Entanglement and disorder: a mean field approach

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Abstract. The entanglement of symmetric mean-field ground states is studied for system of finite size. While a bare mean-field approximation amounts to consider factorized states, the symmetrization allows one to restore an amount of entanglement, which survive if finite systems are considered. For the isotropic XY model in the presence of a random transverse field, we calculated the one-site entanglement entropy, which is zero both for vanishing and maximum disorder and reaches a maximum for an intermediate value of disorder .

1. Introduction

During last years, a new interest in quantum properties of many-body ground states (GS) has grown due to possible applications in quantum computing. In this perspective, a great attention has been devoted to the study of entanglement across phase transitions [1, 2, 3, 4]. The problem of quantification of entanglement in macroscopic systems is discussed in [5]. Due to the issue of possible implementation in real systems, finite-size effects are obviously important [6, 7].

Due to the issue of possible implementation in real systems, finite-size effects are obviously important. Research in the field has considered only exactly solvable models or molecular dynamics approaches. The widely used mean-field (MF) approximation, as it is commonly defined, does not satisfy both requirements because based on a factorised structure of the ground state and gives results meaningful only in the limit of a large system size (LSL). The point is that in the case of systems with a continuous symmetry, MF solution breaks the system symmetry, and thus the generator of the symmetry transformation has non vanishing fluctuations. For instance, in the case of a superconductor, this generator is the particle number operator. This drawback of the approximation is however negligible in the LSL, since fluctuations scale with the inverse of the system size.

It must be recalled that the size problem was considered at least in the case of superconductors already in the original BCS paper, where an alternative formulation was proposed which has been largely applied in nuclear physics and superconductivity at the nanometer scale. In a previous paper, we studied two cases where the mean field theory is exact (the XY chain at the factorizing point and the isotropic Heisenberg model), and discussed how the symmetry of the model influences the disappearing of entanglement [8].

The idea is to define a manifold of equivalent ground states related each other by the symmetry operator and introduce a generalized GS given by the superposition of such states. It is easily shown that, in a suitable superposition, fluctuations vanish for any system size. We shall refer in the following to such state as a symmetric mean field GS (SMF).

Moreover, being the new state a superposition of factorized states, it is expected to be an entangled state at least for finite size systems.

In fact, according to the superselection rules, interference effects disappear in large size limit, and then, even entanglement is expected to vanish in the same limit. Typical quantities like the reduced density matrices become in the LSL the sum of terms which describe the "mixture" state associated to the zero order in the expansion in the inverse system size plus a correction linear in the inverse size which gives the entanglement. Within this framework, we study the role of disorder on the entangled component of the ground-state one-site reduced density matrix (1-RDM) in the case of the homogeneous XY model in the presence of a random field. The dependence of entanglement on the randomness of a spin system has been given in [9, 10]. In [11], the study has been applied to the one-electron Anderson model.

We obtain that, in the parameter region where the symmetry breaking occurs, the averaged over disorder one-site Von Neumann entropy associated to the 1-RDM, shows corrections to LSL which vanish both for zero and maximum disorder, and reaches a maximum at intermediate disorder in agreement with the results of [9]. In [12], the role of the size system in the study of entanglement has been applied to the case of the Dicke model.

2. Large size limit of SMF

LSL is a crucial step in the definition of phase transitions associated to a spontaneous symmetry breaking.

In the standard way to achieve LSL, a symmetry breaking field (SBF) is added to the system which selects one of the possible eigenstates, and then the spontaneous symmetry breaking is obtained considering the limit of vanishing SBF after LSL has been performed.

The drawback of this approach is that relevant fluctuations are out of control for any finite size.

LSL can be studied taking into account that always only a finite number of degrees of freedom $ij \dots k$ are physically accessible, and then what is of interest is the reduced density matrix (RDM) associated to them: $\rho_{ij\dots k} = \text{Tr}_{\{l \neq ij\dots k\}} |\Phi\rangle \langle \Phi|$.

In the case of a mean field approximation, even in the presence of a SBF, the ground state is a product of elements each defined in the Hilbert space of the i -th degree of freedom: $|\Phi(\varphi)\rangle = \otimes_l |\Phi_l(\varphi)\rangle$, or $|\Phi(\varphi)\rangle = U_\varphi |\Phi(\varphi)\rangle$, where φ is the index which determines the particular element under consideration of the MF ground states manifold and U_φ is the symmetry transformation. The RDM becomes the product of the one degree of freedom RDMs associated to $ij \dots k$: $\rho_{ij\dots k}(\varphi) = \rho_i \rho_j \dots \rho_k$, with $\rho_i(\varphi) = |\Phi_i(\varphi)\rangle \langle \Phi_i(\varphi)|$. It is worth to note that only if we consider a quantity which commutes with the symmetry operator, the average of this quantity does not depend on φ . Moreover, a quantity which is the sum of terms associated to different degrees of freedom has an average which is the sum of corresponding average over each degree of freedom. It is immediately seen that interference effects are not present in the factorized GS.

