The homogeneous geometries of real hyperbolic space

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Abstract. We describe the holonomy algebras of all canonical connections of homogeneous structures on real hyperbolic spaces in all dimensions. The structural results obtained then lead to a determination of the types, in the sense of Tricerri and Vanhecke, of the corresponding homogeneous tensors. We use our analysis to show that the moduli space of homogeneous structures on real hyperbolic space has two connected components.

Mathematics Subject Classification (2010). Primary 53C30; Secondary 57S25, 58H15.

Keywords. Real hyperbolic space, homogeneous structure, holonomy.

Contents

1. Introduction

Homogeneous manifolds provide a rich and varied class of spaces on which to study Riemannian geometry. One difficulty that arises is that the same Riemannian manifold (M, g) can admit several different descriptions as a

Partially supported by the Ministry of Economy and Competitiveness, Spain, under grant MTM2011-22528 and by the Danish Council for Independent Research, Natural Sciences.

homogeneous space G/H. It is surprising how little is known about this problem for many well-known spaces.

A substantial attempt to solve this problem was made by Ambrose and Singer [1]. They characterised the property that (M, g) is homogeneous in terms of the existence of a tensor S satisfying a certain set of non-linear differential equations. Each homogeneous description of (M, g) gives rise to a different solution to these equations. These equations were studied further by Tricerri and Vanhecke [10], who introduced a decomposition of S into components under the action of the orthogonal group, and produced a number of examples illustrating the occurrence of different possible classes. In particular, they showed that in dimension 3, the real hyperbolic space $\mathbb{R}\mathrm{H}(3)$ admits homogeneous tensors of two different types. However, they left as an open problem, the determination of all homogeneous tensors on $\mathbb{R}\mathrm{H}(n)$ for $n \geqslant 3$ [10, p. 55].

In [4], we took a different route and used general results of Witte [11] on co-compact subgroups to determine all the groups acting transitively on $\mathbb{R}\mathrm{H}(n)$. This left open the determination of the corresponding homogeneous tensors S and their types. Any homogeneous space G/H with a Lie algebra decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ carries a canonical connection $\widetilde{\nabla}$, characterised by the property that G-invariant tensors are parallel for $\widetilde{\nabla}$. By work of Nomizu [5], the tensor S depends only on the holonomy algebra $\mathfrak{hol} \leqslant \mathfrak{h}$ of $\widetilde{\nabla}$ and $\mathfrak{hol} + \mathfrak{m}$ determines the Lie algebra of a smaller group acting transitively on G/H.

In this paper we answer two questions: what are the holonomy algebras of the canonical connections on $\mathbb{R}\mathrm{H}(n)$? and what are the types of the corresponding homogeneous tensors? Regarding a geometric structure as being given by a collection of tensors that are parallel with respect to some connection, the answer to the first question thus determines which geometric structures may be realised homogeneously on $\mathbb{R}\mathrm{H}(n)$. Our answer to the first question is given by:

Theorem 1.1. *The holonomy algebras of canonical connections on* $\mathbb{R}\mathrm{H}(n)$ *are* $\mathfrak{so}(n)$ *and all the reductive algebras*

$$\mathfrak{k} = \mathfrak{k}_0 + \mathfrak{k}_{ss}$$

of compact type with $\mathfrak{k}_0 \cong \mathbb{R}^r$ *Abelian and* $\mathfrak{k}_{ss}$ *semi-simple such that*

$$3r + \dim \mathfrak{k}_{ss} \leqslant n - 1.$$

The proof includes a description of how this algebra acts on the tangent space of $\mathbb{R}\mathrm{H}(n)$. We then use these results to determine the complete answer to the second question, extending the partial results of [10, 6, 8, 7, 9]. Furthermore, the ideas of our constructions are used to show that the moduli space of homogeneous structures on $\mathbb{R}\mathrm{H}(n)$, $n > 1$, with fixed scalar curvature has exactly two components.

The paper is organised as follows. In Section 2, we briefly recall the results of Ambrose & Singer and Nomizu relating homogeneous spaces to homogeneous tensors. We then specialise to the real hyperbolic space in Section 3 and review our result on the groups that act transitively, establishing notation for the rest of the paper. The determination of the holonomy algebras and their isotropy representations is given in Section 4. We use this in Section 5 to determine the homogeneous tensors and their types. Finally, we use our results to determine the connected components of the moduli space of homogeneous tensors on $\mathbb{R}\mathrm{H}(n)$ and discuss a couple of geometric consequences in Section 6.

