Typical features of the Mintert-Buchleitner lower bound for concurrence

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Despite the great importance of quantum entanglement, a computationally practical, physically motivated, and experimentally implementable way to estimate the amount of entanglement associated with a general (mixed) state of a bipartite quantum system is still lacking. An entanglement lower bound recently advanced by Mintert and Buchleitner [Phys. Rev. Lett. **98**, 140505 (2007)] was proposed as a possible tool for the aforementioned task. Mintert and Buchleitner presented evidence indicating that, due to its tightness, in some important cases the alluded bound can be used as an entanglement measure. From the physical point of view, what makes this bound particularly appealing as an entanglement measure *is that it is based on an observable quantity*. For this last reason, and also because of its other attractive feature, mathematical simplicity, this promising bound and its possible use as an entanglement estimator certainly deserve to be the focus of careful attention. The usefulness of the bound as an entanglement measure, however, depends on its degree of tightness. Here we perform a systematic survey of the behavior of the alluded bound in state space, in order to determine its typical properties (in particular its tightness) and thus assess its value as an entanglement estimator.

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I. INTRODUCTION

One of the most fundamental concepts in the quantum description of nature is that of entanglement [1,2], which in recent years has been the subject of intense research efforts (see, for instance, [1–10] and references therein). A state of a composite quantum system is called "entangled" if it cannot be represented as a mixture of factorizable pure states. Otherwise, the state is called "separable." Entanglement constitutes a physical resource that lies at the heart of important quantum information processes [2] such as quantum teleportation, superdense coding, and quantum computation.

Entanglement is essential for both (1) our basic understanding of quantum mechanics and (2) some of its most revolutionary (actual or possible) technological applications. In the last few years there has been enormous progress not only in the theoretical understanding of quantum entanglement, but also in the development of experimental techniques for the creation and manipulation of entangled states [5]. However, the amount of entanglement associated with a given (mixed) state ρ is notoriously difficult to evaluate. A useful quantitative measure of the entanglement associated with a quantum state is provided by the concurrence. The concurrence of a pure state $|\Psi\rangle$ of a bipartite system is given by [8,9]

$$C(\Psi) = \sqrt{2[1 - \mathrm{Tr}(\rho_r^2)]}, \tag{1}$$

where ρ_r is the marginal density matrix corresponding to either subsystem. It is clear that for separable pure states the concurrence is equal to zero. For mixed states ρ the concurrence is defined as [9]

$$C(\rho) = \inf \sum_i p_i C(\Psi_i), \tag{2}$$

where $\Sigma_i p_i = 1$ and $\{p_i, \Psi_i\}$ are all the possible decompositions of the state ρ into mixtures of pure states $|\Psi_i\rangle$,

$$\rho = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|. \tag{3}$$

Analytical, closed expressions for the concurrence of a mixed state are, in general, not available (excepting the case of two-qubit states). Even if the concurrence of a mixed state of a bipartite system as given by Eq. (2) is a mathematically well-defined quantity that can be computed numerically, its direct numerical evaluation is, for many purposes, unpractical. It is imperative to find practical, easy to evaluate, and physically meaningful procedures to estimate (by recourse to quantitative measures such as the concurrence) the amount of entanglement exhibited by a given quantum state ρ.

In light of the above considerations, it is of clear importance that the recent contribution of Mintert and Buchleitner [10] (MB) provides a possible way (based on an *observable quantity*) to estimate the concurrence of a given mixed state ρ of a bipartite system. The MB proposal is applicable to bipartite systems of arbitrary finite dimension and yields a generalized entanglement witness. MB proved that the square concurrence $C^2(\rho)$ of ρ obeys a simple inequality, i.e., is bounded by below by an easily accessible experimental quantity E_{MB}.

Mintert and Buchleitner [10] assessed the tightness of their bound with reference to quasipure states (that is, states with a small degree of mixedness) and proved that for these states E_{MB} can be regarded as a useful entanglement measure. The case of low mixedness is highly relevant from the experimental point of view. However, it is of interest to explore the behavior of the MB bound taking also into account states exhibiting higher degrees of mixedness. This kind of study will contribute to a better characterization of the range of mixedness values for which the MB bound is a useful entanglement indicator. The aim of the present contribution is to complement the results reported in [10] by means of a systematic survey of the behavior of the MB bound in the full state space of two-qubit systems. Composite systems of larger dimensionality (with a Hilbert space of dimensions $N_1 \times N_2$) constituted by two subsystems with Hilbert spaces of dimensions N_1 and N_2, respectively, are also investigated. In particular, we consider the cases $2\times N_2$, $3\times N_2$, 4×4, and 4×5.

