$ and $t=50\text{s}$.

Figure 13: Exact solutions (—) and numerical solutions for (a) $h + z$ and for (b) hu using the integral procedure in (133) ($-\bullet-$) and in (137) ($-\circ-$). Exact solutions (—) and numerical solutions for constituent distributions using the integrals in (133) and (137) in (c) and (d) respectively. $\Delta x=50\text{m}$ and $t=50\text{s}$.

Table 1: Summary of test cases.

Test	h_L	h_R	u_L	u_R	v_L	v_R	z_L	z_R	$\phi_{1,L}$	$\phi_{1,R}$	R_1
1	4.0	0.779230694	0.0	0.0	0.4	0.1	0.0	1.5	0.8	0	0
2	1.5	0.240807266	2.0	0.0	0.2	0.1	0.0	2.0	0.8	0	0
3	1.1	0.494577290	4.9	6.5	0.1	0.5	0.2	0.0	1.0	0	0
4	4.0	0.970574080	0.0	0.0	0.4	0.1	0.0	1.5	0.5	0	-0.5
5	1.5	0.267834540	2.0	0.0	0.2	0.1	0.0	2.0	0.8	0	-0.05
6	1.1	0.443386909	4.9	6.5	0.1	0.5	0.2	0.0	1.0	0	-0.5
7	4.0	1.418034802	0.0	0.0	0.4	0.1	0.0	1.5	0.5	0	-4.0

542 wave. The density of the mixture changes along the contact wave, producing
543 a discontinuity in the level surface.

544 Figure 14 shows how both numerical solution of the RP using the Go-
545 dunov method and the Reduced Godunov method provide accurate and indis-
546 tinguishable results. The position of the expansion fan, of the contact wave
547 and of the shock are correct and the discharge does not present oscillations
548 in the origin.

549 Figure 15 shows the maximum value of concentration in time given by
550 the Godunov method ($- \circ -$) and by the reduced Godunov method ($- \bullet$
551 $-$). As displayed in Figure 15 the Godunov method is not able to avoid
552 incorrect values of concentration. On the contrary, the Reduced version
553 always guarantees correctly bounded results.

554 Test case 2 is a three shock case with a convergent flow. The initial
555 condition consists of two columns of water of different height, different density
556 and velocity. The solution is presented in Figure 16, and contains a left-
557 moving shock, a stationary shock at the step, a contact wave and a right-
558 moving shock wave. As in the previous example, the step acts as an energy
559 dissipation mechanism. Both the numerical solution of the RP using the
560 Godunov method and the Reduced Godunov method provide accurate and
561 indistinguishable results, as in this case the concentration remains correctly
562 bounded in both cases.

563 In test case 3 a supercritical motion from left to right is considered. The
564 presence of the step introduces no limitation in the signal propagation up-
565 stream, and its effect is in dissipating energy by the stationary shock at the
566 step. The solution, presented in Figure 17, contains a stationary shock at

Figure 14: Test case 1: comparison between exact (—) and numerical solutions at $t=50s$ obtained with the Godunov method (— $\circ$ —) and the Reduced Godunov method (— $\bullet$ —)

Figure 15: Test case 1: numerical solutions the maximum value of ϕ_1 in time obtained with the Godunov method (— $\circ$ —) and the Reduced Godunov method (— $\bullet$ —)

Figure 16: Test case 2: comparison between exact (—) and numerical solutions at $t=50s$ obtained with the Godunov method ($- \circ -$) and the Reduced Godunov method ($- \bullet -$)

Figure 17: Test case 3: comparison between exact (—) and numerical solutions at $t=50s$ obtained with the Godunov method ($- \circ -$) and the Reduced Godunov method ($- \bullet -$)

the step, a right moving rarefaction wave, a contact wave and a right-moving shock wave. Both numerical solution of the RP using the Godunov method and the Reduced Godunov method provide accurate and indistinguishable results, as in this case concentration remains correctly bounded in both cases.