2. The Ambrose-Singer equations

Let (M, g) be a connected, simply-connected complete Riemannian manifold. Suppose S is a tensor of type $(1, 2)$; so for each $X \in TM$, we have that S_X is an endomorphism of TM. Writing ∇ for the Levi-Civita connection of g, we define a new connection $\widetilde{\nabla} = \nabla - S$. In general, $\widetilde{\nabla}$ has non-zero torsion. Ambrose and Singer [1] showed that (M, g) admits a homogeneous structure if and only if there is an S such that

$$\widetilde{\nabla} g = 0, \quad \widetilde{\nabla} R = 0 \quad \text{and} \quad \widetilde{\nabla} S = 0, \tag{2.1}$$

where R is the curvature tensor of ∇. Nomizu [5] gave this homogeneous description as follows. Fix a point p in M. The holonomy algebra $\mathfrak{hol}$ is the subalgebra of the endomorphisms of T_pM generated by the elements $\{ \widetilde{R}_{X,Y} : X, Y \in T_pM \}$, where $\widetilde{R}$ is the curvature of $\widetilde{\nabla}$. Writing $\mathfrak{m} = T_pM$, the vector space

$$\widetilde{\mathfrak{g}} = \mathfrak{hol} + \mathfrak{m}$$

has a Lie bracket defined by

$$[U, V] = UV - VU, \quad [U, X] = U(X), \quad [X, Y] = \widetilde{R}_{X,Y} + (S_X Y - S_Y X)$$

for $U, V \in \mathfrak{hol} \subset \mathrm{End}(T_pM)$ and $X, Y \in \mathfrak{m} = T_pM$. Exponentiating these groups we obtain a reductive homogeneous description of M as $\widetilde{G}/H$, where $\widetilde{G}$ and H have Lie algebras $\widetilde{\mathfrak{g}}$ and $\mathfrak{hol}$ respectively. The connection $\widetilde{\nabla}$ is now the canonical connection of the reductive space $(\widetilde{G}/H, \mathfrak{m})$. Indeed for any homogeneous space $M = G/H$ with reductive description $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, the canonical connection is given at the identity by

$$\widetilde{\nabla}_B C = -[B, C]_{\mathfrak{m}} \tag{2.2}$$

where $C \in \mathfrak{g}$ is extended to the vector field on M whose one-parameter group is $gH \mapsto \exp(tC)gH$. The canonical connection has the property that every left-invariant tensor on M is parallel.

3. Homogeneous descriptions of real hyperbolic space

The description of $\mathbb{R}\mathrm{H}(n)$ as a symmetric space is

$$\mathbb{R}\mathrm{H}(n) = SO(n, 1)/O(n),$$

where we take $SO(n,1)$ to be the set of matrices of determinant $+1$ preserving the form $\operatorname{diag}(\mathrm{Id}_{n-1}, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))$. The connected isometry group has Iwasawa decomposition $SO_0(n,1) = SO(n)\mathbb{R}_{>0}N$ whose Lie algebra is $\mathfrak{so}(n,1) = \mathfrak{so}(n) + \mathfrak{a} + \mathfrak{n}$, given concretely by

$$\mathfrak{so}(n) = \left\{ \begin{pmatrix} B & v & v \\ -v^T & 0 & 0 \\ -v^T & 0 & 0 \end{pmatrix} : B \in \mathfrak{so}(n-1),\ v \in \mathbb{R}^{n-1} \right\},$$
$$\mathfrak{a} = \mathbb{R}\operatorname{diag}(0,\dots,0,1,-1), \quad \mathfrak{n} = \left\{ \begin{pmatrix} 0 & 0 & v \\ -v^T & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : v \in \mathbb{R}^{n-1} \right\}.$$

If G acts transitively on $\mathbb{R}\mathrm{H}(n)$ then $G\backslash\mathbb{R}\mathrm{H}(n)$ is a point, so compact. It follows that $G\backslash SO_0(n,1)$ is an orbit space of the compact group $O(n)$, thus G is a non-discrete co-compact subgroup of $SO_0(n,1)$. Witte's structure theory for co-compact groups [11] then leads to the following result.

Theorem 3.1 ([4]). *The connected groups acting transitively on $\mathbb{R}\mathrm{H}(n)$ are the connected isometry group $SO_0(n,1)$ and the groups $G = F_rN$, where N is the nilpotent factor in the Iwasawa decomposition of $SO(n,1)$ and F_r is a connected closed subgroup of $SO(n-1)\mathbb{R}_{>0}$ with non-trivial projection to $\mathbb{R}_{>0}$.* □

The case $F_r = \mathbb{R}_{>0}$, gives the description $\mathbb{R}\mathrm{H}(n) = \mathbb{R}_{>0}N$ of real hyperbolic space as a solvable group.

4. The holonomy algebras

Assume that $G = F_rN$ acts transitively on $\mathbb{R}\mathrm{H}(n)$ as in Theorem 3.1. This implies that $\mathbb{R}\mathrm{H}(n) = G/H$, with $H = F_r \cap SO(n-1)$. We have immediately that H is reductive, and thus

$$\mathfrak{h} = \mathfrak{h}_0 + \mathfrak{h}_{ss},$$

where $\mathfrak{h}_0$ is Abelian and $\mathfrak{h}_{ss}$ is semi-simple. Let us write

$$\mathfrak{f}_r = \mathfrak{h} + \mathfrak{a}_r, \qquad \mathfrak{g} = \mathfrak{h} + \mathfrak{a}_r + \mathfrak{n},$$

with $\mathfrak{a}_r$ projecting non-trivially to $\mathfrak{a} = \operatorname{Lie}\mathbb{R}_{>0}$. Since $\mathfrak{f}_r$ is a subalgebra of $\mathfrak{so}(n-1) \oplus \mathfrak{a}$, it admits a positive definite invariant metric. This implies that $\mathfrak{f}_r$ is reductive with

$$\mathfrak{f}_r = (\mathfrak{h}_0 + \mathfrak{a}_r) + \mathfrak{h}_{ss}\,.$$

In particular, $[\mathfrak{a}_r, \mathfrak{h}] = 0$ and $\dim \mathfrak{a}_r = 1$.