II. BEHAVIOR OF THE MB BOUND AS AN ENTANGLEMENT INDICATOR

Let us consider the square concurrence $C^2(\rho)$ of a finite-dimensional bipartite state $\rho \in \mathcal{S}$, where $\mathcal{S}$ stands for the full state space (comprising both pure and mixed states). Mintert and Buchleitner [10] advanced the following quantity:

$$E_{\mathrm{MB}} \equiv 2\,\mathrm{Tr}(\rho^2) - \mathrm{Tr}(\rho_1^2) - \mathrm{Tr}(\rho_2^2), \tag{4}$$

with ρ_1 and ρ_2 as the reduced density matrices of the associated subsystems. MB proved that the squared concurrence C^2 of the states ρ is bounded from below by the above quantity,

$$C^2(\rho) \ge E_{\mathrm{MB}}. \tag{5}$$

Thus, strictly speaking E_{MB} is not an entanglement measure: it is just a lower bound for the squared concurrence. However, to the extent that such a bound is tight it also provides a valuable estimation of the concurrence of a state (which, as already explained, constitutes a useful quantitative measure of the state's entanglement). Following [10], we say that the bound E_{MB} is tight (either for general states ρ, or for a particular type of state) if the difference $G=C^2-E_{\mathrm{MB}}$ between the actual value of the squared concurrence $C^2(\rho)$ of a state and the corresponding value of the bound $E_{\mathrm{MB}}(\rho)$ is small. Following MB [10] we use the term "small" in a rather loose sense, meaning that G is small compared to the maximum possible value of C^2 for the system under consideration. The precise meaning of the smallness of G does not affect the main results and conclusions of the present paper. In particular, it does not affect our computations showing that the E_{MB}-based entanglement criterion drastically underestimates the amount of entangled states, and that this problem becomes more acute as the dimensionality of the system increases.

It is of clear relevance to perform a systematic exploration of the tightness of the E_{MB} bound. In order to appraise the value of E_{MB} as an entanglement estimator, it is of particular interest to determine the volume in state space corresponding to states with positive E_{MB}. Those are the states for which the E_{MB} is really informative. Indeed, when $E_{\mathrm{MB}}>0$ the quantity E_{MB} allows us to assert with certainty that the state under consideration is entangled. If, in addition, E_{MB} happens to be a tight bound, it will also provide a good estimation of the actual amount of entanglement exhibited by the state. As a first step to assess the general value of E_{MB} as an entanglement measure, it is then of clear importance to compare the volume of the region in state space corresponding to $E_{\mathrm{MB}}>0$ with the volume of the region corresponding to entangled states. In order for the quantity E_{MB} to provide a useful entanglement measure, the alluded two volumes must have similar values. As we shall see, for states of moderate to large mixedness this is not the case. For these states there turns out to be a large discrepancy between the volume in state space associated with states exhibiting a positive E_{MB}, on one hand, and the volume corresponding to entangled states, on the other hand. Furthermore, the aforementioned disagreement is seen to increase with the dimension of the composite system's Hilbert space.

