SPACE INVERSION SYMMETRY BREAKING AND PATTERN SELECTION IN NONLINEAR OPTICS

Marco Santagiustina, Pere Colet, Maxi San Miguel, Daniel Walgraef⁺

Instituto Mediterraneo de Estudios Avanzados, IMEDEA (CSIC-UIB),*

E-07071 Palma de Mallorca, Spain

Abstract

Pattern formation in nonlinear optical cavities, when an advection-like term is present, is analyzed. This term breaks the space inversion symmetry causing the existence of a regime of convective instabilities, where noise-sustained structures can be found, and changing the pattern orientation and the selected wavevector. The concepts of convective and absolute instability, noise-sustained structures and the selection mechanisms in two dimensions are discussed in the case of the optical parametric oscillators and a Kerr resonators. In the latter case, in which hexagons are the selected structure we predict and observe that stripes are the most unstable structures in the initial linear transient. In the nonlinear regime of the absolute instability these stripes destabilize and hexagons form. Their orientation is dictated by that of the transient stripes and therefore by the advection term. In the convective regime we predict and observe disordered noise-sustained hexagons, preceded in space by noise sustained stripes.

1. Introduction

Spontaneous pattern formation has been predicted and observed in many different nonlinear optical systems such as cavities filled with nonlinear Kerr media¹, Kerr slices in single mirror feedback configuration^{2,3} and optical parametric oscillators^{4,5}. A large variety of patterns has been observed in systems with two transverse dimensions, including stripes, squares, hexagons and honeycombs.

A useful general approach to study pattern formation is to consider which are the symmetries of the system that are broken or relevant in the process. To begin with, in systems which are intrinsically translationally symmetric, pattern formation implies the spontaneous breaking of the continuous space translation symmetry. In addition, a transverse optical pattern is formed with an orientation that breaks the rotational symmetry of the system in the transverse plane. The type of pattern selected is largely dictated by symmetry properties of the nonlinearities involved⁶. Then, the existence of symmetries intrinsically broken is expected to have a strong influence in the process of pattern formation. In this paper we will consider the effect of breaking the space inversion symmetry, i.e. $\vec{r} \rightarrow -\vec{r}$. This situation can occur in optics, although it has not yet been fully investigated. Breaking this symmetry in a two transverse dimension (2D) optical system is associated with an advection-like term of the type $v\partial_y$ (hereafter the advection is taken to occur along the y-axis, for simplicity) in the governing equation of the system.

Here, we will consider the simple case of resonators, filled with media whose nonlinear susceptibility is purely imaginary. In particular two cases will be presented: purely quadratic (i.e. the optical parametric oscillators, OPO's) and purely cubic (Kerr) media. Parameter

conditions will be chosen such that stripes are the selected pattern in the OPO and hexagons in the Kerr cavity. In the OPO's the space inversion symmetry is broken by the crystal birefringence. In fact, in such media, the phenomenon of double refraction takes place, i.e. the first harmonic (FH) signal generated in the cavity spatially walks off the second harmonic (SH) pump beam. For Kerr resonators a similar effect (drifting patterns) arises as soon as the input pump beam is not perfectly aligned with the cavity axis. Both effects are modelled in the governing equations by an advection-like term (where v depends on the walk-off or drift angles).

We will show that the consequences of the presence of the advection-like term are the following: a) the instability turns to be *convective* and not absolute in a certain region of the parameter space; b) in the absolutely unstable regime the selected wavevector of the pattern depends on v , in addition, rotational symmetry in the transverse plane is not spontaneously broken, but rather a preferred orientation exists; c) in the convectively unstable regime patterns exist but they are sustained by noise.

In the 2D OPO's these effects, due to the walk-off, have been theoretically studied in references^{7,8}. In the 1D Kerr resonator drifting patterns were theoretically studied⁹ and experimentally observed²; the existence of convective instabilities and noise-sustained structures was also recently pointed out¹⁰. However, a study of the consequences on pattern formation in a 2D transverse Kerr cavity had not been yet presented.