Let us write

$$\mathfrak{s} = \mathfrak{a} + \mathfrak{n}, \qquad \mathfrak{s}_r = \mathfrak{a}_r + \mathfrak{n},$$

and note that $[\mathfrak{s}, \mathfrak{s}] = \mathfrak{n} = [\mathfrak{s}_r, \mathfrak{s}_r]$. For later use, we remark that $\mathfrak{a}_r$ is not canonically specified, but is any one-dimensional complement to $\mathfrak{h}_0 + \mathfrak{n}$ in

$$\mathfrak{s}_f = (\mathfrak{f}_r)_0 + \mathfrak{n}\,.$$

A homogeneous Riemannian structure on G/H depends on a choice of ad_H-invariant complement $\mathfrak{m}$ to $\mathfrak{h}$ in $\mathfrak{g}$. Such a complement is the graph of an $\mathfrak{h}$-equivariant map

$$\varphi_r\colon \mathfrak{s}_r \to \mathfrak{h}\,. \tag{4.1}$$

Choose an $\mathfrak{h}$-equivariant map $\chi_r\colon \mathfrak{s} \to \mathfrak{s}_r$ extending the identity on $\mathfrak{n}$. Define $\varphi\colon \mathfrak{s} \to \mathfrak{h}$ as

$$\varphi = \varphi_r \circ \chi_r. \tag{4.2}$$

Proposition 4.1. *The Lie algebra* $\mathfrak{hol}$ *of the holonomy group of the canonical connection* $\widetilde{\nabla}$ *associated to the decomposition* $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$*, is*

$$\mathfrak{hol} = \varphi_r(\mathfrak{n}) = \varphi(\mathfrak{n}).$$

Proof. The holonomy algebra is spanned by $[\mathfrak{m},\mathfrak{m}]_{\mathfrak{h}}$. For $A \in \mathfrak{a}$ the standard generator and arbitrary $X \in \mathfrak{n}$, we have $[A, X] = X$. The space $\mathfrak{m}$ is spanned by

$$X_\varphi := X + \varphi X \in \mathfrak{m}, \qquad \text{for } X \in \mathfrak{n},$$

and the element

$$\xi := \chi_r A + \varphi A =: A + A_0.$$

Noting that $[A_0, \mathfrak{f}_r] = 0$, we compute

$$\begin{aligned}[\xi, X_\varphi] &= [A, X] + [A, \varphi X] + [A_0, X] + [A_0, \varphi X] \\ &= X + 0 + [A_0, X] + 0.\end{aligned}$$

This element lies in $\mathfrak{n}$. Moreover, A_0 acts on $\mathfrak{n}$ as an element of $\mathfrak{so}(n-1)$ on $\mathbb{R}^{n-1}$, in particular its characteristic polynomial has no non-zero real roots. This implies that $1 + \mathrm{ad}(A_0)\colon \mathfrak{n} \to \mathfrak{n}$ is invertible and so $\{[\xi, X_\varphi] : X \in \mathfrak{n}\}$ spans $\mathfrak{n}$. For $Y \in \mathfrak{n} \subset \mathfrak{h} + \mathfrak{m}$, we have $Y_{\mathfrak{m}} = Y + \varphi(Y)$ and so $Y_{\mathfrak{h}} = -\varphi(Y)$. We conclude that $\mathfrak{hol}$ contains $\{-\varphi[\xi, X_\varphi] : X \in \mathfrak{n}\} = \varphi(\mathfrak{n})$.

For $X, Y \in \mathfrak{n}$, we have

$$\begin{aligned}[X_\varphi, Y_\varphi] &= [X, Y] + [X, \varphi Y] + [\varphi X, Y] + [\varphi X, \varphi Y] \\ &= 0 + ([X, \varphi Y] + [\varphi X, Y]) + [\varphi X, \varphi Y].\end{aligned}$$

The last term lies in $\mathfrak{h}$, whereas the middle pair lies in $\mathfrak{n}$. Projecting to $\mathfrak{h} \subset \mathfrak{h} + \mathfrak{m}$, the middle pair contributes $-2[\varphi X, \varphi Y]$, since φ is $\mathfrak{h}$-equivariant. Thus

$$[X_\varphi, Y_\varphi]_{\mathfrak{h}} = -[\varphi X, \varphi Y].$$

We find that

$$\mathfrak{hol} = \varphi(\mathfrak{n}) + [\varphi(\mathfrak{n}), \varphi(\mathfrak{n})].$$