In order to accomplish our task we will perform a systematic numerical survey of the $\mathcal{S}$ state space of bipartite systems in order to assess the tightness of Eq. (4) over the whole of $\mathcal{S}$. The space of all (pure and mixed) such states can be regarded as a product space $\mathcal{S}=\mathcal{P}\times\Delta$. Here $\mathcal{P}$ stands for the family of all complete sets of orthonormal projectors $\{\hat{P}\}_i^N$, $\Sigma_i \hat{P}_i = I$ (I is the identity matrix), while Δ is the simplex consisting of all real N-tuples of the form $\{\lambda_1,\ldots,\lambda_N\}$; $\lambda_i \in \mathcal{R}$, $\Sigma_i\lambda_i=1$, and $0\le\lambda_i\le 1$. Any state in $\mathcal{S}$ is of the form $\rho=\Sigma_i\lambda_i\hat{P}_i$. We need to exhaustively explore $\mathcal{S}$. To such an end it is necessary to introduce an appropriate measure μ on this space. Such a measure is needed to compute volumes within $\mathcal{S}$, as well as to determine what is to be understood by a uniform distribution of states on $\mathcal{S}$. The measure that we are going to adopt here is taken from the work of Zyczkowski [11] and Pozniak *et al.* [12]. An arbitrary (pure or mixed) state ρ of a quantum system described by an N-dimensional Hilbert space can always be expressed as the product of three matrices, $\rho=UD[\{\lambda_i\}]U^\dagger$. Here U is an $N\times N$ unitary matrix and $D[\{\lambda_i\}]$ is an $N\times N$ diagonal matrix whose diagonal elements are, precisely, our above-defined $\{\lambda_1,\ldots,\lambda_N\}$. The group of unitary matrices $U(N)$ is endowed with a unique, uniform measure: the Haar measure ν [13]. On the other hand, the N-simplex Δ, consisting of all the real N-tuples $\{\lambda_1,\ldots,\lambda_N\}$ characterizing the diagonal matrices $D[\{\lambda_i\}]$, is a subset of an $(N-1)$-dimensional hyperplane of $\mathcal{R}^N$. Consequently, the standard normalized Lebesgue measure $\mathcal{L}_{N-1}$ on $\mathcal{R}^{N-1}$ provides a measure for Δ. The aforementioned measures on $U(N)$ and Δ lead then to a measure μ on the set $\mathcal{S}$ of all the states of our quantum system [11–13], namely, $\mu=\nu\times\mathcal{L}_{N-1}$. The Haar measure ν is the only sensible measure on the space of $N\times N$ unitary matrix, and the one mostly used by researchers. On the other hand, the measure $\mathcal{L}_{N-1}$ on the simplex Δ is not unique: other possible measures have been proposed in the literature [1,14,15]. Proponents of alternative measures in the simplex Δ do not question the use of the Haar measure on $U(N)$, due to its sound group-theoretical foundations [1,15]. The measure $\mathcal{L}_{N-1}$ is the most simple and, arguably, the most intuitive measure on the simplex Δ. In our numerical computations we randomly generate pure and mixed states uniformly distributed according to the measure μ, in order to determine the fraction of the state space volume associated with states exhibiting a given property [16,17]. Of course, the precise volume of a given region in state space depends on the particular measure used [1]. However, the set of $N_1\times N_2$ separable states becomes nowhere dense in the limit of either subsystem's dimension N_i going to infinity [18], suggesting that when the dimensionality of the bipartite system under consideration goes to infinity the volume corresponding to separable states goes to zero independently of the particular measure used [1].

Here we wish to compare the volume in state space corresponding to states with $E_{\text{MB}}>0$ with the volume occupied by states not complying with the positive partial transpose (PPT) separability criterium by Peres [19]. The Peres criterium provides a necessary condition to be fulfilled by all separable states of bipartite systems (that turns out to be also a sufficient one in the special cases of two-qubit and qubit-qutrit systems [20]). Consequently, the volume corresponding to states not verifying the PPT criterium constitutes a lower bound for the volume of entangled states. In the special cases of two-qubit and qubit-qutrit systems, these two volumes coincide.

III. TIGHTNESS OF THE E_{MB} BOUND FOR TWO-QUBIT STATES

The probability that a random state ρ has a given property is given by the fraction of the volume in state space $\mathcal{S}$ corresponding to states complying with that property (volumes are computed according to the measure μ). Of particular relevance for our present study is to compare the probability of having $E_{\text{MB}}>0$ with the probability of finding a state not having a positive partial transpose. We shall refer to the states with $E_{\text{MB}}>0$ as those verifying the E_{MB} criterion for the detection of entanglement.

