Some Entanglement Features of Highly Entangled Multiqubit States

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Abstract

We explore some basic entanglement features of multiqubit systems that are relevant for the development of algorithms for searching highly entangled states. In particular, we compare the behaviours of multiqubit entanglement measures based (i) on the von Neumann entropy of marginal density matrices and (ii) on the linear entropy of those matrices.

I. INTRODUCTION

The study of quantum entanglement is contributing both to the elucidation of the foundations of quantum mechanics and to the birth of new, revolutionary technologies [1, 2, 3]. A considerable amount of research has recently been devoted to the study of multiqubit entanglement measures defined as the sum of bipartite entanglement measures over all (or an appropriate family of) the possible bi-partitions of the full system [4, 5, 6, 7]. The aim of the present contribution is to explore some basic properties of highly entangled multiqubit states, and also of the "entanglement landscape" in their neighbourhoods. We compare the behaviours of two entanglement measures for multiqubit pure states, one based on the von Neumann entropy of marginal density matrices and the other based upon the linear entropy of those matrices. We also compare the performances of two searching algorithms for highly entangled states, based on different families of bi-partitions of the multiqubit system.

II. MULTIQUBIT ENTANGLEMENT

The genuine multipartite entanglement E of a N-qubit state can be expressed as

$$E = \frac{1}{[N/2]} \sum_{m=1}^{[N/2]} E^{(m)}, \tag{1}$$

$$E^{(m)} = \frac{1}{N^m_{bipart}} \sum_{i=1}^{N^m_{bipart}} E(i). \tag{2}$$

Here, $E(i)$ stands for the entanglement associated with one, single bi-partition of the N-qubits system. The quantity $E^{(m)}$ gives the average entanglement between subsets of m qubits and the remaining $N-m$ qubits constituting the system. The average is performed over the $N^{(m)}_{bipart}$ nonequivalent ways to do such bi-partitions, which are given by

$$N^m_{bipart} = \binom{N}{n} \text{ if } n \neq N/2, \tag{3}$$

$$N^{N/2}_{bipart} = \frac{1}{2}\binom{N}{N/2} \text{ if } n = N/2. \tag{4}$$

Different $E^{(m)}$ represent different entanglement properties of the state. While $E^{(1)}$ can attain its maximum value for a given state, $E^{(2)}$ can be arbitrarily low for such state. This is why all these entanglement measures must be computed to capture all the entanglement properties of the state. The global multiqubit entanglement is given by the average of the $[N/2]$ different $E^{(m)}$ for any state $|\Psi\rangle$.

We will use two types on entanglement measures, E_L and E_{vN}, respectively based on two different measures for the mixedness of the marginal density matrices ρ_i associated with the bi-partitions: (i) the linear entropy $S_L = \frac{2^m}{2^m-1}(1 - Tr[\rho_i^2])$, and (ii) the von Neumann entropy $S_{vN} = -Tr[\rho_i log \rho_i]$. If one uses the linear entropy S_L, $E_L^{(1)}$ turns out to be the well known Meyer-Wallach multipartite entanglement measure [8] that Brennen [9] showed to coincide with the average of all the single-qubit linear entropies. This measure was later generalized by Scott [10] to the case where all possible bi-partitions of the system where considered.

The entanglement measure given in Eq. (1) is maximized by a state which has all its reduced density matrices maximally mixed. Although it is easy to verify that in the 3 qubit case $|GHZ\rangle$ complies with this requirement, the situation becomes much more complicated when systems with four or more qubits are considered. Higuchi and Sudbery proved that there is no 4 qubit state whose two qubit reduced density matrices are all maximally mixed. They conjectured that the 4 qubit state exhibiting the higher entanglement is

$$|HS\rangle = \frac{1}{\sqrt{6}}\Big[|1100\rangle + |0011\rangle + \omega\Big(|1001\rangle + |0110\rangle\Big) + \omega^2\Big(|1010\rangle + |0101\rangle\Big)\Big], \qquad (5)$$

with $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}$. Although it still remains unproven, several analytical [11, 12] and numerical [13, 14] evidences support the aforementioned conjecture. In the cases of 5 and 6 qubits, states have been identified having all their reduced density matrices maximally mixed [13, 14]. Finally, for 7 qubits there is numerical evidence suggesting that a recently discovered state is the one with maximal entanglement although, as in the 4 qubit case, no state of 7 qubits with all its reduced density matrices maximally mixed was found [14].