But $\mathfrak{h}$ is reductive and $\varphi(\mathfrak{n})$ is a sum of $\mathfrak{h}$-modules so $[\varphi(\mathfrak{n}), \varphi(\mathfrak{n})] \subset [\mathfrak{h}, \varphi(\mathfrak{n})] \subset \varphi(\mathfrak{n})$ and $\mathfrak{hol} = \varphi(\mathfrak{n})$ as claimed. $\square$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let us first show that the holonomy algebra has the claimed form. Via φ we have that the $\mathfrak{h}$-module $\mathfrak{hol}$ is isomorphic to a submodule $V_{\mathfrak{hol}}$ of $\mathfrak{n} \cong \mathbb{R}^{n-1}$. Write $\mathfrak{k} = \mathfrak{hol}$ and note that $\mathfrak{k}$ is a subalgebra of $\mathfrak{so}(n-1)$, so of compact type. We may thus split

$$\mathfrak{k} = \mathfrak{k}_0 + \mathfrak{k}_{ss}$$

as a sum of Abelian and semi-simple algebras. This gives a similar splitting $V_{\mathfrak{hol}} = V_0 + V_{ss}$.

Now $\mathfrak{k}$ acts effectively on $\mathfrak{m} \cong \mathfrak{a}_r + \mathfrak{n}$, and trivially on $\mathfrak{a}_r$, and its action preserves the inner product on $\mathfrak{n}$. The action of $\mathfrak{k}_{ss}$ is effective on V_{ss} and trivial on V_0. The action of $\mathfrak{k}_0$ is trivial on all of $V_{\mathfrak{hol}}$.

As $\mathfrak{k}_0 \cong \mathbb{R}^r$ is Abelian, its irreducible metric representations are direct sums of modules of real dimension 2, and an effective representation is of dimension at least $2r$. Thus $\mathfrak{n}$ contains inequivalent modules of dimension $\dim V_{\mathfrak{hol}} = \dim \mathfrak{k}_{ss} + r$ and $2r$. It follows that $n-1 = \dim \mathfrak{n} \geqslant \dim \mathfrak{k}_{ss} + 3r$.

Conversely, given a reductive algebra $\mathfrak{k} = \mathfrak{k}_0 + \mathfrak{k}_{ss}$ of compact type satisfying this constraint on dimensions, we wish to show that it arises as a holonomy algebra for a canonical connection.

Let $V_{\mathfrak{k}}$ be a copy of the $\mathfrak{k}$-module $\mathfrak{k}$ and let V_1 be a minimal effective metric representation of $\mathfrak{k}_0 \cong \mathbb{R}^r$. Then $\dim V_1 = 2r$ and we put

$$\mathfrak{n} = \mathbb{R}^{n-1} = V_{\mathfrak{k}} + V_1 + \mathbb{R}^m,$$

with $\mathbb{R}^m$ a trivial $\mathfrak{k}$-module. This decomposition of $\mathbb{R}^{n-1} = \mathfrak{n}$ admits a $\mathfrak{k}$-invariant inner product extending a bi-invariant metric on $\mathfrak{k} \cong V_{\mathfrak{k}}$ and the invariant inner product on V_1. Such an inner product realises $\mathfrak{k}$ as a subalgebra of $\mathfrak{so}(n-1)$. Let $\psi \colon V_{\mathfrak{k}} \to \mathfrak{k}$ be an isomorphism of $\mathfrak{k}$-modules. Defining φ to be ψ on $V_{\mathfrak{k}}$ and zero on $V_1 + \mathbb{R}^r + \mathfrak{a}$ then realises $\mathfrak{k}$ as the holonomy algebra of a canonical connection $\widetilde{\nabla}$ on $\mathbb{R}\mathrm{H}(n)$ with $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$. □

Note that in the construction of the second part of the proof, the Lie algebra $\mathfrak{k}$ exponentiates to a closed (so compact) subgroup K of $SO(n-1)$, and so $\mathfrak{k}$ is the isotropy algebra of a homogeneous realisation of $\mathbb{R}\mathrm{H}(n)$. Also note that the module $V_1 + \mathbb{R}^r$ may be replaced by any metric representation of $\mathfrak{k}$ on which $\mathfrak{k}_0$ acts effectively, but in this case the corresponding subgroup of $SO(n-1)$ may not be closed.

5. Homogeneous tensors

We now wish to compute the homogeneous tensor $S = \nabla - \widetilde{\nabla}$ associated to a invariant Riemannian structure on G/H.

Let g be the Riemannian metric and let g also denote its restriction to $\mathfrak{m}$. This bilinear form on $\mathfrak{m}$ is ad_H-invariant. At eH, the homogeneous tensor is given by

$$2g(S_B C, D) = g([B,C],D) - g([C,D],B) + g([D,B],C), \qquad (5.1)$$

for $B, C, D \in \mathfrak{m}$. This follows from (2.2) and [2, p. 183]. The description of $\mathbb{R}\mathrm{H}(n)$ as a symmetric space corresponds to $S \equiv 0$. We thus concentrate on the other homogeneous descriptions associated to subgroups F_r of $SO(n)\mathbb{R}_{>0}$, and use the notation of the previous section.