The paper is organized as follows. In the next section we will briefly review what changes are expected in systems presenting an advection term. In particular we will refer to the specific case of the OPO. The aim of this section is to provide a common, general background of the physical concepts which lie behind the phenomena studied. In section 3, guided by

the previous analysis, we will present the study of the 2D Kerr resonator, which shows new interesting features, peculiar of the fact that hexagons are the selected structure. This case may also be particularly interesting in relation with recent experimental results in systems which bear strong similarity with a Kerr resonator^{11–15}. We will show how the dynamics of hexagon selection is affected by the advection-like term. We will devote the final section to the discussion of noise-sustained structures in the convective regime of a 2D Kerr resonator.

2. Effect of advection on 2D stripe pattern selection: the OPO

This section is dedicated to analyze the effects of an advection-like term on a two-dimensional optical pattern forming system, in which stripes are the selected structure. As an example we will refer to the case of the degenerate, type I, OPO^{7,8}.

A first consequence of the presence of advection in a physical system is the change of the nature of the instability in a certain region of the parameter space. Usually, without advection, a steady-state is said to be stable or unstable if an initial perturbation decays or grows with time. Advection splits the instability domain into two: the convectively and the absolutely unstable regimes. A steady-state is convectively unstable if a perturbation grows with time but the speed of advection overwhelms the spreading velocity of the perturbation. Therefore, any initial perturbation eventually leaves the system.

The instability sets up for a value of the control parameter (pump amplitude) that can be calculated as if $v = 0$. If $v \neq 0$, this instability is convective up to a value of the control parameter whose calculation is sketched below (the details can be found elsewhere^{7,10,16}).

However, before giving the guidelines of this calculation it is worth to note that the distinction between convective and absolute instabilities is unambiguous only if there exists

a fixed, preferential, reference frame and the "flow" is open, i.e. no reflection or feedback exist along the direction of the advection. These conditions are easily achieved in nonlinear optical resonators, because the advection-like term (drift or walk-off) acts in the plane orthogonal to the cavity axis and the pump beam has a finite size in that plane. Hence, the fixed reference frame is that of the pump beam and the flow is open because any field decays outside the pumping region. We stress the fact that the advection-like term is transversal in order to distinguish it from the longitudinal advection term which is more commonly known in optics. In the latter case, for beam propagation problems, one finds that the instabilities are always convective and never absolute. In resonators the longitudinal instability becomes absolute due to the reflecting boundary conditions.

Let's consider the specific example of the OPO, in which the equations describing the time evolution of the SH and FH fields $A_0(x, y, t)$, $A_1(x, y, t)$ are:

$$\partial_t A_0 = \gamma_0 [-(1 + i\Delta_0)A_0 + E_0 + ia_0 \nabla^2 A_0 + 2iK_0 A_1^2] \quad (1)$$

$$\partial_t A_1 = \gamma_1 [-(1 + i\Delta_1)A_1 + v\partial_y A_1 + ia_1 \nabla^2 A_1 + iK_0 A_1^* A_0] \quad (2)$$

where x, y are the transversal spatial dimensions, t the time, $\gamma_{0,1}$ the cavity decay rates, $\Delta_{0,1}$ the cavity detunings, $a_{0,1}$ the diffraction coefficients, v the walk-off coefficient, $E_0(x, y, t)$ the input SH pump and K_0 the nonlinear coefficient. The relevant parameters for our discussion are the signal detuning Δ_1 , the pump amplitude E_0 and the advection velocity v . In order to discriminate the nature of the instability, one needs to evaluate the linearized evolution of a perturbation ψ of the steady-state $A_1 = 0$ (the SH component $A_0 = E_0/(1 + i\Delta_0)$ is always stable) i.e. the following integral^{7,16}

$$\psi(x, y, t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dq_x dq_y \tilde{\psi}(q_x, q_y, 0) \exp[i(q_x x + q_y y) + \lambda(q_x, q_y)t] \quad (3)$$