Test case 4 is a dam-break type problem, with a combination of rarefaction and shock waves, as test case 1, but the discontinuity includes the presence of sink term of value $R=-0.5$. The solution, presented in Figure 18, contains a left moving rarefaction wave, a stationary shock at the step, a contact wave and a right-moving shock wave. The presence of the source term in the discontinuity reduces the density of the mixture, generating an extra discontinuity in the value of the concentration. Both the numerical solution of the RP using the Godunov method and the Reduced Godunov method provide accurate results, although only the Reduced version is able to keep bounded values of concentration all the time as shown in Figure 19. If the Reduced Godunov method is combined with a centered discretization of the reaction term, numerical oscillations in the solution appear in the vicinity of the location of the source term, as shown in Figure 20.

Figure 18: Test case 4: comparison between exact (—) and numerical solutions at $t=50$ s obtained with the Godunov method (— $\circ$ —) and the Reduced Godunov method (— $\bullet$ —)

Figure 19: Test case 4: numerical solutions the maximum value of ϕ_1 in time obtained with the Godunov method (— $\circ$ —) and the Reduced Godunov method (— $\bullet$ —)

Figure 20: Test case 4: comparison between exact (—) and numerical solutions at $t=50s$ obtained with the Reduced Godunov method and a centered discretization of the reaction term (— $\circ$ —)

584 Test case 5 is also a three shock case with a convergent flow, where the
 585 discontinuity includes the presence of a sink term of value $R=-0.05$. The solu-
 586 tion is presented in Figure 21, and contains a left-moving shock, a stationary
 587 shock at the step, a contact wave and a right-moving shock wave. As in the
 588 previous example, the presence of the reaction source term in the disconti-
 589 nuity reduces the density of the mixture, generating an extra discontinuity
 590 in the value of the concentration. In this case both numerical solution of the
 591 RP using the Godunov method and the Reduced Godunov method provide
 592 correctly bounded concentration values, but if comparing them, it can be
 593 advertised that the Reduced Godunov method gives more accurate results.

594 In test case 6 is similar to case 3, with a supercritical motion from left
 595 to right but including the presence of sink term of value $R=-0.05$. The pres-
 596 ence of the reaction source term in the discontinuity does not introduce any
 597 limitation in the signal propagation upstream, and its effect is to reduce the
 598 density of the mixture once the flow passes through the step. The solution,
 599 presented in Fig. 23. Both numerical solution of the RP using the Godunov
 600 method and the Reduced Godunov method provide accurate and indistin-
 601 guishable results, as in this case concentration remains correctly bounded in
 602 both cases.

603 Test case 7 is an example of an inaccurate integration of the reaction term
 604 in a dam-break type problem. The reaction term imposed in the numerical
 605 scheme is greater than the reaction limit given by the exact solution, and
 606 in consequence, if it is not limited, negative values of concentration appear.
 607 The solution is presented in Figure 23 where the exact solution predicts the

Figure 21: Test case 5: comparison between exact (—) and numerical solutions at $t=50s$ obtained with the Godunov method ($- \circ -$) and the Reduced Godunov method ($- \bullet -$)

Figure 22: Test case 6: comparison between exact (—) and numerical solutions at $t=50s$ obtained with the Godunov method ($- \circ -$) and the Reduced Godunov method ($- \bullet -$)

Figure 23: Test case 7: comparison between exact (—) and numerical solutions at $t=50s$ obtained with the Godunov method ($- \circ -$) and the Reduced Godunov method ($- \bullet -$)

complete removal of concentration once the flow passes the original discontinuity. The Reduced Godunov method not only provides correctly bounded concentration values, but also makes very easy to identify the correct reduction of the reaction term, providing accurate results. On the other hand, the complete Godunov method does not allow a straightforward way to reduce the reaction term, requiring a detailed analysis of the approximate solution.

7.4. Riemann problems with frictionless discontinuous bed and four constituents

In this section we present comparisons among exact solutions and numerical solutions for test cases 1 and 2 using the Reduced Godunov method. The same constant density in the left initial region of each RP is generated, this time given by the mixture of four dissolved constituents, defined by

$$\begin{aligned}\phi_1(x) &= 0.1 \cos^2\left(\frac{2\pi(-x)}{1000}\right) & \phi_2(x) &= 0.1 \sin^2\left(\frac{2\pi(-x)}{1000}\right) \\ \phi_3(x) &= 0.2 \tanh\left(\frac{2(-x)}{500}\right) & \phi_4(x) &= 0.2 \left(1 - \tanh\left(\frac{2(-x)}{500}\right)\right)\end{aligned}\quad (127)$$

for $x < 0$, and nil in any other case. The relative density of each phase is $\Delta_1 = \Delta_2 = 2.5$, $\Delta_3 = \Delta_4 = 0.75$.

