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GENERALIZED FRACTIONAL HYBRID HAMILTON PONTRYAGIN EQUATIONS

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In this work we present a new approach on the study of dynamical systems. Combining the two ways of expressing the uncertainty, using probabilistic theory and credibility theory, we have investigated the generalized fractional hybrid equations. We have introduced the concepts of generalized fractional Wiener process, generalized fractional Liu process and the combination between them, generalized fractional hybrid process. Corresponding generalized fractional stochastic, respectively fuzzy, respectively hybrid dynamical systems were defined. We have applied the theory for generalized fractional hybrid Hamilton-Pontryagin (HP) equation and generalized fractional Hamiltonian equations. We have found fractional Langevin equations from the general fractional hybrid Hamiltonian equations. For these cases and specific parameters, numerical simulations were done.

Keywords: HP equations,; (generalized) fractional stochastic equations; (generalized) fractional fuzzy differential equations; (generalized) fractional hybrid equations; generalized fractional hybrid Hamiltonian equations; Euler scheme.

1. Introduction

Fractional theory has applicability in many science fields. This approach presents fractional derivatives, fractional integrals, of any real or complex order. Fractional calculus is used when fractional integration is needed. It is used for studying simple dynamical systems, but it also describes complex physical systems. For example, applications of the fractional calculus can be found in chaotic dynamics, control theory, stochastic modeling, but also in finance, hydrology, biophysics, physics, astrophysics, cosmology and so on ([5], [6], [8], [9], [10]). But some other fields have just started to study problems from fractional point of view. In great fashion is the study of fractional problems of the calculus of variations and Euler-Lagrange type equations. There were found Euler-Lagrange equations with fractional derivatives [21], and then Klimek found Euler-Lagrange equations, but with symmetric fractional derivatives [12]. Most famous fractional integral are Riemann-Liouville, Caputo, Grunwald-Letnikov and most frequently used is Riemann-Liouville fractional integral. The study of Euler-Lagrange fractional equations was continued by Agrawal [1], that described these equations using the left, respectively right fractional derivatives in the Riemann-Liouville sense. This fractional calculus has some great problems, such as presence of non-local fractional differential operators, or the adjoint fractional operator that describes the dynamics is not the negative of itself, or mathematical calculus may be very hard because of the complicated Leibniz rule, or the absence of chain rule, and so on. After O.P. Agrawal's formulation [1] of Euler-Lagrange fractional equations, Băleanu and Avkar [2] used them in formulating problems with Lagrangians linear in velocities. Standard multi-variable variational calculus has also some limitations. But in [24] C. Udrişte and D. Opriş showed that these limitations can by broken using the multi-linear control theory. Such limitations can be avoided by considering fractional derivative and fractional integral calculus adapted to nonholonomic distributions. In this way a fractional spacetime geometry is defined, with fundamental geometric objects ([25], [26]).

Another aspect that we use is the stochastic approach. Stochastic concepts were firstly introduced by J.M. Bismut, in his book from 1981, when stochastic Hamiltonian system was introduced. Since then, there has been a need in finding out tools and algorithms for the study of this kind of systems with uncertainty. Bismut's work was continued by Lazaro-Cami and Ortega ([15], [16]), in the sense that his work was generalized to manifolds, stochastic Hamiltonian systems on manifolds extremize a stochastic action on the space of manifold valued semimartingales, the reduction of stochastic Hamiltonian system on cotangent bundle of a Lie group, a counter example for the converse of Bismut's original theorem.

A new way for expressing the uncertainty is given by credibility theory. In this case we are not working on a probability space, like in the stochastic case, but on a credibility one. Credibility theory is based on five axioms from which the notion of credibility measure is defined, and it was introduced in order to measure a fuzzy event. This was first given by Li and Liu in their work [17]. This is a new theory that deals with fuzzy phenomena. Fuzzy random theory and random fuzzy theory can be seen as an extensions of credibility theory. A fuzzy random variable can be seen as a function from a probability space to the set of fuzzy variables, and a random fuzzy variable is a function from a credibility space to the set of random variables [19]. In our actual research, we will use fuzzy differential equations, that were introduced by Liu [18]. This is a type of differential equation, driven by a Liu process, just like a stochastic process is described by a Brownian motion.