where $\tilde{\psi}(q_x, q_y, 0)$ is the initial perturbation in the spatial wave-vector space (q_x, q_y) and $\lambda(q_x, q_y)$ is the linearized eigenvalue. The integral can be estimated by extending the wave-vector (q_x, q_y) to the complex space $(k_x, k_y) = \vec{k}$ using the formal substitution $i\vec{q} \rightarrow \vec{k}$. Mathematically, the extension to the complex space allows us to evaluate (3) easily by means of a saddle point analysis. Physically, the real part of $\vec{k}$, which is not present in the usual analysis where $\vec{q}$ is real, takes into account the existence of fronts (of the type $\exp[-\text{Re}(\vec{k}) \cdot \vec{x}]$) which separate the unstable state and the stable pattern. The speed of this front with respect to the group velocity v determines if perturbations invade or leave the system, i.e. the convective or absolute nature of the instability. The linearized eigenvalue of largest real part for complex wavevectors for the OPO reads⁷:

$$\lambda(\vec{k}) = -1 + vk_y + \sqrt{F^2 - a_1^2(q_c^2 + k_x^2 + k_y^2)^2} \quad (4)$$

where $F = |K_0 E_0 / (1 + i\Delta_0)|$ and $q_c^2 = -\Delta_1 / a_1$. The asymptotic growth rate at a fixed spatial position is given by $\text{Re}[\lambda(k_x^s, k_y^s)]$ where the saddle point $\vec{k}_s$ in the complex wavevector space is defined by:

$$\nabla_{\vec{k}} \lambda(\vec{k})|_{\vec{k}_s} = 0, \quad \text{Re}[\nabla_{\vec{k}}^2 \lambda(\vec{k})|_{\vec{k}_s}] \geq 0. \quad (5)$$

The threshold of the absolutely unstable regime is then found by determining F such that

$$\text{Re}[\lambda(k_x^s, k_y^s)] = 0 \quad (6)$$

In particular, the threshold of the absolutely unstable regime is always larger than for the case $v = 0$. For the OPO, the maximum increase in the value of the threshold is reached for⁷ $\Delta_1 = 0$.

The first of equations (5) applied to (4) yields that $k_x = 0$. This condition implies that

the space inversion symmetry breaking favours the formation of stripes orthogonal to the direction of v . The modulus of the selected wavevector Q_y depends on the advection speed too. The calculation to determine Q_y is formally the same as the case of front propagation into an unstable state¹⁷. In optical cavities with an advection effect, the front which separates the stable and the unstable region is fixed (it corresponds to the pump beam edge) and the field is moving with respect to it (due to the advection). By imposing the conservation of the flux of the field nodes from the linearly unstable regime to the stable state we have^{8,17,18}:

$$q_y = \frac{Im[\lambda(\vec{k}_s)]}{v} \quad (7)$$

This formula is in very good agreement with numerical solutions⁸.

In the convective regime only a continuously applied excitation (noise) can lead to the observation of patterns. These noise-sustained structures bear very peculiar features that allow to distinguish them from the dynamics-sustained structures observed above the absolute instability threshold^{7,10,19}. Mainly: a) in the near field noise-sustained structures are not present in all the system but rather in the part closest to the edge of the outgoing flow; b) in the far field spots are broader with respect to the absolutely unstable case; c) a time spectral analysis of field oscillations can quantitatively determine the nature of the pattern (dynamics- or noise-sustained) and the threshold of the transition from one to the other.

3. Effect of advection on 2D hexagonal pattern selection: Kerr cavities

In this section 2D pattern formation in a Kerr cavity is considered. The equation which models this system is the following^{9,10}:

$$\partial_t A = v \partial_y A + i \nabla^2 A - [1 + i \eta (\Delta - |A|^2)] A + E_0 \quad (8)$$

where Δ represents the cavity detuning, η gives the sign of the Kerr nonlinearity (1 for self-focusing, -1 for self-defocusing, in the following $\eta = 1$), E_0 the pump amplitude and v depends on the angle of incidence of the pump into the cavity. If we linearize this equation around the uniform steady-state A_0 , solution of $A_0[1 + i\eta(\Delta - |A_0|^2)] = E_0$, we obtain the dispersion relation:

$$\lambda(\vec{k}) = vk_y - 1 + \sqrt{F^2 - (2F - \Delta + k_x^2 + k_y^2)^2} . \quad (9)$$

where $F = |A_0|^2$ and where we are considering complex wavevectors $\vec{k}$ as in the previous section.