Figure 24 (a) displays the exact solutions (—) and the numerical solutions for the water surface elevation at $t=60$ s using $(-\triangle-)$ $\Delta x=10$ m, $(-\diamond-)$ $\Delta x=1$ m and $(-\square-)$ $\Delta x=0.1$ m for test case 1. Rarefaction and shock waves are captured more accurately as mesh refinement grows. Figure 24 (b) compares the exact solutions (—) for the four components with the numerical results at $t=60$ s, represented by symbols $(-\circ-)$ for ϕ_1 , $(-\triangle-)$ for ϕ_2 , $(-\diamond-)$ for ϕ_3 and $(-\square-)$ for ϕ_4 , using $\Delta x=10$ m. Figure 24 (c) and (d) represent the same results using $\Delta x=1$ m and $\Delta x=0.1$ m respectively. The solution for the dissolved components converges to the exact solution, leading to more diffusive results ϕ_2 and ϕ_4 , where the initial solute front is discontinuous. The solution for the water depth and velocity is identical to the one provided with one component analyzed in the previous section.

In test case 2, the same analysis is performed. Figure 25 (a) compares the water surface elevation after 100 seconds using $(-\triangle-)$ $\Delta x=10$ m, $(-\diamond-)$ $\Delta x=1$ m and $(-\square-)$ $\Delta x=0.1$ m. All shock waves are well reproduced as mesh is refined. Figure 25 (b) compares the solutions for the four components after 100 seconds, represented by symbols $(-\circ-)$ for ϕ_1 , $(-\triangle-)$ for ϕ_2 , $(-\diamond-)$ for ϕ_3 and $(-\square-)$ for ϕ_4 , using $\Delta x=10$ m. Figure 25 (c) and (d) represent the same results using $\Delta x=1$ m and $\Delta x=0.1$ m respectively. Again, the numerical solution converges to the exact solution, leading to more diffusive results ϕ_2 and ϕ_4 (discontinuous front). The water depth and velocities are identical to those predicted by the case one single component analyzed in the previous section.

7.5. 2D Interaction with obstacles

The performance of the Reduced Godunov scheme for multicomponent flow is analyzed by means of a dam break numerical experiment in a rectangular tank, 7 m long and 4m wide. The tank bed include two obstacles. The bed elevation is given by

Figure 24: Exact solutions (—) and numerical solutions for (a) $h + z$ using $\Delta x = 10, 1, 0.1$ m. Exact solutions (—) and numerical solutions using (b) $\Delta x = 10$ m, (c) $\Delta x = 1$ m and (d) $\Delta x = 0.1$ m. Time $t = 60$ s.

Figure 25: Exact solutions (—) and numerical solutions for (a) $h + z$ using $\Delta x = 10, 1, 0.1$ m. Exact solutions (—) and numerical solutions using (b) $\Delta x = 10$ m, (c) $\Delta x = 1$ m and (d) $\Delta x = 0.1$ m. Time $t = 100$ s.

$$z(x, y) = 0.4 \exp(-a_1^2/0.01) + 0.3 \exp(-a_2^2/0.009) \quad (128)$$

where a_1 and a_2 represent the following distances

$$a_1 = \sqrt{(x - 5.0)^2 + (y - 1.0)^2} \quad a_2 = \sqrt{(x - 5.0)^2 + (y - 3.0)^2} \quad (129)$$

The initial mixture is at rest and with a surface level defined by

$$d(x, y) = \begin{cases} 0.04m & \text{if } x < 2.0 \\ 0.01m & \text{if } x > 2.0 \end{cases} \quad (130)$$

as shown in Figure 26. Also, an initial discontinuity is assumed in the density produced by the following discontinuity in volumetric concentration

$$\phi_1 = \begin{cases} 0.2 & \text{if } x < 2.0 \\ 0.0 & \text{if } x > 2.0 \end{cases} \quad (131)$$

where component ϕ_1 has a relative density $\Delta_1 = 0.2$. The bed roughness is characterized by a Glauckler-Manning number of $n=0.01\text{sm}^{-1/3}$.