In the case when fuzziness and randomness simultaneously appear in a system, we will talk about hybrid process. We have the concept of fuzzy random variable that was introduced by Kwakernaak ([13], [14]). A fuzzy random variable is a random variable that takes fuzzy variable values. More generally, hybrid variable was proposed by Liu [19] to describe the phenomena with fuzziness and randomness. Based on the hybrid process, we will work with differential equations characterized by Wiener-Liu process. This can be computed using Itô-Liu formula [27]. In some situations, there may be more than one Brownian motion (Wiener process) and Liu process in a system, therefore, we can take into consideration also multi-dimensional Itô-Liu formula.

The paper is organized as follows. In Section 2 we present generalization of Riemann-Liouville fractional integral, generalized fractional Wiener process, and we have defined the generalized fractional stochastic equations. To get to a hybrid process, we have defined a generalized Liu process and the generalization of fractional fuzzy equation. The mixture between generalized fractional Wiener process and generalized fractional Liu process results as the generalization of fractional hybrid differential equations. In the third section we used the notions presented in Section 2 for defining the generalized fractional hybrid HP equations. We have defined the generalized fractional Riemann-Liouville, respectively Itô, respectively Liu integrals, and in Theorem 1 we gave the generalized fractional hybrid HP equations. We have also defined generalized fractional hybrid Hamiltonian equations. In Section 4 some numerical simulations were presented, using the first order Euler scheme for particular parameters. At the end, some conclusions are given regarding the presented paper and also some new directions for the future study.

2. Generalized fractional hybrid equations

Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function, $\alpha : \mathbb{R} \to \mathbb{R}$ a C^1 function, and $\rho \in \mathbb{R}$, $\rho > 0$.

A *Riemann-Liouville generalized fractional integral* [10] is defined by

$$ {}_{t_0}I_t^{\alpha} f(t) = \int_{t_0}^{t} \frac{1}{\Gamma_1(\alpha(s-t))} f(s)(t-s)^{\alpha(s-t)-1} e^{-\rho(s-t)} ds, \tag{2.1} $$

where

$$ \Gamma_1(\alpha(s-t)) = \Gamma(\alpha(z))|_{z=s-t}, \tag{2.2} $$

and $\Gamma(\alpha(z))$ is the Euler Gamma function of the form

$$ \Gamma(\alpha(z)) = \int_0^{\infty} (s-t)^{\alpha(z)-1} e^{-(s-t)} dt. \tag{2.3} $$

If $\alpha(z) = a = const$, $0 < a \leq 1$, $\rho = 0$, from (2.1), results that

$$ {}_{t_0}I_t^{a} f(t) = \frac{1}{\Gamma(a)} \int_{t_0}^{t} f(s)(t-s)^{a-1} ds. \tag{2.4} $$

Formula (2.4) is the fractional Riemann-Liouville integral, [11].

Generalized fractional Riemann-Liouville is a mixture between a fractal action used in physical theory, and discount action with rate ρ, [10].

In the relations (2.1) and (2.4), s is called *intrinsic time* and t is called *observed time*, $t > s$.

From (2.1), results that

$$ {}_{t_0}I_t^{\alpha} f(t) = \int_{t_0}^{t} f(s) g_t^{\alpha}(s) ds, \tag{2.5} $$

where

$$ g_t^{\alpha}(s) = \frac{1}{\Gamma_1(\alpha(s-t))} e^{(\alpha(s-t)-1)ln|t-s|-\rho(s-t)}, \quad t > s. \tag{2.6} $$

Let $(\Omega, \mathcal{F}, P_r)$ be a probabilistic space characterized by the usual conditions, and $(W(t))_{t\in\mathbb{R}}$ a 1-dimensional Wiener process.

It is called a *generalized fractional Wiener process*, the process

$$ J^{\alpha}(t) = \int_{t_0}^{t} g_t^{\alpha}(s) dW(s), \quad t > s, \tag{2.7} $$

where g_t^{α} is the function given in (2.6).

If $x(t) = x(t, \omega)$ is a stochastic $n-$dimensional process and $a : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $b : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, are deterministic functions, we will denote by

$$ \begin{aligned} {}_{t_0}I_t^{\alpha} a(t, x(t)) &= \textstyle\int_{t_0}^{t} a(s, x(s)) g_t^{\alpha}(s) ds, \\ {}_{t_0}J_t^{\beta} b(t, x(t)) &= \textstyle\int_{t_0}^{t} b(s, x(s)) g_t^{\beta}(s) dW(s), \end{aligned} \tag{2.8} $$

the generalized fractional Riemann-Liouville integral, respectively generalized fractional Itô integral, where $\alpha : \mathbb{R} \to \mathbb{R}$, $\beta : \mathbb{R} \to \mathbb{R}$ are C^1 functions.