Following a different approach, in the case of a SMF, we introduce a superposition of states

$$|\Phi\rangle = \int d\varphi a(\varphi) |\Phi(\varphi)\rangle \quad (1)$$

In this case, interference effects appear.

The RDMs become

$$\rho_{ij\dots k} = \int d\varphi d\varphi' K(\varphi - \varphi') \rho_i(\varphi, \varphi') \rho_j(\varphi, \varphi') \dots \rho_k(\varphi, \varphi'), \quad (2)$$

with

$$\rho_i(\varphi, \varphi') = \frac{|\Phi_i(\varphi)\rangle \langle \Phi_i(\varphi')|}{\langle \Phi_i(\varphi') | \Phi_i(\varphi) \rangle}, \quad (3)$$

and $K(\varphi - \varphi') = a^*(\varphi') a(\varphi) \langle \Phi(\varphi') | \Phi(\varphi) \rangle$.

The superselection principle implies asymptotic orthogonality of different $|\Phi_i(\varphi)\rangle$, and then vanishing of interference for the RDMs associated with any arbitrarily large but finite number of degrees of freedom:

$$\lim_{N \rightarrow \infty} K(\varphi - \varphi') = \delta(\varphi - \varphi') |a(\varphi)|^2, \quad (4)$$

$$\lim_{N \rightarrow \infty} \rho_{ij\dots k} = \int d\varphi |a(\varphi)|^2 \rho_i(\varphi) \rho_j(\varphi) \dots \rho_k(\varphi). \quad (5)$$

In terms of RDM properties, we obtain that, in the thermodynamic limit, the SMF properties are the same of a mixed state as far as any finite number degrees of freedom are concerned. Entanglement is related to correction with respect to LSL.

Expanding close to $\varphi - \varphi' \simeq 0$, the non diagonal RDMs read

$$\begin{aligned} \rho_{ij\dots k}(\varphi, \varphi') &= \rho_{ij\dots k}^{(0)}(\varphi, \varphi) + \rho_{ij\dots k}^{(1)}(\varphi, \varphi)(\varphi - \varphi') \\ &+ \rho_{ij\dots k}^{(2)}(\varphi, \varphi)(\varphi - \varphi')^2 + \dots, \end{aligned} \quad (6)$$

and

$$\rho_{ij\dots k}^{(n)} = \int d\varphi d\varphi' K(\varphi - \varphi') \rho_{ij\dots k}^{(n)}(\varphi, \varphi) (\varphi - \varphi')^n. \quad (7)$$

The information content is measured by the Von Neumann entropies, defined as

$$S_{ij\dots k} = -\frac{1}{\log 2} \text{Tr} [\rho_{ij\dots k} \log \rho_{ij\dots k}], \quad (8)$$

and conveniently calculated through the identity

$$S_{ij\dots k} = -\frac{1}{\log 2} \left(\frac{d}{dn} \text{Tr} \rho_{ij\dots k}^n \right)_{n=1} \quad (9)$$

Assuming $\rho_{ij\dots k} = \rho_{ij\dots k}^{(0)} + \delta\rho_{ij\dots k}$, we obtain

$$\left(\frac{d}{dn} \text{Tr} \rho_{ij\dots k}^n \right)_{n=1} \simeq \text{Tr} \rho_{ij\dots k}^{(0)} \log \rho_{ij\dots k}^{(0)} + \text{Tr} [\delta\rho_{ij\dots k} \log \rho_{ij\dots k}^{(0)}] \quad (10)$$

Here we have taken into account that the unitarity of trace beeing satisfied to the zero order implies that the correction matrix is traceless.

We obtain the Von Neumann entropies as the sum of the mixture entropies and the entanglement entropies

$$S_{ij\dots k}^{mix} = -\frac{1}{\log 2} \text{Tr} \rho_{ij\dots k}^{(0)} \log \rho_{ij\dots k}^{(0)}, \quad (11)$$

$$S_{ij\dots k}^{ent} = -\frac{1}{\log 2} \text{Tr} [\delta\rho_{ij\dots k} \log \rho_{ij\dots k}^{(0)}]. \quad (12)$$

As we shall see, the leading contribution to the entanglement entropy is of the order of the inverse system size. Thus, there is no definite size scale which separates classical (the mixed state entropies) from quantum behaviour (entangled entropies).