The domain is assumed bounded by solid walls and has been discretized using two structured triangular meshes, M_1 and M_2 with $140 \times 80 \times 2$ and $280 \times 160 \times 2$ cells respectively.

The results have been computed using the reduced Godunov scheme and a dynamic time step based on $CFL = 0.5$. Figure 27 shows four snapshots of the solution at $t=2.5\text{s}$, $t=3.5\text{s}$, $t=4.5\text{s}$ and $t=9.5\text{s}$ where 3D plant view of the mixture level surface and the module of the velocity are plotted together with the module of the velocity. Subfigures (a),(c),(e) and (g) on the left correspond to the computation on mesh M_1 whereas subfigures (b),(d),(f) and (h) on the right correspond to the computation on the finer mesh M_2 . The results show that the initial discontinuities produce a complex pattern of shocks and rarefactions that interact with the bed irregularities following the same tendency on both grids. The waves travel at the same speed and the areas covered or uncovered by the wetting/drying fronts are almost identical. The observed surface waves from M_1 are more smeared than in the results from M_2 as expected. This has not a great influence on the velocity fields obtained that are very similar.

Figure 28 shows 3D plant view of the mixture level surface and contour plots of the volumetric concentration at times $t=6.5\text{s}$, $t=7.5\text{s}$, $t=8.5\text{s}$ and

Figure 26: 2D dam break problem: initial water level surface

675 $t=9.5s$. Again the left and right subfigures stand for the results obtained on
 676 the M_1 and M_2 grids respectively. These figures are useful to see the influence
 677 of the density gradients on the surface waves (subfigures (e)-(f) and (g)-(h))
 678 when the solution is far from the initial condition and the shock waves are
 679 weaker.

680 Finally, Figure29 shows a comparison between the numerical solutions of
 681 the mixture level in time computed on mesh M_1 and M_2 at $P_1 = (4.5, 3)$,
 682 $P_2 = (5.5, 3)$ on the left and right of the upper obstacle respectively, and
 683 $P_3 = (4.5, 1)$ and $P_4 = (5.5, 1)$ on the left and right of the lower obstacle.

684 8. Conclusions

685 A conservative and robust numerical scheme able to handle two-dimensional
 686 shallow flows with horizontal density variations in complex situations has
 687 been presented. It can be useful to provide mathematical formulation of
 688 shallow flows in realistic environmental analysis. The system of equations is
 689 formed by the 2D shallow water equations, retaining horizontal density vari-
 690 ation for mass and momentum of the mixture, supplemented by equations for
 691 the mass or volume fraction of the mixture constituents. The original Roe's
 692 average values for clean water have been extended to include the variation of
 693 density, written in terms of volumetric concentration of the different compo-
 694 nents. The resulting approximate Jacobian matrix satisfies the consistency
 695 condition, which is a prior requirement when defining approximate solvers.

696 The augmented Riemann solver previously proposed to deal with source
 697 terms is applied in the present work. This tool has proved successful to define

Figure 27: 2D dam break problem: 3D plant view of the mixture level surface and plots of the module of the velocity at time $t=2.5$ using meshes (a) M_1 and (b) M_2 , at time $t=3.5$ using meshes (c) M_1 and (d) M_2 , at time $t=4.5$ using meshes (e) M_1 and (f) M_2 and at time $t=9.5$ using meshes (g) M_1 and (h) M_2 .

Figure 28: 2D dam break problem: 3D plant view of the mixture level surface and contour plots of the volumetric concentration at time $t=2.5$ using meshes (a) M_1 and (b) M_2 , at time $t=3.5$ using meshes (c) M_1 and (d) M_2 , at time $t=4.5$ using meshes (e) M_1 and (f) M_2 and at time $t=9.5$ using meshes (g) M_1 and (h) M_2 .

Figure 29: Test case 7: comparison between the numerical solutions of the mixture level in time, computed on mesh M_1 ($-\circ-$) and M_2 ($-\bullet-$) at (a) $P_1 = (4.5, 3)$, (b) $P_2 = (5.5, 3)$, (c) $P_3 = (4.5, 1)$ and (d) $P_4 = (5.5, 1)$.

the transition of the conserved variables along each RP. As a result, it has been possible to define correction rules over the source term integrals.