We call *generalized fractional stochastic differential equation*, the functional Volterra type equation given by

$$ x(t) = x(t_0) + \int_{t_0}^{t} a(s, x(s)) g_t^{\alpha}(s) ds + \int_{t_0}^{t} b(s, x(s)) g_t^{\beta}(s) dW(s). \tag{2.9} $$

Using the notations from (2.8), it results that

$$ x(t) = x(t_0) +_{t_0} I_t^{\alpha} a(t, x(t)) +_{t_0} J_t^{\beta} b(t, x(t)). \tag{2.10} $$

The equation (2.10) can be written formally in the following way

$$ dx = a(s, x(s)) g_t^{\alpha}(s) ds + b(s, x(s)) g_t^{\beta}(s) dW(s). \tag{2.11} $$

Let us consider $a(t, x(t)) = \mu(t) x(t)$, $b(t, x(t)) = \sigma(t) x(t)$, where $\mu : \mathbb{R} \to \mathbb{R}$, $\sigma : \mathbb{R} \to \mathbb{R}$, $x : \mathbb{R} \to \mathbb{R}$ and $\alpha(z) = \alpha_1$, $\beta(z) = \frac{1+\alpha_1}{2}$. Then equation (2.9) becomes

$$ x(t) = x(t_0) + \frac{1}{\Gamma(\alpha_1)} \int_{t_0}^{t} \frac{\mu(s) x(s)}{(t-s)^{1-\alpha_1}} ds + \frac{1}{\Gamma(\frac{1+\alpha_1}{2})} \int_{t_0}^{t} \frac{\sigma(s) x(s)}{(t-s)^{(1-\alpha_1)/2}} dW(s). \tag{2.12} $$

The equation (2.12) is called *fractional differential equations that governs the stock model (Black-Scholes),* ([3], [8], [18], [20], [22]).

Let $(\Theta, \mathcal{C}, Cr)$ be the credibility space with the usual conditions, and $(L_t)_{t\in\mathbb{R}}$ a 1-dimensional Liu process [18].

We call a *generalized Liu process*, the following process

$$K^{\alpha}(t)=\int_{t_0}^{t} g_t^{\alpha}(s)dL(s), \quad t\neq s. \tag{2.13}$$

If $x(t)=x(t,\theta)$ is an $n-$dimensional fuzzy process and $a_1:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$, $b_1:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ are deterministic functions, we define the generalized Riemann-Liouville integral, respectively the generalized Liu integral by

$$\begin{aligned} {}_{t_0}H_t^{\alpha}a_1(t,x(t)) &= \textstyle\int_{t_0}^{t} a_1(s,x(s))g_t^{\alpha}(s)d(s),\\ {}_{t_0}L_t^{\beta}b_1(t,x(t)) &= \textstyle\int_{t_0}^{t} b_1(s,x(s))g_t^{\beta}(s)dL(s), \end{aligned} \tag{2.14}$$

where $\alpha:\mathbb{R}\to\mathbb{R}$, $\beta:\mathbb{R}\to\mathbb{R}$ are C^1 functions, and g_t^{α}, g_t^{β} are given by (2.6).

We call *generalized fractional fuzzy differential equation*, the functional Volterra type equation given by

$$\begin{aligned} x(t) &= x(t_0)+\textstyle\int_{t_0}^{t} a_1(s,x(s))g_t^{\alpha}(s)ds+\int_{t_0}^{t} b_1(s,x(s))g_t^{\beta}(s)dL(s)\\ &= x(t_0)+_{t_0}H_t^{\alpha}a_1(t,x(t))+_{t_0}L_t^{\beta}b_1(t,x(t)). \end{aligned} \tag{2.15}$$

Formally, equation (2.15) can be written as

$$dx=a_1(s,x(s))g_t^{\alpha}(s)ds+b_1(s,x(s))g_t^{\beta}(s)dL(s). \tag{2.16}$$

If $a_1(t,x(t))=\mu(t)x(t)$, $b_1(t,x(t))=\sigma(t)x(t)$, $\mu:\mathbb{R}\to\mathbb{R}$, $\sigma:\mathbb{R}\to\mathbb{R}$, $x:\mathbb{R}\to\mathbb{R}$, $\alpha(z)=1$, $\beta(z)=\beta_1$, from (2.16) results the following equation

$$x(t)=x(t_0)+\int_{t_0}^{t}\mu(s)x(s)ds+\frac{1}{\Gamma(\beta_1)}\int_{t_0}^{t}\frac{\sigma(s)x(s)}{(t-s)^{1-\beta_1}}dL(s). \tag{2.17}$$

The equation (2.17) is called *fuzzy equation of a stock model* [22].