We shall start by considering two-qubit states. Two-qubit states are highly relevant on account of conceptual, historic, practical, and didactic reasons. In the two-qubit case we have at our disposal a closed analytical expression for the concurrence: the concurrence formula of Wootters [21]. Using the methods described above to randomly generate two-qubit states ρ, we now compare the global behavior of the exact value of C^2 (as computed by the Wootters formula) to the behavior of the E_{MB} lower bound, with emphasis on the difference $G(\rho)=C^2(\rho)-E_{\text{MB}}$. In Fig. 1 we plot, against G, the probability density function for $G(\rho)$ associated with the full state space $\mathcal{S}$ (solid line) and compare it with the probability density functions for $G(\rho)$ obtained by considering only those special states which have $E_{\text{MB}}>0$ (dashed line), on one hand, and only those states with $E_{\text{MB}}<0$ (dotted line), on the other hand. When determining these distributions the quantity E_{MB} is set to zero whenever $E_{\text{MB}}<0$. The statistical error bars corresponding to the results presented in Fig. 1 are negligible at the scale of the figure (the same happens with the rest of the figures in this paper, excepting the region of small S_L values in Fig. 4). The distribution function corresponding to states with $E_{\text{MB}}>0$ (dashed line) differs appreciably from the other two distribution functions depicted in Fig. 1. When restricting our considerations to states having $E_{\text{MB}}>0$ (which is when the quantity E_{MB} is really useful for detecting entanglement), E_{MB} turns out to be a somewhat poorer indicator of entanglement than when considering the full state space $\mathcal{S}$. However, even in this case the typical mismatch between E_{MB} and C^2 is around 0.1, which indicates that E_{MB} provides a reasonable estimation of the amount of entanglement exhibited by these states. It is also interesting to notice that the distribution function for $G(\rho)$ associated with states with $E_{\text{MB}}<0$ is very similar to the one corresponding to the full state space S. This is due to two reasons: (i) a large fraction of the two-qubit states correspond to states exhibiting zero entanglement and (ii) only a small fraction of all the two-qubit states do have $E_{\text{MB}}>0$. Actually, *only a small fraction of the entangled states of two qubits have* $E_{\text{MB}}>0$. In point of fact, our systematic numerical survey of the state space indicates that for these systems we have $E_{\text{MB}}>0$ for just 4% of the states, while the actual percentage of entangled states is ~37%. This means that, even for two-qubit states, a large fraction of the entangled states have $E_{\text{MB}}<0$ and, therefore, cannot be detected using the E_{MB} bound. As we shall see, the situation does not get any better when considering systems of larger dimensionality.

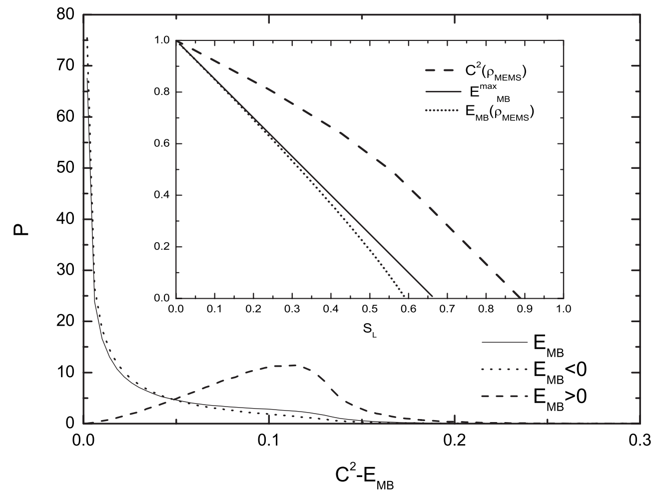

FIG. 1. Probability densities of finding two-qubit states with given values of $G=C^2-E_{\text{MB}}$ when considering arbitrary states ρ (solid line), only states entangled according to the E_{MB} concurrence bound (dashed line), and only states with $E_{\text{MB}}<0$ (dotted line). When determining these distributions the quantity E_{MB} is set to zero each time that $E_{\text{MB}}<0$. Inset: the dashed line stands for MEMS's squared concurrence, while the dotted line corresponds to E_{MB} for the same family of states. The solid line represents the maximum possible value for E_{MB}. All depicted quantities are dimensionless.

But before discussing systems of larger dimensionality, let us consider a particularly interesting class of two-qubit states, the so-called maximally entangled mixed states (MEMSs) [22], which are those states that have the maximum possible value of the concurrence for a given degree of mixture. We remark on the fact that MEMSs have recently been encountered in the laboratory [23,24]. The associated density matrix is written in terms of a variable x that ranges in [0,1]. In the computational basis their representative matrices read

$$\rho_{\text{MEMS}}=\begin{pmatrix} g(x) & 0 & 0 & x/2 \\ 0 & 1-2g(x) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ x/2 & 0 & 0 & g(x) \end{pmatrix}, \tag{6}$$

with $g(x)=1/3$ for $0\le x\le 2/3$, and $g(x)=x/2$ for $2/3\le x\le 1$. The inset of Fig. 1 refers to MEMS. The upper curve (dashed line) is a plot of C^2 against S_L for MEMS, where $S_L[\rho]=\frac{N}{N-1}(1-\text{Tr}[\rho^2])$ stands for the linear entropy of