Note that g induces $\mathfrak{h}$-invariant inner products $g_r = (1+\varphi_r)^* g$ on $\mathfrak{s}_r$ and $g_\varphi = \chi_r^* g_r = (\chi_r^* + \varphi^*) g$ on $\mathfrak{s}$. As we remarked above, the space $\mathfrak{a}_r$ is not canonical. The module $\mathfrak{s}$ splits g_φ-orthogonally as a sum of a trivial $\mathfrak{h}$-module $\mathfrak{s}_0$ and a module $\mathfrak{s}_1 \subseteq \mathfrak{n}$ that decomposes as a sum of non-trivial $\mathfrak{h}$-modules. The space $\mathfrak{a}$ is contained in $\mathfrak{s}_0$ and is any complement to $\mathfrak{s}_0 \cap \mathfrak{n}$. In particular, we can take $\mathfrak{a}$ to be g_φ orthogonal to $\mathfrak{n}$ and take $\mathfrak{a}_r = \chi_r\, \mathfrak{a}$.

As above, let A be the generator of $\mathfrak{a}$ that satisfies $\operatorname{ad}(A)|_{\mathfrak{n}} = +1$. An arbitrary element B of $\mathfrak{m}$ may be written as

$$B = \lambda_B \xi + N_B,$$

where $N_B = (X_B)_\varphi$, for some $X_B \in \mathfrak{n}$. By our choice of $\mathfrak{a}$, we see that

$$\lambda_B = g(B,\xi)/g(\xi,\xi).$$

Lemma 5.1. *Let S be a homogeneous tensor on $\mathbb{R}\mathrm{H}(n)$ associated to module maps φ and φ_r as in* (4.2) *and* (4.1)*. Then*

$$\begin{aligned} g(S_B C, D) = {} & -\lambda_C g(B,D) + \lambda_D g(B,C) + g([B',C],D) \\ & + \tfrac{1}{2}\Big(\lambda_B\big(h_r(C,D) - h_r(D,C)\big) \\ & \qquad - \lambda_C\big(h_r(B,D) + h_r(D,B)\big) \\ & \qquad + \lambda_D\big(h_r(B,C) + h_r(C,B)\big)\Big), \end{aligned} \tag{5.2}$$

where $B' = \varphi(\lambda_B A + X_B) = \varphi(B_{\mathfrak{s}}) \in \mathfrak{h}$ and $h_r(B,C) = g([A_1, X_B]_\varphi, C)$, $A_1 = \chi_r A - A \in \mathfrak{so}(n-1)$.

Proof. To see this, let us compute

$$\begin{aligned} [B,C]_{\mathfrak{m}} &= (\lambda_B[\xi,N_C] - \lambda_C[\xi,N_B] + [N_B,N_C])_{\mathfrak{m}} \\ &= \lambda_B(N_C + [\varphi A, X_C] + [A_1, X_C]_\varphi) \\ &\qquad - \lambda_C(N_B + [\varphi A, X_B] + [A_1, X_B]_\varphi) \\ &\qquad + [\varphi X_B, N_C] - [\varphi X_C, N_B], \end{aligned}$$

where we have used that $\varphi[\varphi A, X_C] = [\varphi A, \varphi X_C] = 0$. This gives

$$\begin{aligned} g([B,C],D) &= \lambda_B\big(g(C,D) - \lambda_C g(\xi,D) + g([\varphi A, C],D) + h_r(C,D)\big) \\ &\qquad - \lambda_C\big(g(B,D) - \lambda_B g(\xi,D) + g([\varphi A, B],D) + h_r(B,D)\big) \\ &\qquad + g([\varphi X_B, C],D) - g([\varphi X_C, B],D) \\ &= \lambda_B g(C,D) - \lambda_C g(B,D) + g([B',C],D) - g([C',B],D) \\ &\qquad + \lambda_B h_r(C,D) - \lambda_C h_r(B,D) \end{aligned}$$

and the result (5.2) follows from (5.1) and the fact that g is $\mathfrak{h}$-invariant. □

We now wish to determine the possible types of S in the sense of Tricerri and Vanhecke [10]. The first of the Ambrose-Singer equations (2.1) implies that at each point S_x preserves g, so S is a section of $T^*M \otimes \mathfrak{so}(n) \cong TM \otimes \Lambda^2 TM$. As a representation of $\mathfrak{so}(n)$, this space decomposes as

$$\Gamma(TM \otimes \Lambda^2 TM) \cong \mathscr{T}_1 \oplus \mathscr{T}_2 \oplus \mathscr{T}_3$$

with $\mathscr{T}_1 \cong \Gamma(TM)$ and $\mathscr{T}_3 \cong \Gamma(\Lambda^3 TM)$. One says that S is of type $\mathscr{T}_i$ if S lies in $\mathscr{T}_1$, and correspondingly S is of type $\mathscr{T}_{i+j}$ if $S \in \mathscr{T}_i + \mathscr{T}_j$.