Let $(\Theta,\mathcal{C},Cr)\times(\Omega,\mathcal{F},P_r)$ be the chance space [18], with the usual conditions, and $(W_t)_{t\in\mathbb{R}}$ a 1-dimensional Wiener process and $(L_t)_{t\in\mathbb{R}}$ a 1-dimensional Liu process. Let $x(t)=x(t,\omega,\theta)$ an $n-$dimensional hybrid process and $a_2:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$, $b_2:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$, $c_2:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$, deterministic functions.

It is called a *generalized fractional hybrid differential equation*, the functional Volterra type equation given by

$$\begin{aligned} x(t) &= x(t_0)+\textstyle\int_{t_0}^{t} a_2(s,x(s))g_t^{\alpha}(s)ds+\int_{t_0}^{t} b_2(s,x(s))g_t^{\beta}(s)dW(s)\\ &\quad+\textstyle\int_{t_0}^{t} c_2(s,x(s))g_t^{\gamma}(s)dL(s), \end{aligned} \tag{2.18}$$

where $\alpha:\mathbb{R}\to\mathbb{R}$, $\beta:\mathbb{R}\to\mathbb{R}$, $\gamma:\mathbb{R}\to\mathbb{R}$, are C^1 functions.

With the notations given in (2.8) and (2.13), equation (2.18) can be written as

$$x(t)=x(t_0)+_{t_0}I_t^{\alpha}a_2(t,x(t))+_{t_0}J_t^{\beta}b_2(t,x(t))+_{t_0}K_t^{\gamma}c_2(t,x(t)). \tag{2.19}$$

Formally, the equation (2.18) can be expressed as

$$dx=a_2(s,x(s))g_t^{\alpha}(s)ds+b_2(s,x(s))g_t^{\beta}(s)dW(s)+c_2(s,x(s))g_t^{\gamma}(s)dL(s). \tag{2.20}$$

3. Generalized fractional hybrid HP equation

Let Q be the paracompact configuration manifold and $\mathcal{P} = \mathbb{R} \times TQ$, T^*Q the associated bundle of Q. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(W(t), \mathcal{F}_t)_{t\in[a,b]}$, where $[a,b] \subset \mathbb{R}$, $W(t)$ is a real-valued Wiener process, and $\mathcal{F}_t$ is the filtration generated by the Wiener process [4]. The HP principle unifies the Hamiltonian and Lagrangian description of a mechanical system. The classical HP integral action will be perturbed using deterministic function $\gamma : Q \to \mathbb{R}$.

Let $\mathcal{L} : \mathcal{P} \to \mathbb{R}$ be a C^2 function, called *Lagrangian* for the mechanical system and $\gamma_1, \gamma_2 : Q \to \mathbb{R}$ two functions of class C^1. It is called *generalized fractional action* of $\mathcal{L}$, with respect to the processes $(W(t))_{t\in\mathbb{R}}$ and $(L(t))_{t\in\mathbb{R}}$, the function $\mathcal{A}^\alpha : \Theta \times \Omega \times C(\mathcal{P}) \to \mathbb{R}$ defined by

$$\begin{aligned}\mathcal{A}^\alpha(t,q,v,p) = \int_a^b (\mathcal{L}(s,q(s),v(s)) + < p(s), \tfrac{dq}{ds} - v(s) >) g_t^\alpha ds + \int_a^b \gamma_1(q(s)) g_t^\alpha dW(s) \\ + \int_a^b \gamma_2(q(s)) g_t^\alpha dL(s).\end{aligned} \tag{3.1}$$

The first integral in (3.1) is called *generalized Riemann-Liouville fractional integral*, the second one is *generalized fractional Itô integral* and the third is *generalized fractional Liu integral.* Moreover,

$$C(\mathcal{P}) = \{(t,q,v,p) \in C^0([a,b],\mathcal{P}),\ q \in C^1([a,b],Q),\ q(a) = q_a,\ q(b) = q_b\},$$

$[a,b] \subset \mathbb{R}$, $q_a, q_b \in Q$.