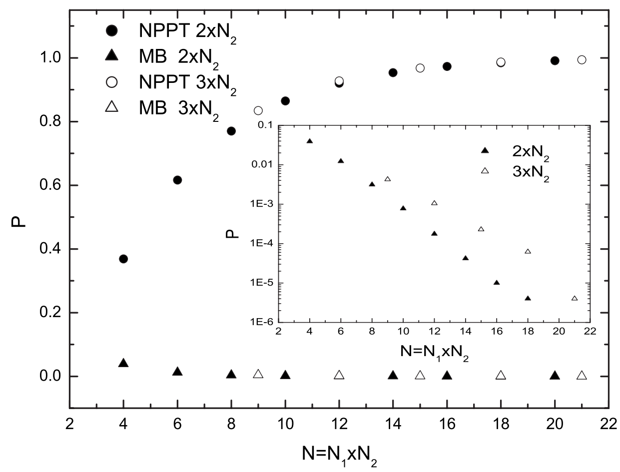

FIG. 2. Probabilities that a state ρ of a composite quantum system with Hilbert-space dimensions $N_1 \times N_2$ does not have a positive partial transpose (referred to in the figure by the abbreviation NPPT) (circles) or has $E_{\rm MB}>0$ (triangles). These probabilities are plotted against $N_1 \times N_2$. The inset shows the probabilities corresponding to states with $E_{\rm MB}>0$, against $N_1 \times N_2$, plotted in a logarithmic scale. All depicted quantities are nondimensional.

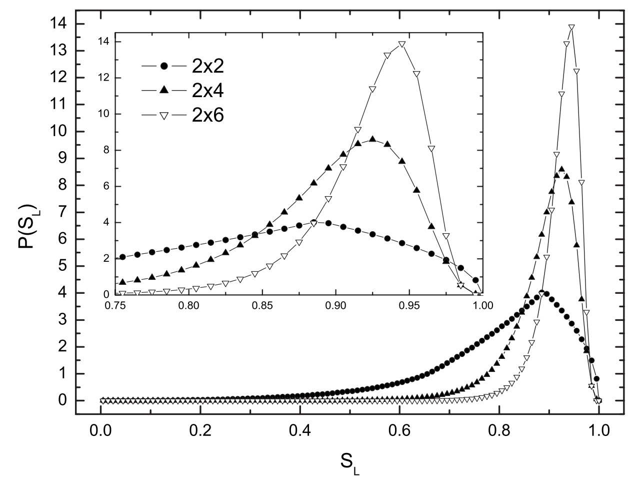

FIG. 3. Probability $P(S_L)$ of finding states with a given degree of mixture S_L for different Hilbert-space dimensions. Inset: probability of finding states with $0.75<S_L<1$. All depicted quantities are dimensionless.

a mixed state ρ describing a system with an N-dimensional Hilbert space. We have here the maximum possible concurrence for a given S_L value. The dotted line in the inset displays MEMS's $E_{\rm MB}$ values $[E_{\rm MB}(\text{MEMS})]$ in that S_L range for which $E_{\rm MB}(\text{MEMS})>0$, i.e., $2/3 \le x \le 1$, corresponding to the S_L range of $0 \le S_L \le \frac{16}{27}$. One has $E_{\rm MB}(\text{MEMS})=\frac{1}{8}(4+2\sqrt{4-6S_L-9S_L})$. The MEMS states clearly exhibit the general trend already mentioned: $E_{\rm MB}$ provides a good estimation of C^2 for states of low mixedness, but the quality of this estimation deteriorates quickly as the degree of mixedness increases. It is illustrative to notice that the dotted curve closely approaches the straight line $E_{\rm MB}^{\max}=1-\frac{3}{2}S_L$, obtained by simply setting in Eq. (4) both $\text{Tr}[(\rho_r^{(i)})^2]$ equal to their minimum possible values (which are 1/2).