Tricerri and Vanhecke [10] showed that if (M, g) is connected, simply-connected and complete, then it admits a homogeneous structure of type $\mathscr{T}_1$ if and only if (M, g) is isometric to the standard metric on $\mathbb{R}\mathrm{H}(n)$. The corresponding homogeneous description is that of $\mathbb{R}\mathrm{H}(n)$ as a solvable group. Furthermore, [8] showed that structures on $\mathbb{R}\mathrm{H}(n)$ of type $\mathscr{T}_{1+3}$ correspond to semi-simple isotropy groups. We can now describe all the types of homogeneous structures on $\mathbb{R}\mathrm{H}(n)$.

Theorem 5.2. *Let S be a non-zero homogeneous tensor for $\mathbb{R}\mathrm{H}(n)$ with holonomy algebra $\mathfrak{hol}$. Then S always has a non-trivial component in $\mathscr{T}_1$ and S is of type $\mathscr{T}_1$ if and only if $\mathfrak{hol}$ is* 0.

The structure is of strict type $\mathscr{T}_{1+3}$ if and only if $\mathfrak{a} \subset \ker\varphi$ and $\mathfrak{hol}$ is a non-zero semi-simple algebra acting trivially on $\ker\varphi$, in the notation of Section 4.

Otherwise S is of general type.

Proof. From (5.2), we have $S = S^1 + S^2$, with

$$S^1_B C = g(\xi, \xi)^{-1}(g(B, C)\xi - g(C, \xi)B), \tag{5.3}$$

which is of type $\mathscr{T}_1$, and

$$\begin{aligned} S^2_B C &= [B', C] + S^r_B C, \\ 2S^r_B C &= (\lambda_B(Z_r - Z_r^*)(X_C) - \lambda_C(Z_r + Z_r^*)(X_B))_\varphi \\ &\qquad + (h_r(B, C) + h_r(C, B))g(\xi, \xi)^{-1}\xi, \end{aligned}$$

where $Z_r \colon \mathfrak{n} \to \mathfrak{n}$ is $Z_r = \mathrm{ad}(A_1)|_{\mathfrak{n}}$.

We claim that S^2 is of type $\mathscr{T}_{2+3}$. This means that $\sum_{i=1}^n S^2_{e_i} e_i = 0$ for an orthonormal basis of $\mathfrak{m}$. Noting that this condition is independent of the choice of orthonormal basis, we deal with the two terms of S^2 separately.

Let us show that the $(1, 2)$-trace $\sum_{i=1}^n S^r_{e_i} e_i$ is zero. Write g_0 for the metric on $\mathfrak{n}$ preserved by $\mathfrak{so}(n-1)$; this metric is unique up to scale. Then Z_r is skew-adjoint with respect to g_0. Let $E_1, \dots, E_{n-1}$ be a g_φ-orthonormal basis diagonalising g_0, so $g_0(E_i, E_i) = t_i > 0$. Then the matrix (z_{ij}) of Z_r satisfies $t_i z_{ji} + t_j z_{ij} = 0$ so $z_{ii} = 0$. Putting $e_i = (E_i)_\varphi$ and $e_n = \xi / g(\xi, \xi)^{1/2}$, we obtain an orthonormal basis for all of $\mathfrak{m}$. For $i = 1, \dots, n-1$, we have that $S^r_{e_i} e_i$ is $g(\xi, \xi)^{-1}\xi$ multiplied by the factor $h_r(E_i, E_i) = g((Z_r(e_i)_\varphi, E_i) = g_\varphi(Z_r(e_i), e_i) = z_{ii} = 0$, so $S^r_{e_i} e_i = 0$ in these cases. Moreover, $S^r_\xi \xi = 0$, and thus we have the claimed vanishing of the $(1, 2)$-trace of S^r.

For the remaining terms $\sum_{i=1}^n [e_i', e_i]$ of the $(1, 2)$-trace of S^2 we choose a different basis e_i. Write $\mathfrak{n} = V_{\mathfrak{hol}} + \ker\varphi|_{\mathfrak{n}}$, in such a way that these are $\mathfrak{h}$-modules whose images in $\mathfrak{m}$ are orthogonal. Choose a compatible orthonormal basis e_i for $\mathfrak{m}$ with $e_i = X_i + \varphi(X_i)$, $i = 1, \dots, n-1$, such that $X_i \in V_{\mathfrak{hol}}$,

$i = 1, \dots, k$, and $X_j \in \ker \varphi|_{\mathfrak{n}}$, $j = k+1, \dots, n-1$ and with e_n proportional to ξ. Then for $i = 1, \dots, n-1$ we have

$$[e_i', e_i] = [\varphi(X_i), X_i + \varphi(X_i)] = [\varphi(X_i), X_i].$$

This is clearly zero for $i = k+1, \dots, n-1$. For $i = 1, \dots, k$, the fact that $V_{\mathfrak{hol}}$ is an $\mathfrak{h}$-module implies $[\varphi(X_i), X_i] \in V_{\mathfrak{hol}}$, so $[e_i', e_i] = \psi^{-1}(\varphi[\varphi(X_i), X_i]) = \psi^{-1}[\varphi(X_i), \varphi(X_i)] = 0$. Finally $[e_n', e_n]$ is proportional to $[A_0, \xi] = [A_0, A] = 0$. Thus in all cases $[e_i', e_i] = 0$ and S^2 has no $\mathscr{T}_1$ component.

