Phase Transitions and Complex Systems

Simple, nonlinear models capture complex systems at the edge of chaos

RICARD V. SOLÉ, SUSANNA C. MANRUBIA, BARTOLO LUQUE, JORDI DELGADO, AND JORDI BASCOMPTE

PHASE TRANSITIONS AND ORDER PARAMETERS

How to define complexity is not a trivial question. Several definitions have been proposed, and all of them share the intuitive idea that complexity is neither complete order nor complete disorder [1-5]. But this statement, though fairly intuitive, is far from satisfactory. A quantitative characterization of complexity is necessary. Which "universal" features share apparently different complex systems? The behavior of physical systems close to critical points may answer this question. It is well known, from the theory of phase transitions, that a given system (possibly made of many subsystems) can undergo strong qualitative changes in its macroscopic properties if a suitable control parameter is adequately tuned and that close to these critical points some key characteristic constants (the so-called critical exponents) are the same for very different systems [6]. At critical points, fractal structures, complex dynamical patterns and optimal information transfer appear in a spontaneous way. Observing such properties in those systems which we call "complex," we can conjecture that *complexity tends to appear close to instability points.*

In this paper, two main types of theoretical approximations are considered[1]: (i) continuous dynamical systems, possibly including stochastic terms, defined as [1,7-8]

$$dX/dt = f(X) + \sigma(X)\xi \qquad (1)$$

where $X = (x_1, ..., x_n) \in \mathcal{R}^n$, f is a given nonlinear function and $\sigma(X)\xi$ a noise term (see below); and (ii) cellular automata (CA) models, given by a set of transition rules [4],

$$S_{t+1}(\mathbf{r}) = T[\{c_j\}; \{S_t(\mathbf{r}')\}] \qquad (2)$$

where a d-dimensional lattice $\Lambda_d(L) = \{\mathbf{k}\}$ ($\mathbf{k} = (k_1,...,k_d); 1 \leq k_j \leq L$) is used. For the CA approach, space and time are discrete as are, typically, the number of states: $S \in \Sigma = \{0, 1, ..., s-1\}$. Here T indicates a set of transition rules which can be deter-

ministic or stochastic. $\{c_j\}$ is a set of parameters and $\mathbf{r}'$ are a given set of nearest neighboring positions of $\mathbf{r}$. Typically, the set of neighboring points, $\mathcal{N}(\mathbf{r})$, is symmetrically distributed around the site $\mathbf{r}$, and those CA able to simulate real systems often verify some particular constraints.

In this section, we summarize the most relevant phase transitions (PHTs). To begin with, we use the approach of (1), where spatial degrees of freedom and explicit local properties are neglected.[2] We will assume that these are *nonequilibrium phase transitions*, which are achieved by means of energy or matter inputs into the system.

First-Order Transitions

Sudden changes in the behavior of nonlinear systems as a consequence of a continuous change in a given parameter are well known [1,8]. An example is given by the one-dimensional nonlinear system

$$dx/dt = f(x) = x^3 + \mu x + \beta. \qquad (3)$$

Ricard V. Sole is an associate professor in the Department of Physics of the Universitat Politecnica de Catalunya and head of the Complex Systems Research Group, to which all the authors belong. Susanna C. Manrubia is an assistant professor at UPC who is completing her Ph.D. in the physics of complex systems, particularly the critical phenomena of living systems. Bartolo Luque is working on his Ph.D. in the physics of complex systems at UPC and specializing in Kauffman networks and cellular automata. Jordi Delgado is an assistant professor in the Department of Theoretical Computer Science and Software at UPC and is working on a Ph.D. in artificial intelligence and fluid neural networks. Jordi Bascompte is a biologist with a Ph.D. in theoretical ecology from the University of Barcelona and is currently a post-doctoral fellow in the Department of Ecology and Evolutionary Biology at the University of California/Irvine.