We do the following notations $q(t,\theta,\omega) = q(t)$, $v(t,\theta,\omega) = v(t)$, $p(t,\theta,\omega) = p(t)$.

Let $c = (q,v,p) \in C([a,b],q_a,q_b)$ be curves on $\mathcal{P}$ between q_a and q_b, and $B = (q,v,p,\delta q,\delta v,\delta p) \in C^0([a,b],\mathcal{P}\times\mathcal{P})$ such that $\delta q(a) = \delta q(b) = 0$, and q, δq are of class C^1.

Let $(q,v,p)(\cdot,\epsilon) \in C(\mathcal{P})$ be a family of curves on $\mathcal{P}$ such that they are differentiable with respect to ϵ. The differential of the action $\mathcal{A}^\alpha$ is defined by

$$d\mathcal{A}^\alpha(\delta q,\delta v,\delta p) = \frac{\partial}{\partial\epsilon}\mathcal{A}^\alpha(\omega,\theta,q(t,\epsilon),v(t,\epsilon),p(t,\epsilon))\Big|_{\epsilon=0},$$

where

$$\begin{aligned}\delta q(t) &= \tfrac{\partial}{\partial\epsilon}q(t,\epsilon)\Big|_{\epsilon=0},\ \delta q(a) = \delta q(b), \\ \delta v(t) &= \tfrac{\partial}{\partial\epsilon}v(t,\epsilon)\Big|_{\epsilon=0},\ \delta p(t) = \tfrac{\partial}{\partial\epsilon}p(t,\epsilon)\Big|_{\epsilon=0}.\end{aligned} \tag{3.2}$$

Using (3.1), by direct calculus, we get the following theorem.

Theorem 1. *Let $\mathcal{L} : \mathcal{P} \to \mathbb{R}$ be a Lagrangian, a C^2 function with respect to t, q and v and the first order derivatives are Lipschitz functions with respect to t, q, v. Let $\gamma_1, \gamma_2 : Q \to \mathbb{R}$ be functions of class C^2, and with the first order derivatives Lipschitz functions. Then, the curve $c = (q,v,p) \in C(\mathcal{P}\times Q)$ satisfies the generalized* *fractional hybrid HP equations a.s.*

$$
\begin{aligned}
&dq^i = v^i ds,\\
&dp_i = (\tfrac{\partial \mathcal{L}}{\partial q^i} - p_i h(s,t))ds + \tfrac{\partial \gamma_1}{\partial q^i} dW(s) + \tfrac{\partial \gamma_2}{\partial q^i} dL(s),\\
&p_i = \tfrac{\partial \mathcal{L}}{\partial v^i},\ i = 1, ..., n,\ t \neq s,
\end{aligned} \tag{3.3}
$$

where

$$h(s,t) = \frac{d(\alpha(s-t))}{ds} ln|t-s| + \frac{\alpha(s-t)-1}{s-t} + \rho - \frac{1}{\Gamma_1(\alpha(s-t))}\frac{d\Gamma_1(\alpha(s-t))}{ds},$$

if and only if it is a critical point of the function $\mathcal{A}^\alpha : \mathcal{P} \times Q \to \mathbb{R}$, *i.e.* $d\mathcal{A}^\alpha(c) = 0$.

□

From (3.3) we have:

(i) If $\alpha(z) = 1$, $\rho = 0$, then

$$
\begin{aligned}
&dq^i = v^i ds,\\
&dp_i = \tfrac{\partial \mathcal{L}}{\partial q^i} ds + \tfrac{\partial \gamma_1}{\partial q^i} dW(s) + \tfrac{\partial \gamma_2}{\partial q^i} dL(s),\\
&p_i = \tfrac{\partial \mathcal{L}}{\partial v^i},\ i = 1, ..., n,\ t \neq s;
\end{aligned} \tag{3.4}
$$

(ii) If $\alpha(z) = a = const$, $0 < a \leq 1$, $\rho = 0$, then

$$
\begin{aligned}
&dq^i = v^i ds,\\
&dp_i = (\tfrac{\partial \mathcal{L}}{\partial q^i} - p_i \tfrac{a-1}{s-t})ds + \tfrac{\partial \gamma_1}{\partial q^i} dW(s) + \tfrac{\partial \gamma_2}{\partial q^i} dL(s),\\
&p_i = \tfrac{\partial \mathcal{L}}{\partial v^i},\ ,\ i = 1, ..., n,\ t \neq s.
\end{aligned} \tag{3.5}
$$

For $\gamma_2 = 0$, we get the fractional stochastic HP equations [7].