IV. $E_{\rm MB}$-BASED ENTANGLEMENT CRITERION FOR SYSTEMS OF TWO QUBITS AND SYSTEMS OF LARGER DIMENSIONALITY

We now investigate the behavior of bipartite systems consisting of one subsystem with a Hilbert space of dimension N_1 and another subsystem with Hilbert space of dimension N_2 (the Hilbert space of the composite system having dimensions $N_1 \times N_2$). In particular, we considered systems comprising a qubit and a subsystem described by an N_2-dimensional Hilbert space, and systems consisting of a qutrit and a subsystem with Hilbert space of dimension N_2. Some relevant features of these systems are summarized in Fig. 2. In this figure we plotted, against the Hilbert space's dimensions $N_1 \times N_2$ of the composite system, the fraction of state space volume corresponding to states not having a positive partial transpose and the fraction of state space volume associated with states having $E_{\rm MB}>0$. The most remarkable feature of Fig. 2 is that most states have $E_{\rm MB}<0$. This means that the $E_{\rm MB}$ bound is not, in most cases, a useful entanglement indicator (excepting, of course, quasipure states). The fraction of state space corresponding to states not having a positive partial transpose constitutes a lower limit for the fraction of volume corresponding to entangled states. It is clear from Fig. 2 that the volume corresponding to entangled states is much larger than the volume corresponding to $E_{\rm MB}>0$. This means that in most cases the lower bound based on the quantity $E_{\rm MB}$ is not useful in detecting entangled states, let alone in estimating the amount of entanglement exhibited by those states. This feature of $E_{\rm MB}$ becomes more accentuated the larger is the dimensionality of the system under study. It transpires from Fig. 2 that the dependence on N_2 of the behavior of composite systems consisting of a qubit and a system of dimension N_2 is essentially the same as the behavior exhibited by composite systems comprising a qutrit and a system of dimension N_2. We have also computed the alluded volumes in state space for composite systems of dimensions 4×4 and 4×5. The results obtained were basically the same as those obtained, respectively, in the cases 2×8 and 2×10. Consequently, our numerical results suggest that the main trends observed in Fig. 2 are not restricted to the cases $2\times N_2$ and $3\times N_2$ but, on the contrary, are verified by general bipartite systems with finite Hilbert spaces. Our findings suggest that the fractions of state space volume corresponding to states with $E_{\rm MB}>0$ and to states not exhibiting a positive partial transpose depend only on the total dimensions $N_1 \times N_2$ of the Hilbert space describing the composite system. The behavior depicted in Fig. 2, as a function of the system's dimensionality, of the fraction of state space associated with states not having a positive partial transpose is fully consistent with the known fact that states of infinite-dimensional bipartite systems are generically entangled [18]. On the other hand, the behavior exhibited in Fig. 2 by the fraction of state space corresponding to $E_{\rm MB}>0$ indicates that those entangled states are generically nondetectable by the $E_{\rm MB}$-based entanglement criterion.

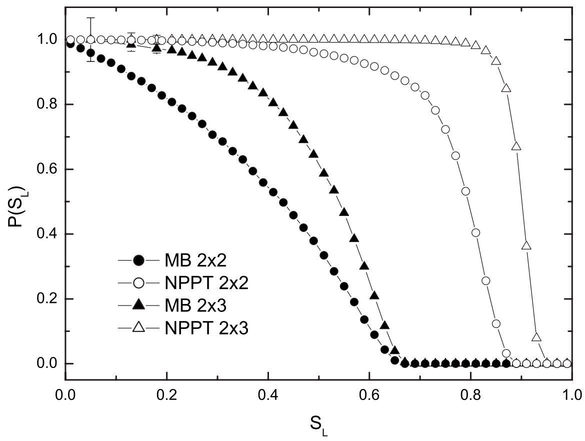

FIG. 4. Probability of finding an entangled state ρ in 2×2 (circles) and 2×3 (triangles) systems according to PPT criteria (empty symbols) and the E_{MB} concurrence lower bound (full symbols) as a function of the linear entropy S_L. All depicted quantities are dimensionless.