A new Reduced Godunov scheme has been proposed to control the bounds of the volumetric concentration variables in the simplest way whilst retaining the coupled formulation and without increasing numerical diffusion.

The technique has been used to simulate both steady and unsteady problems of variable density with exact solution as well as academic two-dimensional dam break problems with variable bed. The obtained results point out that the new method is able to predict faithfully the overall behavior of the solution and of any type of waves.

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10. Appendix A. Integration of the bed and friction source terms.

The source term $\mathbf{S}$ can be expressed in the normal direction as

$$\bar{\mathbf{S}}_k = \left(0, \bar{S}n_x, \bar{S}n_y, 0, \dots, 0 \right)_k^T, \quad \bar{\mathbf{S}} = \bar{\mathbf{S}}_z + \bar{\mathbf{S}}_f \quad (132)$$

where $\bar{\mathbf{S}}_z$ refers to the bed slope source term, and $\bar{\mathbf{S}}_f$ refers to friction effects near the bed. Assuming a piecewise representation of the bed level, the bed slope source term in the normal direction (6) can be integrated as

$$\bar{S}_z^a = -g\tilde{h}\tilde{r}\delta z \quad (133)$$

Another possibility is to use the physical definition of the hydrostatic force exerted over a bed discontinuity. Calling T the thrust term exerted over the bed step [7], the bed slope source term is written as

$$\bar{S}_z^b = -\frac{T}{\rho_w} \quad (134)$$

where the thrust term T is written in a general form as:

$$T = g\rho_w r_l \left(h_l - \frac{|\delta z'|}{2} \right) \delta z' \quad (135)$$

with

$$l = \begin{cases} i & \text{if } \delta z \geq 0 \\ j & \text{if } \delta z < 0 \end{cases} \quad \delta z' = \begin{cases} h_i & \text{if } \delta z \geq 0 \text{ and } d_i < z_j \\ h_j & \text{if } \delta z < 0 \text{ and } d_j < z_i \\ \delta z & \text{otherwise} \end{cases} \quad (136)$$

721 where $d = (h + z)$ is the water surface level.

722 Both approaches are then blended to define $\bar{S}_z$ in (132) as:

$$\bar{S}_z = \begin{cases} -T_{max}/\rho_w & \text{if } \delta d \delta z \geq 0 \text{ and } \tilde{u}\delta z > 0 \\ -T/\rho_w & \text{otherwise} \end{cases} \quad (137)$$

723 and

$$T_{max} = \begin{cases} g\tilde{h}(\delta z) & \text{if } |g\tilde{h}(\delta z)| \geq |T| \\ T & \text{otherwise} \end{cases} \quad (138)$$

724 that considers the problems associated to flow across an upward step in
725 overtopping waves.

726 Regarding the friction term, the discretization based on [29] is applied

$$\bar{S}_f \approx C_f \mathbf{u} \mathbf{n} |\mathbf{u}_{min}| d_n \quad (139)$$

727 with $|\mathbf{u}_{min}| = \min(|\mathbf{u}|_i, |\mathbf{u}|_j)$ and d_n the normal distance between center cells.

728 11. Appendix B. Well-balanced scheme.

729 In cases of still water with an initial continuous water level surface, the
730 approximations a in (133) and b in (134) of the thrust term provide correct
731 solutions for all values when constructing the approximate solution $\hat{\mathbf{U}}(x, t)$.
732 Straightforward manipulations of the approximate solution in (92,93) show
733 that in this particular case the inner solution of the RP guarantees $d_i^n = d^{-\downarrow} =$
734 $d^{+\downarrow} = d_{i+1}^n$ in all the regions.

735 In a more general case of still water, where a discontinuity in the bed ele-
736 vation produces a discontinuity in the water surface elevation and a wet/dry
737 interface appears, option a fails, while option b ensures a correct approximate
738 solution retaining $d_i^n = d^{-\downarrow}$ and $d^{+\downarrow} = d_{i+1}^n$.

739 In cases of non zero velocity, more differences between the solutions pro-
740 vided by the bed level source term integral a and b arise. The evaluation
741 of the thrust term in (137) provides a correct tracking of wetting advance
742 fronts and ensures an energy dissipating mechanism in steady shocks due to
743 bed level elevation [7].