It is expected that entanglement calculated in the SMF is relevant when MF is accurate. In this case corrections due the size can be more important in a suitable size range than corrections due to fluctuations with respect to MF [13] (RPA see Rossignoli)

3. XY model in the presence of a random transverse field

The general framework of the previous section is now applied to the study of the relation between entanglement and disorder in a very simple case. The Hamiltonian of the model reads

$$H = -\frac{J}{2D} \sum_{j, i \in I_j} [\sigma_i^+ \sigma_j^- + h.c.] + \sum_i h_i \sigma_i^z \quad (13)$$

Here I_j is the set of D sites next neighbours of site j , and σ denotes the Pauli matrices. The random transverse magnetic field is distributed dicotomically following

$$P(h_i) = \frac{1}{2} [\delta(h - h_i) + \delta(h + h_i)] \quad (14)$$

The model can be also considered as a particles model on a lattice: the empty site is associated to the spin down state the occupied site to the spin up state. Double multiple occupation of sites is allowed.

The first term is the kinetic energy associated to the transfer of particles from one site to the D next neighbors sites, while the second one introduces a random energy associated to empty or occupied sites. Quantum information properties of the model have been recently studied by Kinoshita [9].

The MF approximation is obtained introducing a local state as superposition of spin up e down states:

$$|\Phi_l\rangle = \cos \theta_l |\uparrow_l\rangle + \sin \theta_l |\downarrow_l\rangle \quad (15)$$

For the calculation of entangled entropies it is enough to use the variational results in the LSL. We obtain

$$\cos 2\theta_l = \frac{h_l}{\sqrt{h_l^2 + J^2 \Delta_l^2}} \quad (16)$$

$$\sin 2\theta_l = \frac{J \Delta_l}{\sqrt{h_l^2 + J^2 \Delta_l^2}} \quad (17)$$

Taking into account that $h_l^2 = h^2$, because of the assumed dicotomic probability distribution of the transverse field, the self consistency equation for the gap parameter is satisfied by a site independent quantity Δ such that

$$\Delta_l = \frac{J}{D} \sum_{l' \in I_l} \frac{\Delta_{l'}}{\sqrt{h^2 + J^2 \Delta_{l'}^2}}. \quad (18)$$

A site independent non vanishing solution for the “gap” parameter is obtained if $(h/J) < 1$. In this case

$$\Delta = \sqrt{1 - \frac{h^2}{J^2}} \quad (19)$$

and

$$\cos 2\theta_l = \frac{h_l}{J}, \quad (20)$$

$$\sin 2\theta_l = \sqrt{1 - \frac{h^2}{J^2}}. \quad (21)$$

The symmetry of the model is the rotation around the z axis $U_\varphi = e^{-i\varphi \sigma^z}$, with $\sigma^z = \sum_l \sigma_l^z$. The rotation however modifies the MF ground state.

$$U_l(\varphi) |\Phi_l\rangle = \cos \theta_l e^{-i\varphi} |\uparrow_l\rangle + \sin \theta_l e^{i\varphi} |\downarrow_l\rangle \quad (22)$$

This is the broken symmetry associated to the MF.

Each element of the manifold of states obtained by the symmetry operation $|\Phi(\varphi)\rangle = \otimes_l U_\varphi |\Phi_l\rangle$ obviously satisfies the variational principle

$$\langle \Phi(\varphi) | H | \Phi(\varphi) \rangle = \langle \Phi | U_\varphi^\dagger H U_\varphi | \Phi(\varphi) \rangle = \langle \Phi | H | \Phi \rangle. \quad (23)$$

The symmetry breaking of MF implies non vanishing fluctuations of σ^z which in the particles interpretation imply a fluctuating particles number. Introducing the intensive quantity $\tilde{\sigma}^z = \sigma^z / N$, we obtain

$$\langle (\tilde{\sigma}^z)^2 \rangle - \langle \tilde{\sigma}^z \rangle^2 = \frac{v^2}{N}, \quad (24)$$

with

$$v^2 = \frac{1}{N} \sum_l \sin^2 2\theta_l. \quad (25)$$

Then, fluctuations disappear only in the LSL.