To see when S^2 is in $\mathscr{T}_3$, consider

$$\begin{aligned} S_2^2(B, C) &\coloneqq S_B^2 C + S_C^2 B \\ &= [C', X_B] + [B', X_C] - (Z_r^*(\lambda_B X_C + \lambda_C X_B))_\varphi \\ &\qquad + (h_r(B, C) + h_r(C, B)) g(\xi, \xi)^{-1} \xi, \end{aligned}$$

which is proportional to its projection to $\mathscr{T}_2$. For S^2 to belong to $\mathscr{T}_3$ we need this expression to be zero for all B and C. First, consider $C = \xi$ and B orthogonal to ξ, then $S_2^2(B, \xi) = [\varphi A, X_B] - (Z_r^* X_B)_\varphi = 0$. This implies that $g(S_2^2(B, \xi), D) = g_\varphi(X_B, [A_0, X_D]) = 0$, but the representation of $\mathfrak{so}(n-1)$ on $\mathfrak{n}$ is faithful, so $A_0 = 0$. Thus $\mathfrak{m}$ and hence $\mathfrak{g}$ contains $\mathfrak{a}$ and we may take $\mathfrak{s}_r = \mathfrak{s}$, giving $A_1 = 0$ and so $\varphi A = 0$. We now have $S_2^2(B, C) = [C', X_B] + [B', X_C]$. Second, suppose $X_B \in \ker \varphi$, then we must have $[C', X_B] = [\varphi(C_{\mathfrak{s}}), X_B] = 0$ for all $C_{\mathfrak{s}} \in \mathfrak{s}$. Thus $S \in \mathscr{T}_{1+3}$ requires $\ker \varphi$ to be a trivial $\mathfrak{hol}$-module. By the proof of Theorem 1.1, this is the case if and only if $\mathfrak{hol}$ is semi-simple and $\mathfrak{n} = V_{\mathfrak{hol}} + \mathbb{R}^s$, with the trivial module $\mathbb{R}^s$ lying in $\ker \varphi$. Moreover, in this situation, if B, C have $X_B, X_C \in V_{\mathfrak{hol}}$ then $S_2^2(B, C) = [\varphi(X_C), X_B] + [\varphi(X_B), X_C] = \psi^{-1}([C', B'] + [B', C']) = 0$, so $S \in \mathscr{T}_{1+3}$.

Furthermore, the $\mathscr{T}_3$-component is non-zero exactly when $\mathfrak{hol}$ is non-trivial. Indeed, in general the $\mathscr{T}_3$-component is proportional to

$$\begin{aligned} S_3^2(B, C) &\coloneqq S_B^2 C - S_C^2 B \\ &= [B', C] - [C', B] + (Z_r(\lambda_B X_C - \lambda_C X_B))_\varphi. \end{aligned}$$

Suppose this tensor S_3^2 is zero. Considering $B = \xi$ and C orthogonal to ξ, we have $S_3^2(\xi, C) = [\varphi A, C] + Z_r(X_C)_\varphi = [A_0, X_C]_\varphi$, since $[C', A] = 0 = [C', \varphi A] = [C', A_1]$ as $C' \in \mathfrak{h}$. This gives $A_0 = 0$, and we may write

$$S_3^2(B, C) = 2[B', C'] + [B', X_C] - [C', X_B].$$

For general B, C, the component of $S_3^2(B, C)$ in $\mathfrak{h}$ is $2[B', C']$. This implies that $\mathfrak{hol}$ is Abelian. Finally for $X_C \in \ker \varphi$ and $\lambda_C = 0$, we have $S_3^2(B, C) = [B', X_C]$, so $\mathfrak{hol}$ acts trivially on $\ker \varphi$. By the proof of Theorem 1.1 we conclude that $\mathfrak{hol} = 0$. Thus the $\mathscr{T}_3$-component is zero exactly when $\varphi = 0$. □

6. Consequences

Our description of splittings via graphs in Section 4 yields the following statement.

Theorem 6.1. *For $n > 1$, the moduli space of homogeneous tensors on $\mathbb{R}\mathrm{H}(n)$ with fixed scalar curvature, consists of two connected components.*

Proof. Any non-zero S is homotopic to the S of type $\mathscr{T}_1$ on AN via a scaling of φ to 0. So there are at most two components in the moduli space. We need to show that $\{S = 0\}$ is a separate component.