The stationary solutions of (1) are given by $f(x_s) = 0$ and are represented in Figure 1 in terms of the parameter space (μ, β), forming a smooth surface with a fold. If a single parameter, β, is smoothly changed from negative to positive (from i to f in Figure 1), the steady state x_s is also continuously changed, but, for a given β_s, a sudden transition takes place. The state variable jumps from the upper branch to the lower one. In the language of catastrophe theory, it corresponds to a "cusp catastrophe" for the x_s solutions. Many physical and biological systems show this type of behavior, known as *first-order* PHTs. These types of PHTs are not relevant for our discussion on complex systems as they reflect a transient phenomenon between two different states.

Second-Order Transitions

A different type of PHT takes place in the dynamical system (DS)

$$dx/dt = f(x) = -\mu x - \beta x^3, \qquad (4)$$

which can be written as deriving from a scalar potential $\phi(x)$,

$$dx/dt = -(\partial/\partial x)\varphi(x), \qquad (5)$$

where $\phi(x) = \mu x^2/2 + \beta x^4/4$, also known as the ϕ^4 model [8]. The stationary solutions of (4) are $x_s^0 = 0$ and $x_s^\pm = \pm\sqrt{-\mu/\beta}$. When $\mu, \beta > 0$, the single real solution, x_s^0, is stable (i.e., $\partial_x f(x_s) < 0$). But for $\mu > \mu_c = 0$, a bifurcation takes place, x_s^0 becomes unstable, and two new solutions $(x_s^\pm)$ emerge. Any minimal perturbation

First-order phase transition. This surface is obtained from the stationary solutions of eq. (3), defined over the (μ, β) parameter space. As a given parameter is varied continuously from an initial value (i) to a new one (f), the system undergoes a sudden shift from the upper branch to the lower one.

from x_s^0 will lead to a dynamical evolution towards one of the new solutions. In Figure 2(a), the bifurcation diagram for x_s is shown; this is a characteristic *second-order* PHT. Two stable solutions are available when $\mu < 0$, and the system must choose one of them; we then say that *symmetry breaking* takes place [1,8]. Figure 2(b) is a mechanical illustration of this process.

Fluctuations can be included in this formal approach by means of the so-called Fokker-Planck equation (FPE) [7,8]. This approach is very important in order to analyze real systems, where fluctuations are inevitable and where probability distributions are the observed data. Starting from a general dynamical system $dx/dt = f(x)$, a stochastic term can be included as

$$dx/dt = f(x) + \sigma(x)\xi. \qquad (6)$$

where ξ is a white noise with Gaussian distribution and $\langle\xi\rangle = 0$. It can be shown that the time-dependent probability, $f(x, t)$, of finding $x \in (x, x + dx)$ at the time, t, is given by the FPE:

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\sigma^2(x)f\right] + \frac{\partial f}{\partial x}\left[G(x)f\right]. \qquad (7)$$

where $G(x)$ is a given function defined from $f(x)$ and the noise term $\sigma(x)$.[3] The stationary solution $f_s(x)$, if it exists, is obtained from

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\sigma^2(x)f_s\right] + \frac{\partial f}{\partial x}\left[G(x)f_s\right] = 0 \qquad (8)$$
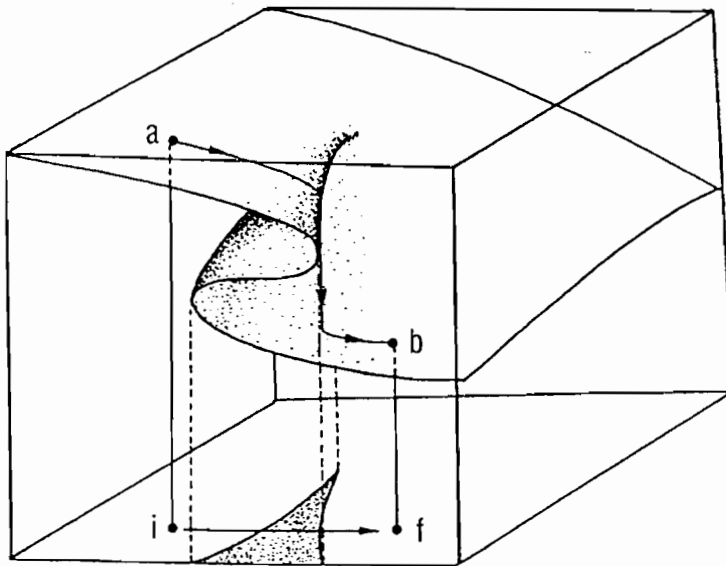
and it reads

$$f_s(x) = \frac{K}{\sigma^2(x)}\exp\left\{\int_{x_0}^{x}2\frac{G(z)}{\sigma^2(z)}dz\right\}, \qquad (9)$$

K being the normalization constant. Once the distribution $f_s(x)$ is obtained, the moments of the distribution can be calculated.