The hamiltonian symmetry can be restored by introducing a SMF, defined through the superposition of all possible symmetry-broken MF states:

$$|\Phi\rangle = \frac{1}{\Lambda} \int_0^{2\pi} d\varphi' e^{i\langle\sigma^z\rangle\varphi'} |\Phi(\varphi')\rangle \quad (26)$$

In fact, it is easily shown

$$e^{-i\sigma^z\varphi} |\Phi\rangle = e^{-i\langle\sigma^z\rangle\varphi} |\Phi\rangle \quad (27)$$

The first point is the study of the transition from quantum behaviour in the present model. Let us consider the overlap kernel

$$K_N(\varphi - \varphi') = \frac{1}{\Lambda} e^{i\langle\sigma^z\rangle(\varphi' - \varphi)} \prod_l \langle\Phi_l(\varphi) | \Phi_l(\varphi')\rangle, \quad (28)$$

where Λ is a normalization factor. Because of the structure of $|\Phi_l(\varphi)\rangle$

$$\begin{aligned} K_N(\varphi - \varphi') &= \frac{1}{\Lambda} e^{i\langle\sigma^z\rangle(\varphi' - \varphi)} \\ &\times \prod_l (\cos(\varphi - \varphi') + i \sin(\varphi - \varphi') \cos 2\theta_l) \end{aligned} \quad (29)$$

In the integration region where $(\varphi - \varphi') \simeq 0$,

$$\begin{aligned} K_N(\varphi - \varphi') &\simeq \frac{1}{\Lambda} e^{-i\langle\sigma^z\rangle(\varphi' - \varphi)} \\ &\times \exp \left[\sum_l \left(-\frac{1}{2} (\varphi' - \varphi)^2 \sin^2 2\theta_l + i (\varphi' - \varphi) \cos 2\theta_l \right) \right] \end{aligned} \quad (30)$$

Taking into account that $\langle\sigma^z\rangle = \sum_l \cos 2\theta_l$, we obtain

$$K_N(\varphi - \varphi') \simeq \frac{1}{\Lambda} \exp \left[-\frac{N}{2} (\varphi' - \varphi)^2 v^2 \right] \quad (31)$$

3.1. The variational principle for SMF

The new ground state implies different variational conditions. It is however easily shown that corrections disappear in the LSL. In fact,

$$\langle\Phi|H|\Phi\rangle = \frac{1}{\Lambda} \sum_l h_l \int_0^{2\pi} d\varphi d\varphi' K(\varphi - \varphi') \frac{\langle\Phi_l(\varphi) | \sigma_l^z | \Phi_l(\varphi')\rangle}{\langle\Phi_l(\varphi) | \Phi_l(\varphi')\rangle} \quad (32)$$

$$\begin{aligned} &+ \frac{J}{D} \sum_l \int_0^{2\pi} d\varphi d\varphi' K(\varphi - \varphi') \\ &\times \left[\sum_{l' \in I_l} \Delta_{l'} \frac{\langle\Phi_l(\varphi) | \sigma_l^- | \Phi_{l'}(\varphi')\rangle}{\langle\Phi_l(\varphi) | \Phi_{l'}(\varphi')\rangle} + c.c. \right], \end{aligned} \quad (33)$$

with

$$\Delta_l(\varphi - \varphi') = \frac{1}{D} \sum_{l' \in I_l} \frac{\langle\Phi_{l'}(\varphi) | \sigma_{l'}^+ | \Phi_{l'}(\varphi')\rangle}{\langle\Phi_{l'}(\varphi) | \Phi_{l'}(\varphi')\rangle}. \quad (34)$$

It can be shown that

$$\begin{aligned} \Delta_l(\varphi - \varphi') &= \frac{1}{D} \sum_{l' \in I_l} \sin 2\theta_{l'} + 2i(\varphi - \varphi') \frac{1}{D} \sum_{l' \in I_l} \sin^2 \theta_l \sin 2\theta_{l'} \\ &+ (\varphi - \varphi')^2 \frac{1}{D} \sum_{l' \in I_l} \sin^2 \theta_l \sin 4\theta_{l'} \end{aligned} \quad (35)$$

As we have seen, the kernel K selects, for large N , only diagonal elements in the integral. Moreover, K is symmetric, and thus only second order corrections to the diagonal contribution in $(\varphi - \varphi')$ are present in the variational equation. As we shall see these corrections can be neglected in the study of the entangled component of the GS entropy.