From equations (5) and (6) we can find the absolute instability threshold. It is interesting to note that equation (5) gives again $k_x = 0$; therefore the threshold corresponds to that calculated for a 1D Kerr cavity¹⁰. Note that in two-dimension the linear analysis is valid in the limit of small detunings. In fact in two dimensions the hexagons are sub-critical and this implies that nonlinear front propagation should be considered and hence the threshold of the absolute instability can slightly differs from that determined by a linear analysis. However, for small detunings the subcriticality is weak and the linear analysis is still a good approximation as we tested in our numerical simulations. Moreover, we want to show that in the limit of validity of the linear approximation a very reliable and simple formula to determine the threshold can be furnished. In fact, if we Taylor expand the eigenvalue λ close to the most unstable mode, say $(0, q_y^m)$, and truncate the expansion at the second order, we find that the integral (3) can be solved analytically. The condition for the absolute instability is then given by:

$$Re[\lambda(0, q_y^m)] + \frac{(y_0 + vt)^2}{2Re[\lambda_{yy}(0, q_y^m)]t^2} = 0 \quad (10)$$

where $\lambda_{yy} = d^2\lambda/dq_y^2$. Finally, upon substitution of λ_{yy} calculated through (9) and as $t \rightarrow \infty$, we obtain

$$-1 + F - \frac{v^2 F}{8(2F - \Delta)} = 0 \quad (11)$$

where we also used the relation¹⁰ $q_y^m = 2F - \Delta$. Finally, solving for F we get:

$$F = \frac{1}{4} \left\{ 2 + \Delta + \frac{v^2}{8} + \left[\left(2 + \Delta + \frac{v^2}{8} \right)^2 - 8\Delta \right]^{1/2} \right\} \quad (12)$$

In figure 1 we present the approximated and exact solutions; the agreement is very good for different values of v and different detunings $\Delta < 1$. Therefore, for a practical calculation of the absolute instability threshold, equation (12) is very useful.

Let us now consider the influence of the drift term on the pattern selection process. The condition $k_x = 0$ means that also in this case the modes which spread faster are stripes orthogonal to the advection direction. In OPO's the nonlinearity eventually selects stripes; therefore the absolutely unstable modes which spread faster will grow, i.e. horizontal stripes appear and invade all the system⁷. In Kerr media another mechanism comes into play, i.e. the nonlinearity favours the formation of triads of wavevectors and thus of hexagonal structures. Numerical solutions of eq. (8) with different initial random conditions, pump amplitudes (in particular we used super-Gaussian beams whose largest amplitude is above the absolute instability threshold) indicate that, in the initial transient, a stripe pattern appears. The stripes are oriented orthogonally to the advection direction as indicated by the analysis. We show this transient state in figure 2. Figure 2b clearly shows that in the far field the fastest growing modes of the ring are those for which $k_x = 0$. However, as soon as the pattern reaches larger amplitudes the nonlinear terms come up so that the stripes become unstable and hexagons are formed. In figure 3 we show the beginning of this process.

Due to the fact that the most rapidly spreading modes at the initial stage satisfy $k_x = 0$, the hexagons are preferentially oriented as in figure 4.

Slightly inclined configurations were also obtained especially in smaller systems. This stems from the fact that the transient stripes bend close to the boundary of the pumping region. If the initial conditions are such that the hexagons develop first on the bended part of the stripes then the final structure will be slightly tilted with respect to the most favourable configuration shown in figure 4a. An example of slightly tilted orientation is shown in figures 5, which can be compared with figures 3a and 4a. Once formed, the structure does not apparently change its orientation for times attainable by numerical integrations (a few thousands units) as shown in figure 5b. This is probably due to the strong forcing of the hexagonal lattice: a new bright spot of the structure, which is formed at the pump edge of the incoming flow (the bottom in our figures), is strongly forced by the nonlinearity to preserve the orientation of the existing lattice.