We have $\mathbb{R}\mathrm{H}(n) = SO(n,1)/O(n)$, $SO_0(n,1) = KAN$ and Theorem 3.1 tells us that A_rN acts transitively on any homogeneous description of $\mathbb{R}\mathrm{H}(n)$. Now A_rN is isomorphic to AN as a group, and any metric on AN is hyperbolic, indeed the isomorphism may be chosen to be an isometry of left-invariant metrics, cf. [4]. With fixed scalar curvature, we may assume that this is an isometry to one fixed choice of left-invariant hyperbolic metric on AN. If S is a homogeneous tensor on $\mathbb{R}\mathrm{H}(n)$ it gives a left-invariant tensor on A_rN and hence on AN. The equation $\widetilde{\nabla} S = 0$ may be rewritten as

$$\nabla S = S.S, \tag{6.1}$$

where $(S_X.S)_Y Z = S_X(S_Y Z) - S_{S_X Y} Z - S_Y(S_X Z)$, and ∇ is the Levi-Civita connection. On the set of left-invariant tensors on AN, equation (6.1) is a set of polynomial equations in the components of S. We thus see that the set of homogeneous tensors for $\mathbb{R}\mathrm{H}(n)$ may be regarded as a real algebraic variety $\mathscr{S}$ in $\mathbb{R}^n \otimes \Lambda^2\mathbb{R}^n = \mathbb{R}^N$. The moduli space is a quotient of $\mathscr{S}$ by the relation of isomorphism of homogeneous structures; in particular tensors S with different holonomy groups give rise to different points of the moduli space. Once we have shown that $\{S = 0\}$ is a separate component of $\mathscr{S}$, we will have that the components of $\mathscr{S}$ are preserved by the equivalence relation and so give distinct components of the moduli space.

Now for any real algebraic variety $\mathscr{S} \subset \mathbb{R}^N$ and any point S in $\mathscr{S}$ there is an analytic path S_t, $t \in [0,1]$, with $S_0 = S$ and $S_t \neq S$, for $t \in (0,1]$. Indeed such a path may be taken to be a Nash function, see Bochnak et al. [3, Proposition 8.1.17]. Combining this with [3, Définition et Proposition 2.5.11] one has that the connected components of $\mathscr{S}$ are analytically path-connected.

Suppose S_t is an analytic path of homogeneous structures with $S_0 = 0$. Then equation (6.1), gives that for $\dot{S} = dS_t/dt|_{t=0}$, we have $\nabla\dot{S} = \dot{S}.S_0 + S_0.\dot{S} = 0$, so $\dot{S}$ is parallel for the Levi-Civita connection. But any parallel tensor on $\mathbb{R}\mathrm{H}(n)$ is holonomy invariant.

The holonomy representation of ∇ on the tangent space of $\mathbb{R}\mathrm{H}(n)$ is $U = \mathbb{R}^n$ as the standard representation of $SO(n)$. The tensor $\dot{S}$ lies in

$$U \otimes \Lambda^2 U \cong U \oplus \Lambda^3 U \oplus W,$$

with W an irreducible representation of $SO(n)$ of dimension $\frac{1}{3}n(n-2)(n+2)$. This decomposition contains an invariant submodule only when $n = 3$. So for $n \neq 3$, we conclude that $\dot{S} = 0$.

We may repeat this argument for the higher derivatives of S_t at $t = 0$. When $n \neq 3$, this gives that S_t has Taylor expansion 0 around $t = 0$, and thus that S_t is the constant path. So for $n \neq 3$, we have that $\{S = 0\}$ is a connected component of the moduli space.

For $n = 3$, we may argue more directly. By Theorem 1.1, the holonomy algebras of homogeneous connections on $\mathbb{R}\mathrm{H}(3)$ are $\mathfrak{so}(3)$ and 0, since the only other possibility is $\mathfrak{so}(n-1) = \mathfrak{so}(2)$, which is Abelian, but has $3 \dim \mathfrak{so}(2) = 3 > n-1 = 2$. Thus there are only two homogeneous structures on $\mathbb{R}\mathrm{H}(3)$, one with $S = 0$, and the other of type $\mathscr{T}_1$ by Theorem 5.2. For structures of type $\mathscr{T}_1$, the tensor S is the S^1 of equation (5.3). The scalar curvature of the corresponding metric is determined by $\|\xi\|^2$, so for fixed scalar curvature, there is no path to $S = 0$, and the moduli space again has two components. □

Note that the final of the proof part confirms the determination of homogeneous structures on $\mathbb{R}\mathrm{H}(3)$ by Tricerri and Vanhecke [10].

The proof of the main Theorem 1.1 yields the following information about the action of the holonomy group.

Corollary 6.2. *Suppose $\widetilde{\nabla}$ is a homogeneous canonical connection on $\mathbb{R}\mathrm{H}(n)$ whose holonomy algebra $\mathfrak{hol}$ is not $\mathfrak{so}(n)$. Then the isotropy representation of $\mathfrak{hol}$ on $\mathfrak{m}$ contains at least three disjoint modules: the first isomorphic to $\mathfrak{hol}$, the second an effective representation of the centre of $\mathfrak{hol}$ and the third a one-dimensional trivial module. If $\mathfrak{hol}$ is semi-simple the second module is zero.*

Proof. In the notation of the proof of Theorem 1.1 the three modules are $V_{\mathfrak{hol}}$, V_1 and $\mathfrak{a}_r$. □

Let us regard a geometric structure as any collection of tensors preserved by some connection, not necessarily torsion-free. We say this geometry is homogeneous if can be realised on a reductive homogeneous space with the connection being the canonical connection.

Corollary 6.3. *Any homogeneous geometry on $\mathbb{R}\mathrm{H}(n)$ that is not invariant under the connected isometry group $SO_0(n,1)$, admits a nowhere vanishing parallel vector field.* □

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