Figure 2(a) shows the trajectory of a dynamical system as described by (4) but with a small noisy perturbation of constant strength σ. i.e., we have $dx/dt = f(x) + \sigma\xi$. It can be shown that these small fluctuations are amplified close to the instability point. At μ_c they become very large. This phenomenon is known as *critical slowing down* [6,8] and is a characteristic fingerprint of the onset of a second-order PHT. These kinds of transitions are more interesting for several reasons, and they are linked, in spatially distributed systems, with the appearance of self-similarity and optimal information transfer. Typically, for an arbitrary dynamical system $dx/dt = F_\mu(x)$, which shows symmetry breaking for $\mu > \mu_c$, a macroscopic measure O_μ (the so-called order parameter) can be defined over the system (here μ means a given [set of] parameter[s]) in such a way that this quantity is positive for $\mu > \mu_c$ and vanishes at μ_c. For a

FIGURE 2

second-order PHT, O_μ grows close to μ_c (when $\mu > \mu_c$) as $O_\mu \approx (\mu - \mu_c)^{1/2}$ and the relaxation time of fluctuations behaves near μ_c, like $\tau \approx |\mu - \mu_c|^{-1}$.

The concept of order parameter has been widely generalized [1,8] as a cornerstone of the theory of nonequilibrium PHTs. In this sense, when an arbitrary system (from a laser to an ant colony) crosses an instability point, μ_c, only just a few variables become relevant and serve as order parameters which define the macroscopic pattern. At the same time, these macroscopic variables govern the behavior of the microscopic parts. This is the core of what we call "emergent properties."

Noise-Induced Transitions

A further example of PHTs is linked with the response of some nonlinear systems to multiplicative external noise, i.e., when $\sigma = \sigma(x)$ as in (7). These fluctuations, coupled in this system-dependent way, can induce new nonequilibrium transitions, i.e., qualitative changes in the steady state of the system, so the system no longer adjusts its macroscopic behavior to the average properties of the environment but responds in a more active way. To be specific, we again treat a nonlinear system,

$$dx/dt = \alpha - x + \lambda x(1 - x), \qquad (10)$$

with $x \in [0, 1]$, i.e., the genetic model [7]. In a constant environment (λ constant), no instability occurs; there is no transition phenomena; and, with $\lambda = 0$ and $\alpha = 0.5$, for example, the steady state is $x^* = 0.5$.

If $\lambda = \lambda + \sigma\xi_t$, we have a stochastic process where ξ_t fulfill the previous properties. It can be shown that the previous equation can be written similarly to equation (6), being $\sigma(x) = x(1 - x)$. In Figure 3, the stationary density $f_s(x)$ is shown for different intensities of noise. A noise-induced transition is observed where a *qualitative change in the shape of the stationary probability density* is generated. In the case of the genetic model, we can find, for the above-mentioned parameters $\lambda = 0$ and $\alpha = 0.5$, a change in the number of extrema of $f_s(x)$: as we increase the noise intensity σ, $f_s(x)$ goes from a maximum in $x^* = 0.5$, i.e., the most probable value will be the deterministic one, to the appearance of three extrema for $\sigma^2 > 4$, one minimum at $x^* = 0.5$, and two maxima $x_{m\pm}$, so the system now displays bistable behavior. This is a *pure noise-induced* phenomenon. It shows that, sometimes, *the knowledge of the average environmental state is insufficient to predict the macroscopic behavior of the system*. In other cases, external noise can lead to even more profound modifications.[4]