III. BEHAVIOUR OF E_L AND E_{VN} FOR HIGHLY ENTANGLED STATES OF 4 QUBITS.

The multipartite entanglement measures E_L based on the averaged linear entropies of the reduced density matrices are widely used, but sometimes it is more convenient to use an entanglement measure E_{vN} based on the von Neumann entropy of the reduced density

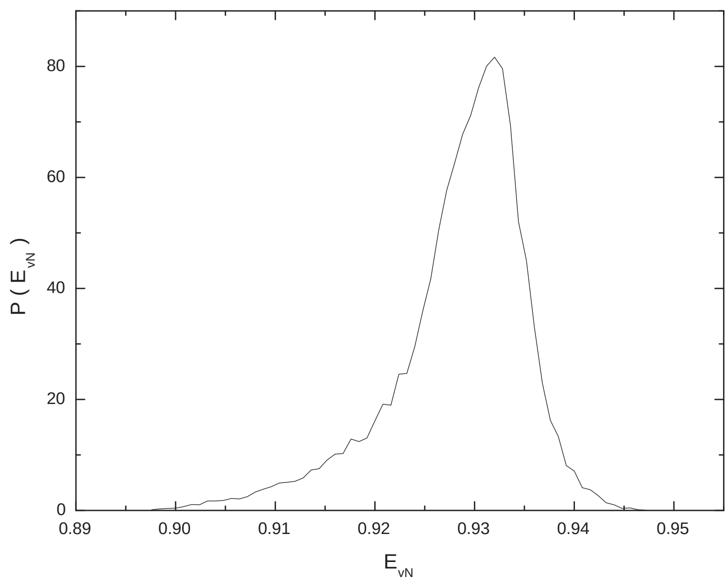

FIG. 1: Probability Density Function for E_{vN} among those states that maximize E_L.

matrices, although its computation is not as straightforward as the computation of E_L. Here we compare the behaviour of the entanglement measures E_L and E_{vN} when searching highly entangled states of 4 qubits. Our results indicate that E_{vN} is the best measure to use. As shown in Fig. 1, most states that maximize E_L are not maximally entangled according E_{vN}, even though they are all highly entangled states (most of these states have a value of E_{vN} around 0.935).

The study of the set of highly entangled 4 qubits states is of considerable interest because they represent the lowest dimensional system for which the non existence of the theoretically maximally entangled state has been proved. Brown *et al.* [13] developed a numerical algorithm to search highly entangled states of multi-qubit systems and found a maximally entangled state of 5 qubits. However, when applied to 4 qubit systems their algorithm yielded a state (which we here call $|BSSB4\rangle$) less entangled than the $|HS\rangle$ state previously discovered by Higuchi and Sudbery. This state is $|BSSB4\rangle = \frac{1}{2}(|0000\rangle + |{+}011\rangle + |1101\rangle + |{-}110\rangle)$, where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

A new and slightly different numerical algorithm was recently developed by us [14] that

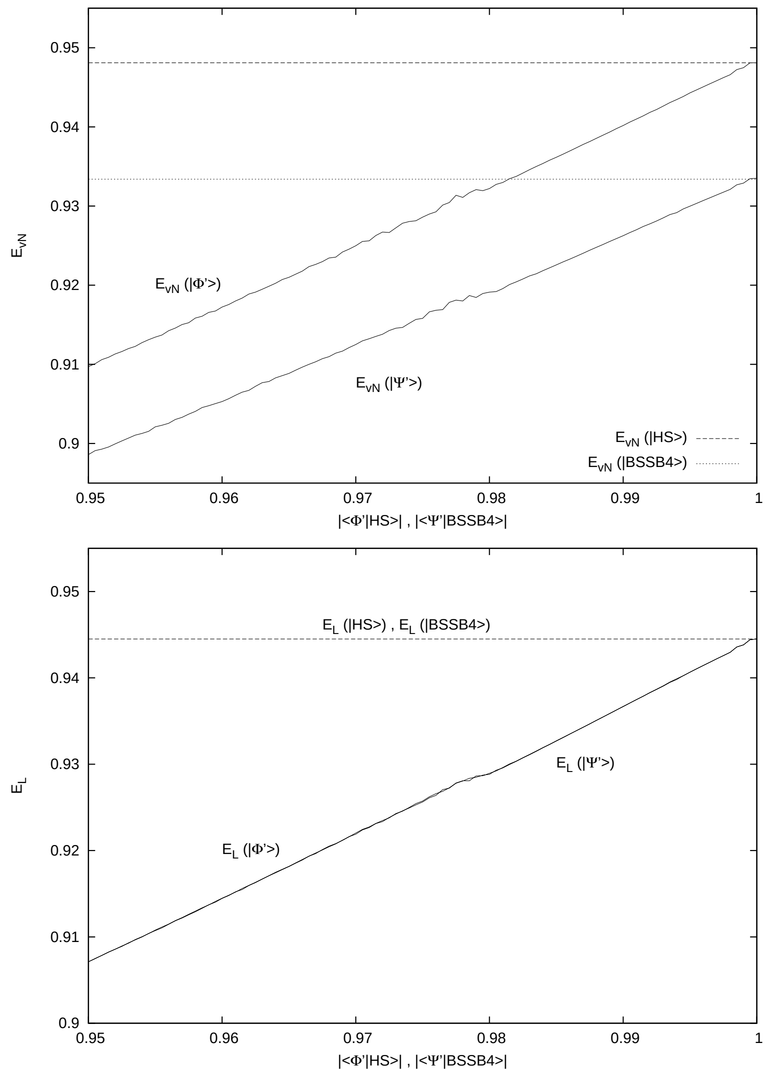

FIG. 2: Entanglement values for those states in the neighbourhood of $|HS\rangle$ and $|BSSB4\rangle$ as a function of the overlap with them.

