Reply to “Comment on ‘Connection between entanglement and the speed of quantum evolution’ and on ‘Entanglement and the lower bounds on the speed of quantum evolution’”

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Chau’s Comment deals with the properties exhibited by a particular set of two-qubit states in connection with entanglement and the quantum evolution of some low-dimensional composite systems. However, there is an important aspect of the previously mentioned two-qubit states and of the role they play in relation with the “speed” of quantum evolution that was not mentioned by Chau and deserves to be pointed out: for the two-qubit system under consideration, these states require the longest possible absolute time to evolve to an orthogonal state.

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In Ref. [1] Giovannetti, Lloyd, and Macconne (GLM) discovered that there is an interesting connection between entanglement and quantum evolution: Entanglement helps to “speed up” the time evolution of certain states of composite quantum systems. The alluded “speed” of quantum evolution is given by the time τ required by the system to evolve to a state orthogonal to the initial one. This time τ admits a lower bound,

$$\tau_{\min} = \left(\frac{\pi\hbar}{2E}, \frac{\pi\hbar}{2\Delta E} \right), \quad (1)$$

determined by the state’s mean energy and energy’s uncertainty, $E = \langle H \rangle$ and $\Delta E = \sqrt{\langle H^2 \rangle - \langle H \rangle^2}$, where H is the Hamiltonian governing the system’s evolution. Optimal, “highest speed” evolution occurs when the previously mentioned bound is saturated, that is, when $\tau = \tau_{\min}$. In this case, the system evolves at the “highest speed” compatible with its energy resources. In this sense, the speed of evolution is given by the quotient $\frac{\tau}{\tau_{\min}}$: The smaller the value of this quantity, the higher the speed.

Giovannetti, Lloyd, and Macconne pointed out in Ref. [1] that, except for highly asymmetric initial states where all the energy resources are concentrated in one subsystem, composite systems evolving under local Hamiltonians cannot achieve the speed limit in the case of initial separable pure states. On the contrary, it was shown in Ref. [1] that (some) initial maximally entangled states of those systems do achieve the speed limit. This interesting finding by GLM was based on the study of the behavior of a few particular states. GLM themselves acknowledged in Ref. [1] that the connection between entanglement and speed of evolution is not a strict one, in the sense that it is not always true that, when comparing two particular initial states, the state of larger entanglement is the one with smaller value of $\frac{\tau}{\tau_{\min}}$. Indeed, it is not difficult to find particular examples of pairs of states such that the state with higher entanglement also has the largest value of $\frac{\tau}{\tau_{\min}}$. This raises a natural question: What is the *typical, generic behavior* of states of multipartite quantum systems with regard to the connection between entanglement and speed of quantum evolution? This is the question that we addressed in Refs. [2,3]. In these two works we found that, in fact, when one considers

the *typical* properties of two-qubit states evolving under a local Hamiltonian $H = H_1 \otimes H_2$ there is evidence indicating that larger amounts of entanglement allow some states to reach higher “speed,” in the sense of having a lower value of $\frac{\tau}{\tau_{\min}}$. In Ref. [2] we study the case of pure states, and in Ref. [3] extended the study to the *generic* features exhibited by mixed states.

What Chau discusses in his note is a particular family of initial states not complying with the general tendencies observed in the *generic case*. This particular case was overlooked in Ref. [2]. The comment by Chau constitutes a valuable contribution to the subject discussed in Refs. [2,3]. However, there is an important aspect of the states discussed by Chau, and of the role they play in connection with the “speed” of quantum evolution, that was not mentioned by Chau and deserves to be pointed out.

In Refs. [2,3] we study the behaviors of two noninteracting qubits, each one governed by a Hamiltonian with eigenvalues 0 and $\epsilon > 0$. For an initial state $|\Psi\rangle = c_0|00\rangle + c_1|01\rangle + c_2|10\rangle + c_3|11\rangle$ of the two-qubit system to evolve to an orthogonal state, the quadratic equation,

$$|c_3|^2 z^2 + (|c_2|^2 + |c_1|^2)z + |c_0|^2 = 0, \quad (2)$$

must have a root of modulus equal to 1 [2]. If we consider the generic case of states complying with this condition, it turns out that for these states a connection between entanglement and speed of evolution is observed [2]. Chau focuses on the case where the leading term in the quadratic equation vanishes, $|c_3| = 0$, observing that within this case there are nonmaximally entangled states that saturate the speed limit. Chau’s case includes some factorizable states evolving at the speed limit. However, the overwhelming majority of initial factorizable pure states do not reach this limit. Within the special case pointed out by Chau there are also nonfactorizable pure states of low entanglement that saturate the speed limit. These states verify

$$|c_0|^2 = |c_2|^2 + |c_1|^2 = \frac{1}{2}, \quad (3)$$

with both $|c_2|$ and $|c_1|$ different from zero. As already mentioned, these states saturate the quantum speed limit in the sense that $\tau = \tau_{\min}$. However, for these states we have

$$\tau = \frac{\pi\hbar}{\epsilon}, \quad (4)$$

which, for the two-qubit system under consideration, happens to be *the longest possible (absolute) time to evolve to an*

orthogonal state. Summing up, these states do saturate the speed limit without much entanglement, but they require the longest possible absolute time to evolve to an orthogonal state, that is, these states are actually the slowest ones in terms of absolute time. Therefore, the “speed up” in relative time ($\tau/\tau_{\min}$) achieved by entanglement in this particular case is accompanied by a severe “slow down” in absolute time.

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