4. Noise-sustained structures in 2D Kerr resonators

In the previous section we have dealt with the absolutely unstable regime, in which patterns are dynamics-sustained. Now, we briefly turn to the case of the convectively unstable regime; there, patterns can be observed only if noise is present.

A typical example of a noise-sustained pattern in a Kerr cavity is shown in figure 6. The noise-sustained pattern can be easily distinguished from a dynamics-sustained structure by comparing figure 6 with figure 4 in the absolutely unstable regime.

In the near field the pattern does not occupy all the pumping region, but rather it appears at a certain position well inside it. This is due to the fact that noise is amplified during

the advection and thus it needs some space to grow. By observing other snapshots, at later times, we noticed that the position at which the structure reaches a saturated value is fluctuating in time. Noise intensity could be in principle characterized by measuring the average and variance of the space delay fluctuations. Whatever small noise can be amplified up to the macroscopic value of the pump field if the system is large enough along the direction of the advection. It is also interesting to note that just before the formation of the bright spots, stripes orthogonal to the advection direction appear. This confirms once more the previous analysis which indicates that these modes are the most unstable. When their growth saturates, nonlinearities lead to a noisy hexagonal pattern.

The far field shows peculiar features too. Note that there is less evidence of an hexagonal lattice as it appears in the absolutely unstable regime (figure 4b). This is due to the fact that for that value of the pump amplitude some degree of disorder is usually observed also in the absolute regime, as shown in 4a. In the convective regime this disorder is enhanced because noise is exciting all possible orientations (note the ring in figure 6b) and the pattern can be re-generated by noise with a different orientation.

5. Conclusions

In conclusion we have analyzed the changes that occur in the transverse, two dimensional pattern formation in nonlinear optical cavities when an advection-like effect is present which breaks the space inversion symmetry of the system. We considered two cases: the optical parametric oscillator, when stripe structures are selected and the Kerr oscillator, when hexagons are preferred. In both cases the advection induces the existence of a convectively unstable regime, preceding the absolutely unstable one, where no dynamics-sustained

pattern can exist but rather noise-sustained structures. The absolute and convective instability thresholds can be calculated by means of a linear stability analysis which includes the advection-like term. This analysis requires the use of complex wave-vectors and is formally the same type of analysis of front propagation of stable into unstable states. For the OPO and the Kerr cavity all thresholds obtained agree with the findings of the numerical solutions of the governing equations. A simplified approximated formula, valid for small detunings, is also given for the absolute threshold of the Kerr case. The analysis also gives quantitative and qualitative informations about the change in the selected pattern. In particular, in the OPO, stripes are forced to be orthogonal to the advection direction and the pattern wavelength increases by increasing the advection term. For the Kerr cavity the analysis also explains the orientation of the hexagons in the absolutely unstable regime. The stripe modes orthogonal to the advection are again the most rapidly spreading in the system and the initial transitory confirm clearly this effect. However, when the nonlinearity comes into play stripes destabilize and hexagons form. In the convective regimes we show evidence of the existence of noise-sustained structures which for the Kerr resonators are highly disordered hexagons. This is due to the fact that noise excites all possible orientations and the newly-generated pattern has no relation with the already formed pattern because advection is overwhelming. The disordered hexagons are spatially preceded by a region where stripes can be observed, in agreement with the analysis. These structures might represent a paradigmatic example of a macroscopic effect induced by microscopic noise, which in this context can be of quantum nature.

This work is supported by QSTRUCT (Project ERB FMRX-CT96-0077). Financial support

from DGICYT (Spain) Projects PB94-1167 and PB97-0141 are also acknowledged.