In Fig. 3 we depict the probability density associated with the degree of mixture (as measured by the linear entropy), for different dimensions of the pertinent Hilbert space. Clearly, probabilities of finding states with given mixedness grow with the degree of mixing, peak, and then diminish as there exists only one totally mixed state. To understand this behavior it is instructive to consider the case of two-qubit systems, which live in a four-dimensional Hilbert space. In this case all the possible 4-tuples of eigenvalues of the density matrix $(\lambda_1,\ldots,\lambda_4)$ constitute a four-simplex having as its natural representation a regular tetrahedron contained in $\mathbb{R}^3$. The vector position $\mathbf{r}$ of a point in the tetrahedron associated with the eigenvalues $(\lambda_1,\ldots,\lambda_4)$ is given by $\mathbf{r}=\Sigma_{i=1}^{4}\lambda_i\mathbf{r}_i$, where $\mathbf{r}_i$ are the positions of the vertices of the tetrahedron (which is assumed to be centered at the origin $\mathbf{r}=\mathbf{0}$). Density matrices ρ with a given constant value of $\mathrm{Tr}(\rho^2)$ (and, consequently, a constant value of $S_L[\rho]$) correspond to points in the tetrahedron lying on a sphere concentric with the tetrahedron. For the two-qubit case, the probability density depicted in Fig. 3 is proportional to the surface area of the subregion of the alluded sphere that is constrained within the tetrahedron. It is clear that the value of this area vanishes for $S_L=1$ (which corresponds to the maximally mixed state represented by the center of the tetrahedron) and it also vanishes in the case $S_L=0$ (associated with pure states, which are represented by the vertices of the tetrahedron). The peak of $P(S_L)$ corresponds to the S_L value associated with a sphere tangent to the faces of the tetrahedron. Summing up, the extreme cases $S_L=0,1$ where $P(S_L)=0$ correspond to sets of density matrices of zero measure, and the peak of $P(S_L)$ occurs at an intermediate value of S_L (see [25] and references therein for details). These geometrical considerations can be extended straightforwardly to systems of higher dimensionality.

It transpires from Fig. 3 that typical degrees of mixing tend to increase as the dimension of ρ grows. Since the E_{MB}-based criterion for the detection of entangled states seems to worsen for high degrees of mixing [10], it is to be expected that it has to work in poorer fashion in cases in which typical states become more mixed, i.e., when dimensionality becomes larger. For a better understanding of this fact, see Fig. 4. There, the fractions of state space volume (plotted against S_L) corresponding to states not complying with the PPT separability criterion and to states with $E_{\mathrm{MB}}>0$ are compared. Two-qubit systems and qubit-qutrit systems are considered. In both these cases the PPT criterion is a necessary and sufficient separability one. Consequently, a state is entangled if and only if it does not comply with the PPT separability criterion. It is clear from Fig. 4 that the E_{MB}-based criterium for detecting entanglement significantly subestimates the fraction of state space corresponding (for a giving degree of mixing) to entangled states.

V. CONCLUSIONS

In this work we have assessed how well the MB bound functions as an entanglement indicator for bipartite quantum systems. Particular attention was paid to systems consisting of a qubit or a qutrit, on one hand, and a second subsystem described by an N-dimensional Hilbert space, on the other hand. The cases of 4×4 and 4×5 systems were also considered. Our main conclusion is that, when considering states with arbitrary degrees of mixedness, the quantity E_{MB} becomes a poor estimator of entanglement. The E_{MB}-based criterion for detecting entangled states tends (in some cases substantially) to underestimate entanglement. More specifically, if we randomly pick up a state of such a composite system, we will get a state with $E_{\mathrm{MB}}>0$ with a probability much smaller than the probability of finding a state not passing the PPT criterion (which in turn, constitutes only a lower bound of the true probability of finding an entangled state). In other words, the fraction of state space volume corresponding to those entangled states that can be identified as such by the E_{MB} bound is much smaller than the actual volume of entangled states. This discrepancy becomes more serious as N increases. When considering (for two-qubit and qubit-qutrit systems) the behavior of E_{MB} for states of a given degree of mixedness (see Fig. 4), our findings are fully consistent with the results reported by MB in [10]. For states of low degree of mixedness E_{MB} constitutes a good entanglement indicator. The quality of this indicator decreases rapidly as the degree of mixedness increases.

For two-qubit systems more detailed treatments are accessible, given that C can be explicitly evaluated by recourse to the Wootters formula. For these systems E_{MB} provides a reasonable estimation of the squared concurrence of a state. However, even for two-qubit systems, a large fraction of the entangled states are not detected by the E_{MB}-based entanglement criterion. Our present results, of course, do not call into question the utility of E_{MB} as a good entanglement indicator for the special (but experimentally very relevant) case of quasipure states.

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