744 12. Appendix C. Stability region and source fix

745 Only in cases of motionless steady state it is possible to integrate exactly
 746 source terms, ensuring a well balanced scheme. In general the time lineariza-
 747 tion of the source terms in (79) and (111) does not ensure positive values
 748 of the mixture depth in all regions of the approximate solution $\hat{\mathbf{U}}(x', t)$ in
 749 subcritical flow [7]. Therefore when cell averaging the solution, the stability
 750 condition in (76) must be reduced to ensure positive values of the mixture
 751 depth in the updated solution.

752 Instead of revisiting the stability condition (76), the positivity of the
 753 solution can be preserved by modifying the source strength coefficients. Only
 754 subcritical RP have to be analyzed. Supercritical cases are not considered
 755 as water depth is not affected by the source terms in the cell averaging.
 756 Also cases that require an entropy fix are not considered. Considering that
 757 the numerical scheme is completely linearized and explicit, this can be done
 758 easily and with a low computational cost.

759 The procedure is based in the construction of the states $(hr)_i^{-\downarrow}$ and $(hr)_j^{+\downarrow}$,
 760 in the left and right sides of the solution, that following (92,93) are given by

$$\begin{aligned} (hr)_i^{-\downarrow} &= (hr)_i + (\theta_1 \alpha \tilde{e}_1)_k^1 \\ (hr)_j^{+\downarrow} &= (hr)_j - (\theta_1 \alpha \tilde{e}_3)_k^3 \end{aligned} \quad (140)$$

761 followed by the conditions

$$\beta_S^1 = \begin{cases} (hr)_i^* \tilde{\lambda}_k^1 & \text{if } (hr)_i^{-\downarrow} < 0 \\ \beta_S^1 & \text{otherwise} \end{cases}, \quad \beta_S^3 = -\beta_S^1 \quad (141)$$

762 and

$$\beta^3 = \begin{cases} -h_i^* \tilde{\lambda}_k^3 & \text{if } (hr)_j^{+\downarrow} < 0 \\ \beta^3 & \text{otherwise} \end{cases}, \quad \beta^1 = -\beta^3 \quad (142)$$

763 where $h_i^* = h_i^n + (\alpha \tilde{e}_1)_{i+1/2}^1 > 0$, $(hr)_i > 0$ and $(hr)_j > 0$.

764 Additionally, in cases of wet/dry fronts, with $(hr)_i = 0$ or $(hr)_j = 0$ it is
 765 possible to formulate condition (141), but in this case the solution indicates
 766 that the edge acts as a solid wall [7].

767 Then, when a negative solution of the mixture depth appears in the ini-
 768 tially dry region, the flux functions $\mathbf{M}_{1,k}^{-\downarrow}$ and $\mathbf{M}_{1,k}^{+\downarrow}$ have to be corrected, and
 769 considering that in this particular case the edge acts as an inner boundary
 770 condition, it is necessary to define a ghost cell.

771 The definition of the inner ghost cells can be avoided if, instead of using
 772 the flux formulation, the Godunov first order method is written splitting the
 773 flux $\delta \mathbf{M}_k$ in right-going and left-going wave propagations. Fluxes $\delta \mathbf{M}_{1,i,k}^+$ and
 774 $\delta \mathbf{M}_{1,j,k}^+$ in (103) are corrected as follows

- 775 • If $(hr)_j^n = 0$ and $(hr)_j^{***} < 0$ set

$$\delta \mathbf{M}_{1,i,k}^- = \delta \mathbf{M}_{1,k} \quad \delta \mathbf{M}_{1,j,k}^- = 0 \quad (143)$$

- 776 • If $(hr)_i^n = 0$ and $(hr)_i^* < 0$ set

$$\delta \mathbf{M}_{1,j,k}^- = \delta \mathbf{M}_{1,k} \quad \delta \mathbf{M}_{1,i,k}^- = 0 \quad (144)$$

777 If the k edge is considered as a temporal solid wall, in this case the normal
 778 component velocity to the edge must be set zero, which guarantees accurate
 779 results for Riemann problems with reflection waves [7]. This fix can be ap-
 780 plied directly to the Reduced Godunov scheme ensuring positive values of
 781 mixture depth and correctly bounded values of volumetric concentration.

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