REFERENCES

- +. Permanent address: Center for Nonlinear Phenomena and Complex Systems, Université Libre de Bruxelles, Campus Plaine, Blv. du Triomphe B.P 231, 1050 Bruxelles.
- *. <http://www.imedeo.uib.es/PhysDept>
1. Lugiato L A and Lefever R 1987 Phys. Rev. Lett. **58** 2209
 2. Grynberg G 1994 Opt. Commun. **109** 483; Petrossian A, Dambly L and Grynberg G 1995 Europhys. Lett. **29** 209
 3. Residori S, Ramazza P L, Pamploni E, Boccaletti S and Arecchi F T 1996 Phys. Rev. Lett. **76** 1063
 4. Staliunas K 1992 Opt. Comm. **91** 82; 1995 J. Mod. Opt. **42** 1261
 5. Oppo G-L, Brambilla M, Lugiato L A 1994 Phys. Rev. A, **49**, 2028; Oppo G-L, Brambilla M, Camesasca D, Gatti A and Lugiato L A 1994 J. Mod. Opt. **41** 1151
 6. Cross M C and Hohenberg P C 1993 Rev. Mod. Phys. **65** 851
 7. Santagiustina M, Colet P, San Miguel M and Walgraef D 1998 Phys. Rev. E. **58** 3843
 8. Santagiustina M, Colet P, San Miguel M and Walgraef D 1998 Optics Letters **23** 1167
 9. Haelterman M and Vitrant G 1992 J. Opt. Soc. Am. B **9** 1563; Vitrant G and Danckaert J 1994 Chaos, Solitons and Fractals **4** 1369
 10. Santagiustina M, Colet P, San Miguel M and Walgraef D 1997 Phys. Rev. Lett. **79** 3633
 11. Ramazza P L, Boccaletti S, Pamploni E, Soria S and Arecchi F T 1996 Phys. Rev. A

12. Castaldo F, Paparo D and Santamato E 1997 Opt. Commun. **143** 57
13. Honda T 1995 Opt. Lett. **20** 851
14. Mamaev A V and Saffman M 1997 Opt. Lett. **22** 283
15. Seipenbusch J P, Ackermann T, Schäpers B and Lange W 1997 Phys. Rev. A **56** R4401
16. Deissler R J 1985 J. Stat. Phys. **40** 376; 1989 **54** 1459; 1992 Physica D **56** 303
17. Dee G T and Van Sarloos W 1988 Phys. Rev. Letts. **60** 2641
18. Montagne R, Amengual A, Hernandez-Garcia E and San Miguel M 1994 Phys. Rev. E, **50** 377; San Miguel M, Montagne R, Amengual A and Hernandez-Garcia E 1996 Instabilities and Nonequilibrium Structures V, 85 (E. Tirapegui, W. Zeller Eds., Kluwer Ac., Amsterdam)
19. Babcock K L, Ahlers G and Cannell D S 1991 Phys. Rev. Lett. **67** 3388; 1994 Phys. Rev. E **50** 3650; Tsameret A and Steinberg V 1991 Phys. Rev. Lett. **67** 3392; 1994 Phys. Rev. E **49** 1291

Figure 1: The solid line represents the threshold of the absolute instability for the Kerr cavity as determined by the approximated formula (12) for $\Delta = 1$; the asterisks represent the numerically calculated exact solutions. The dashed line and the diamonds represents the same for $\Delta = 0.5$.

Figure 2: Near (a) and far (b) field for $t = 1200$; $\Delta = 0.5$, $v = 0.1$, $E_0(x, y, t) = E_m \exp(-((x^2 + y^2)/\sigma_0^2)^m/2)$, with $E_m = 1.01$, $m = 5$ and $\sigma_0 = 56$. In the near field (a) we show only the central part of the beam ($-36 < x < 36$, $-36 < y < 36$). In the far field the central spot represents the average super-Gaussian pump.

Figure 3: The same of figure 2 for $t = 1400$.

Figure 4: The same of figure 2 for $t = 2000$. In figure a) the whole system is shown.

Figure 5: a) The near field for $t = 1250$; the parameters are the same of figures 2,3 and 4, but with different random initial conditions. Differently from figure 3a hexagons start to form on stripes slightly bended because of boundary conditions. The pattern at $t = 2000$ is shown in figure b).

Figure 6: Near (a) and far (b) field for $t = 500$; $\Delta = 1$, $v = 0.55$, $E_m = 1.03$, $\sigma_0 = 112$, the other parameters are the same of figure 2. A Gaussian, white noise as defined in reference¹⁰ was applied; its amplitude is 7 orders of magnitude smaller than that of the pump field.