If $\mathcal{L} : \mathcal{P} \to \mathbb{R}$ is hyperregular, that means $det\left(\frac{\partial^2 \mathcal{L}}{\partial v^i \partial v^j}\right) \neq 0$, from (3.3) results the following proposition.

Proposition 2. *(Generalized fractional hybrid Hamiltonian equations)*
The equations (3.3) *are equivalent with the equations*

$$
\begin{aligned}
&dq^i = \tfrac{\partial H}{\partial p_i} ds,\\
&dp_i = (-\tfrac{\partial H}{\partial q^i} - p_i h(s,t))ds + \tfrac{\partial \gamma_1}{\partial q^i} dW(s) + \tfrac{\partial \gamma_2}{\partial q^i} dL(s),
\end{aligned} \tag{3.6}
$$

where

$$H(t,q,p) = p_i v^i - \mathcal{L}(t,q,v),$$

$$h(s,t) = \frac{d\alpha(s-t)}{ds} ln|t-s| + \frac{\alpha(s-t)-1}{s-t} + \rho - \frac{1}{\Gamma_1(\alpha(s-t))}\frac{d\Gamma_1(\alpha(s-t))}{ds}.$$

□

The equations (3.6) are called *generalized fractional hybrid Langevin equations* and can be written to describe the movement equations for relativistic particles with white noise and Liu process.

Proposition 3. *If $\mathcal{L}(q,v)=\frac{1}{2}g_{ij}v^iv^j$, where g_{ij} are the components of a metric on a manifold Q, then the equations (3.3) take the form*

$$dq^i = v^i ds,$$
$$dv^i = -(\Gamma^i_{jk}v^jv^k - h(s,t)v^i)ds + g^{ij}\tfrac{\partial\gamma_1}{\partial q^j}dW(s) + g^{ij}\tfrac{\partial\gamma_2}{\partial q^j}dL(s),\ i,j=1,...,n, \tag{3.7}$$

where Γ^i_{jk} are Christoffel coefficients associated to the considered metric and $h(s,t)$ is given above.

The equations (3.6) become

$$\begin{aligned} dq^i &= g^{ij}p_j ds,\\ dp_i &= (\tfrac{1}{2}\tfrac{\partial g_{kl}}{\partial q^i}p^kp^l - h(s,t)p_i)ds + \tfrac{\partial\gamma_1}{\partial q^i}dW(s) + \tfrac{\partial\gamma_2}{\partial q^i}dL(s),\ i=1,...,n. \end{aligned} \tag{3.8}$$

□

Proposition 4. *Let $\mathcal{L}: J^1(\mathbb{R},\mathbb{R})\to\mathbb{R}$ be given by*

$$\mathcal{L}(q,v)=\frac{1}{2}v^2 - V(q),$$

and $V,\gamma_1,\gamma_2:\mathbb{R}\to\mathbb{R}$. The equations (3.8) are given by

$$\begin{aligned} dq &= pds,\\ dp &= (-\tfrac{\partial V}{\partial q} - h(s,t)p)ds + \tfrac{\partial\gamma_1}{\partial q}dW(s) + \tfrac{\partial\gamma_2}{\partial q}dL(s). \end{aligned} \tag{3.9}$$

□

4. Numerical simulations

In this section we will present some numerical simulations, by considering particular cases for the functions $V,\gamma_1,\gamma_2:\mathbb{R}\to\mathbb{R}$, given in the above propositions.

If $V(q)=\cos(q)$, $\gamma_1(q)=\alpha_1\sin(q)$ and $\gamma_2(q)=\frac{1}{2}\alpha_2q^2$, the first order Euler scheme for the equations (3.9) is given by

$$\begin{aligned} q(n+1) &= q(n)+Kp(n),\\ p(n+1) &= p(n)+K(\sin(q(n))-h(nK,t))+\alpha_1\cos(q(n))G+L(n,z_2), \end{aligned} \tag{4.1}$$

where $n=0,...,N-1$, $K=\frac{T}{N}$, G and $L(n,z)$ are simulations of Wiener and Liu processes and

$$h(nK,t)=\dot{\alpha}(nK,t)ln|t-nK|+\frac{\alpha(nK-t)-1}{nK-t}+\rho-\frac{1}{\Gamma_1(\alpha(nK-t))}\dot{\Gamma}_1(\alpha(nK-t)),$$

with $\dot{\alpha}(s,t) = \frac{d\alpha(s-t)}{ds}$, $\quad \dot{\Gamma_1}(\alpha(s,t)) = \frac{d\Gamma_1(\alpha(s-t))}{ds}$, and

$$G = random[normald[0, \sqrt{h}](1),\ L(n,z_2) = \frac{2}{1+e^{\pi|z_2|/(h\sigma\sqrt{6}S_2(n))}},\ S_2(n) = \sum_{k=0}^{n-1} \alpha_2 q(k).$$