3.2. Entanglement entropies

From the expansion of the one-site density matrix, we obtain a diagonal structure, since the angular integration cancels non diagonal elements in the 2×2 matrix associated to $|i\rangle\langle k|$ with $i, k = 1(\uparrow), 2(\downarrow)$. We have

$$\rho_l = \rho_l^{11} |\uparrow_l\rangle\langle\uparrow_l| + \rho_l^{22} |\downarrow_l\rangle\langle\downarrow_l|, \quad (36)$$

with $\rho_l^{11} = a_N^{(+)} \cos^2 \theta_l$, $\rho_l^{22} = a_N^{(-)} \sin^2 \theta_l$, and

$$a_N^{(\pm)} = \int_0^{2\pi} d(\varphi - \varphi') K_N(\varphi - \varphi') \frac{1 \pm i \tan(\varphi - \varphi')}{1 + i \tan(\varphi - \varphi') \cos 2\theta_l}. \quad (37)$$

To the leading order in $1/N$,

$$a_N^{(+)} \simeq 1 + (1 - \cos 2\theta_l) \frac{\cos 2\theta_l}{Nv^2}, \quad (38)$$

$$a_N^{(-)} \simeq 1 - (1 + \cos 2\theta_l) \frac{\cos 2\theta_l}{Nv^2}. \quad (39)$$

The following identifications can be made: $\rho_l^{(0)11} = \cos^2 \theta_l$, $\rho_l^{(0)22} = \sin^2 \theta_l$, and

$$\rho_l^{(1)11} = \frac{1}{2} [\sin^2 2\theta_l]_0 [\cos 2\theta_l]_0 \left(\frac{1}{Nv^2} \right), \quad (40)$$

$$\rho_l^{(1)22} = -\frac{1}{2} [\sin^2 2\theta_l]_0 [\cos 2\theta_l]_0 \left(\frac{1}{Nv^2} \right). \quad (41)$$

The corresponding von Neumann entanglement entropy is

$$\begin{aligned} S_l^{ent} &= -\frac{1}{2} \left(\frac{1}{Nv^2} \right) [\sin^2 2\theta_l]_0 [\cos 2\theta_l]_0 \\ &\times [\log(1 + [\cos 2\theta_l]_0) - \log(1 - [\cos 2\theta_l]_0)], \end{aligned} \quad (42)$$

where $\cos(2\theta_l)$ and $\sin(2\theta_l)$ are calculated from the variational principle to the zeroth order in the inverse size: $[\cos 2\theta_l]_0 = h/J$ and $[\sin 2\theta_l]_0 = \sqrt{1 - (h/J)^2}$. The final result is

$$S^{ent} = \frac{1}{2} \left(\frac{1}{Nv^2} \right) \left(1 - \frac{h^2}{J^2} \right) \frac{h}{J} \left[\log \left(1 + \frac{h}{J} \right) - \log \left(1 - \frac{h}{J} \right) \right]. \quad (43)$$

We see that $S^{ent}(0) = 0$ and $S^{ent}(1) = 0$. The maximum amount of entanglement is reached for an intermediate value of h/J .

Figure 1. Entanglement entropy as a function of h/J

4. Conclusion

In conclusion, a simple analytic theory for the relation between entanglement and disorder is derived for the isotropic XY model in the presence of a transverse random magnetic field. Using a mean-field approach, we built a state which respects the Hamiltonian symmetry for a system of finite size. Within this approximation, we derived the entanglement content of this state. As expected, the entanglement entropy tends to disappear once the thermodynamic limit is made. We studied the corrections due to finiteness of the size. The same approach can be easily generalized to the study of many-site reduced density matrices.

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References

- [1] Osborne T J and Nielsen M A 2002 *Phys. Rev. A* **66** 032110
- [2] Osterloh A, Amico L, Falci G and Fazio R 2002 *Nature (London)* **416** 608–10
- [3] Vidal G, Latorre J I, Rico E and Kitaev A 2003 *Phys. Rev. Lett.* **90** 227902
- [4] Amico L, Fazio R, Osterloh A and Vedral V 2008 *Rev. Mod. Phys.* **80** 517–76
- [5] Vedral V *Nature* 2008 **453** 1004–7
- [6] de Pasquale F, Giorgi G L and Paganelli S 2005 *Phys. Rev. A* **71** 042304
- [7] Paganelli S, de Pasquale F and Giorgi G L 2006 *Phys. Rev. A* **74** 012316
- [8] de Pasquale F, Giorgi G L and Zannetti M 2009 *Fortschr. Phys.* **57** 1111–20
- [9] Fujinaga M and Hatano N 2007 *J. Phys. Soc. Jpn.* **76** 094001
- [10] Saguia A, Sarandy M S, Boechat B and Continentino M A 2007 *Phys. Rev. A* **75** 052329
- [11] Li H and Wang X G 2005 *Mod. Phys. Lett. B* **19** 517–27
- [12] Tsyplatyev O and Loss D 2009 *J. Phys.: Conf. Ser.* **193** 012134
- [13] Rossignoli R, Canosa N and Matera J M 2008 *Phys. Rev. A* **77** 052322