MEASURING COMPLEXITY: THE ISING MODEL

Many examples of second-order or noise-induced PHTs are known from classical and quantum physics: the PHTs when magnets are heated up (see below); superconductors which become simple conductors; inflationary models of the early evolution of the universe, and so forth. In biology, threshold
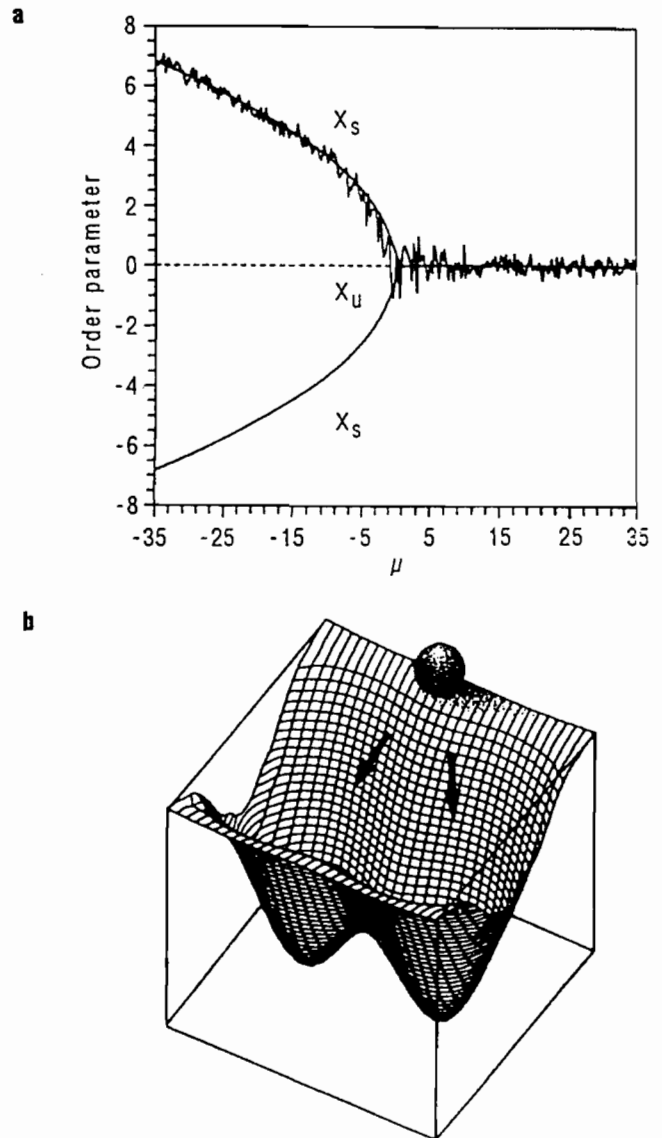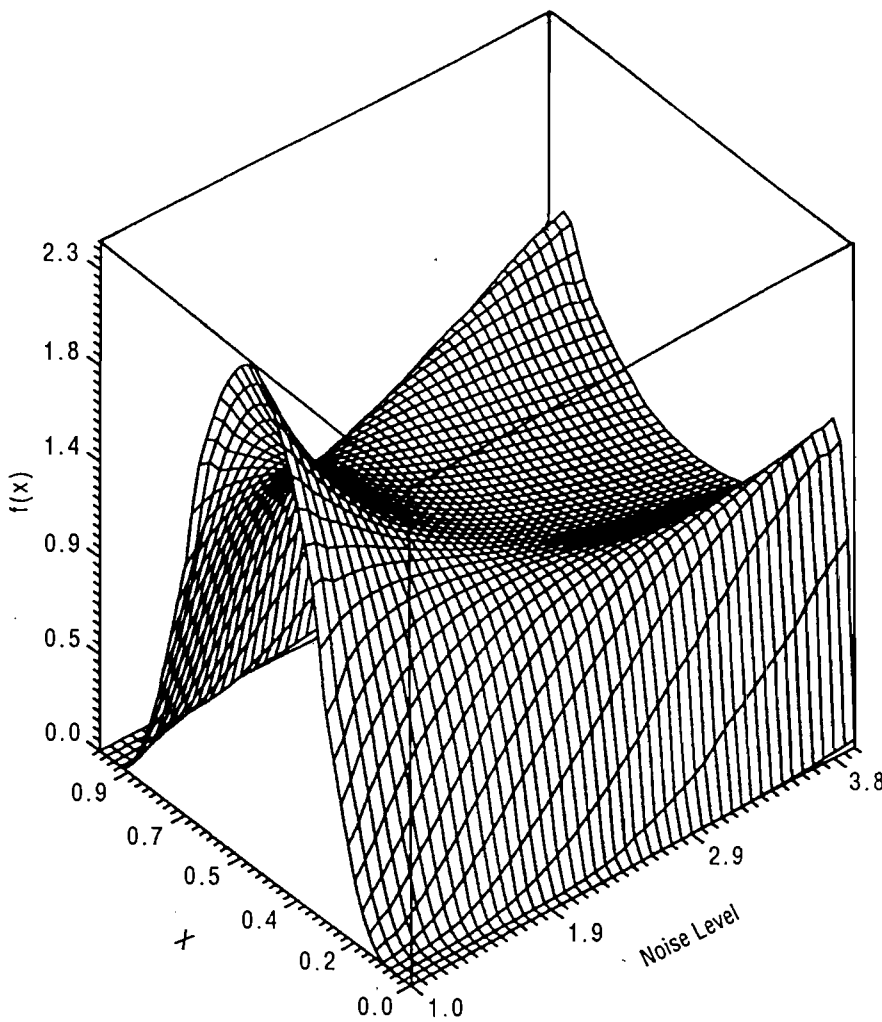
Second-order phase transitions. (a) As defined from eq. (4), in a second order phase transitions, two new branches appear symmetrically from the old one, which becomes unstable. A symmetry-breaking phenomenon takes place, and one of the two possible solutions is chosen, as shown by the stochastic trajectory. (b) A mechanical illustration of symmetry-breaking phenomena. A potential $\phi(x)$ can be represented by a surface with two minima. As the ball moves down (starting from a symmetric initial condition), any small fluctuation will decide the final state.