has been successfully applied to find maximally entangled states in systems up to 7 qubits, including the 4 qubit $|HS\rangle$ state. Here we compare the behaviour of E_L and E_{vN} as multiqubit entanglement measures for highly entangled states of 4 qubits, through the study of the entanglement properties of the states living in the neighbourhoods of $|BSSB4\rangle$ and $|HS\rangle$. To such an end we first compute the average entanglement of states having given overlaps with $|BSSB4\rangle$. We considered, in total, a family of 15 000 000 states $|\Psi'\rangle$, with $0.95 \leq |\langle\Psi'|BSSB4\rangle| \leq 1$. A similar computation is done with a second family of 15 000 000 states $|\Phi'\rangle$ (this time close to $|HS\rangle$) with $0.95 \leq |\langle\Phi'|HS\rangle| \leq 1$. The results are summarized in Fig. III. While both entanglement measures identify all the alluded states as highly entangled, E_L does not succeed in distinguishing the neighbours of $|BSSB4\rangle$ from the neighbours of $|HS\rangle$. On the other hand, the averaged E_{vN} measure successfully distinguishes both families of states, and identifies the states in the neighbourhood of $|HS\rangle$ as more entangled than those related to the $|BSSB4\rangle$. Interestingly, the slopes of both curves depicted in the upper part of Fig.2 (indicating the rate of decrease in entanglement as we consider states with decreasing overlaps with $|BSSB4\rangle$ or $|HS\rangle$) are approximately the same. This suggests that the "entanglement landscapes" in the neighbourhoods of $|BSSB4\rangle$ or $|HS\rangle$ share some basic features.

IV. ALTERNATIVE APPROACH TO THE NUMERICAL SEARCH ALGORITHM

In Ref. [14] we proposed a numerical search algorithm that was able to find maximally entangled states in systems up to 7 qubits, starting from an initial separable state. To find the maximally entangled state, the coefficients of the initial state are slightly modified to obtain a new one. The entanglement of the new state is computed, if it is larger than the entanglement of the previous state the new state is kept. Otherwise, the new state is rejected and a new, tentative state is generated. This iterative process is repeated until it converges to a maximally entangled state. At each iteration the entanglement given by Eq. (1) must be computed. Consequently, at each step we must evaluate as many $E^{(m)}$ measures as non-equivalent bi-partitions the system has. This implies an exponential increase with the number of qubits of the computational resources needed to find the final state. Consequently, it is highly desirable to develop schemes to decrease the number of iterations needed to obtain the convergence and the time needed to perform each iteration.

Number of qubits	3	4	5	6	7
E_L	1.0000	0.9445	1.0000	1.0000	0.9961
E_{vN}	1.0000	0.9481	1.0000	1.0000	0.9948

If a multiqubit state has highly mixed reduced density matrices corresponding to subsystems of $[N/2]$ qubits, it is reasonable to expect that the same will happen with the reduced density matrices describing smaller subsystems. For this reason, in order to optimize our algorithm, we have tried a modified scheme based on the maximization of $E_{vN}^{[N/2]}$. The results of this experiment have been reasonably successful. For systems of 3, 4, 5, and 6 qubits the final highly entangled states obtained maximizing E_{vN} are the same as those obtained maximizing $E_{vN}^{[N/2]}$. This is a big improvement in our numerical algorithm, because in each iteration the number of bi-partitions to be considered is roughly reduced to the half, and the total number of iterations needed to reach the convergence are usually considerably reduced as well. For 7 qubits the E_{vN} entanglement values of the states yielded by the $E_{vN}^{[N/2]}$-based algorithm differ in the sixth decimal digit from the E entanglement value of the optimum state obtained maximizing E_{vN}. For 8 qubits the optimization algorithms, based either on balanced bi-partitions or on the global entanglement measure, do not converge always to the same state. The entanglement values (for different number of qubits) of the multiqubit states of highest entanglement considered in the present work are given in Table 1.

V. CONCLUSIONS

We have compared the behaviours of the multiqubit entanglement measures E_L and E_{vN} based, respectively, on the linear and the von Neumann entropies. Our results indicate that E_{vN} is better than E_L for the search of highly entangled states, because it discriminates between states that, while exhibiting the same value of E_L, have different degrees of entanglement. We also found evidence that search algorithms based upon balanced bi-partitions are almost as efficient as those based on the complete set of bi-partitions.

Acknowledgments

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