Using Maple 13, for the values of the parameters, $\alpha = 0.6$, $t = 0.8$, $\alpha_1 = 0.1$, $\alpha_2 = 0.3$, $z_2 = 15$, we get the orbits that describe the dynamics of the state variables q and p, and also the dynamics on the phase space for the fractional hybrid dynamical system. These orbits are represented in Fig1, Fig 2 and Fig3.

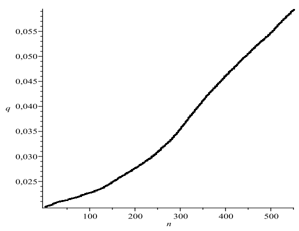

Fig1: $(n, q(n, z_2, \omega))$

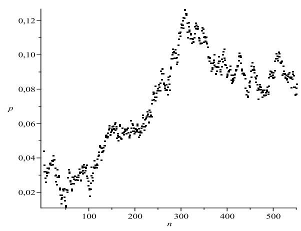

Fig2: $(n, p(n, z_2, \omega))$

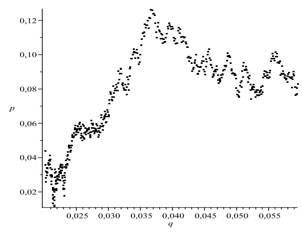

Fig3: $(q(n, z_2, \omega), p(n, z_2, \omega))$

In the classical case, with $\alpha = 1$, the dynamics of the state variables q and p are described in Fig4 and Fig5, and the dynamics on the phase space, in the classical case, is represented in Fig6.

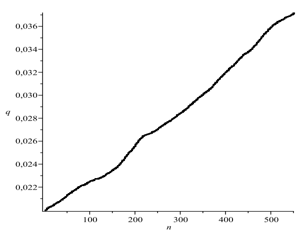

Fig4: $(n, q(n, z_2, \omega))$

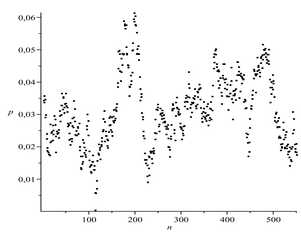

Fig5: $(n, p(n, z_2, \omega))$

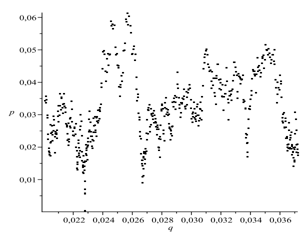

Fig6: $(q(n, z_2, \omega), p(n, z_2, \omega))$

For $\alpha = 0.8$, $t = 0.8$, $\alpha_1 = 0.$, $\alpha_2 = 0.3$, $z_2 = 15$, we get the orbits for the state variables q and p (Fig7 and Fig8), and in Fig9 it is given the dynamics on the phase space. If $\alpha = 1$, we get Fig10, Fig11 and Fig12, that illustrate the dynamics of the variables q and p and the phase space, in the classical case.

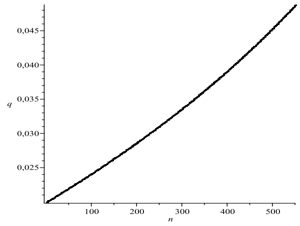

Fig7: $(n, q(n, z_2))$

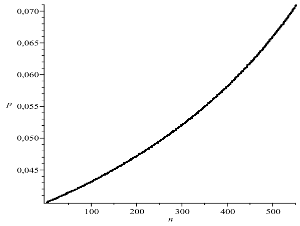

Fig8: $(n, p(n, z_2))$

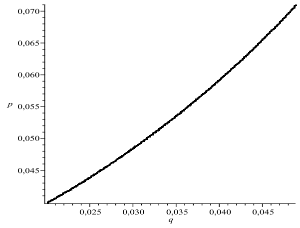

Fig9: $(q(n, z_2), p(n, z_2))$

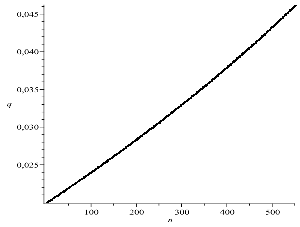

Fig10: $(n, q(n, z_2))$

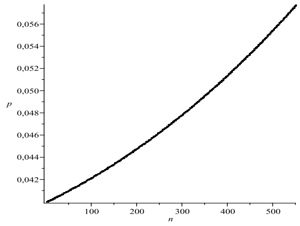

Fig11: $(n, p(n, z_2))$

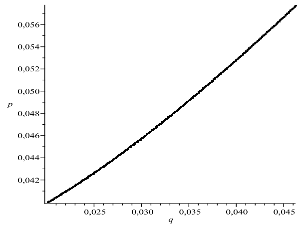

Fig12: $(q(n, z_2), p(n, z_2))$

The orbits for $(q(n, \omega), p(n, \omega))$, $(q(n, z_2), p(n, z_2))$ and $(q(n, z_2, \omega), p(n, z_2, \omega))$, for the values of the parameters $\alpha = 0.8$, $t = 3$, $z_3 = 3$, $\alpha_1 = 4$, $\alpha_2 = 0$, respectively $\alpha_1 = 0$, $\alpha_2 = 2$, respectively $\alpha_1 = 4$, $\alpha_2 = 2$, are represented in Fig13, Fig14 and Fig15.

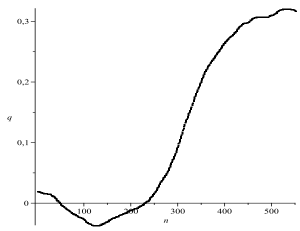

Fig13: $(q(n, \omega), p(n, \omega))$

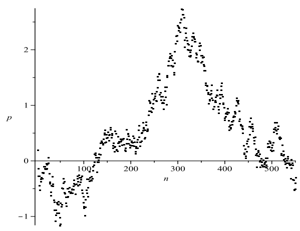

Fig14: $(q(n, z_2), p(n, z_2))$

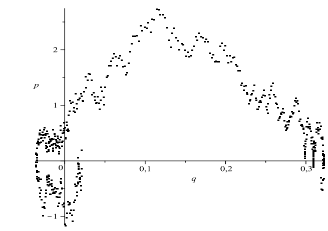

Fig15: $(q(n, z_2, \omega), p(n, z_2, \omega))$

For $\alpha = 1$, $t = 3$, $z_2 = 3$, $\alpha_1 = 4$, $\alpha_2 = 0$, respectively $\alpha_1 = 0$, $\alpha_2 = 2$, respectively $\alpha_1 = 4$, $\alpha_2 = 2$, we get Fig16 for $(q(n, \omega), p(n, \omega))$, Fig17 for $(q(n, z_2), p(n, z_2))$, and Fig18 for $(q(n, z_2, \omega), p(n, z_2, \omega))$.

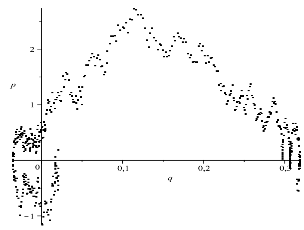

Fig16: $(q(n, \omega), p(n, \omega))$

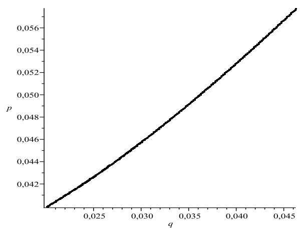

Fig17: $(q(n, z_2), p(n, z_2))$

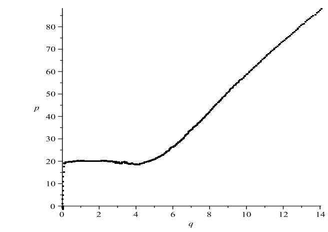

Fig18: $(q(n, z_2, \omega), p(n, z_2, \omega))$

5. Conclusions

In this paper we have presented a generalization of fractional Riemann-Liouville integral, Wiener process, and we have defined the generalized fractional stochastic, Liu and hybrid equations. The mixture between generalized fractional Wiener process and generalized fractional Liu process results as the generalization of fractional hybrid differential equations. We defined generalized fractional hybrid HP equations and generalized fractional hybrid Hamiltonian equations. The first order Euler scheme is presented and implemented for particular parameters. In the future work, we will consider other problems that deal with stochastic and hybrid fractional HP principle.

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