phenomena in cellular dynamics, the collective behavior of insect societies, or the large-scale dynamics of the brain are just some examples. In all these examples, the system undergoes dramatic changes in its qualitative properties when the critical point is reached. But before proceeding further in our study of PHTs, let us consider the best known model of them, the Ising model [6], as a reference nonlinear (closed) system formed by many elements with local interactions. Several relevant measures of general interest for complex systems will be defined.

FIGURE 3

Noise-induced phase transition. Here the deterministic dynamical system given by eq. (10) is perturbed by means of a multiplicative noise term. The stationary probability density (as given by eq. (9)) is shown. As the noise level is increased, the single maximum is replaced by two new estrema.

Here k_B is the Boltzmann constant and T the temperature. The temperature is a very important control parameter: if we start from a system at high temperatures and we cool slowly our system, a sudden change in the behavior of this system takes place at a critical point T_c the so-called Curie temperature. In order to understand the nature of this change, we can define a new quantity: the global magnetization M given by $M = \Sigma_i S_i$. In Figure 2(a) we see how this quantity behaves. For $T > T_c$, $M = 0$ as a consequence of the effect of thermal excitations on the interaction of nearest spins. When $T << T_c$, interactions are not perturbed and the spins are aligned (so M becomes maximum). Formally, the magnetization per spin, $m = M/N$, is shown to behave as $m = 0$ for $T > T_c$ and as

$$m = \left[1 - (\sinh(2\beta\tau))^{-4}\right]^{1\ 8}$$

for $T < T_c$. Close to T_c (for $T < T$), m behaves as $m \approx |\tau|^\alpha$ where τ is the so-called reduced temperature, i.e., $\tau = (T - T_c)/T_c$. The exponent α is named a *critical exponent*. Generally, the critical exponent λ associated with a function $F(t)$ is defined as $\lambda = \lim_{\tau \to 0}[\log|F(t)|/\log|\tau|]$, i.e., $F(t) \approx |\tau|^\lambda$.

Correlations and Percolation

At the critical point, something happens. What? The behavior at the critical point, T_c, can be visualized by means of the simulation model. In Figure 4 we see three examples of this system for three different points: for (a) $T = 1.2\ T_c$, for (b) $T = T_c$, and for (c) $T = 0.95\ T_c$. For temperatures greater than the Curie temperature, entropy dominates over energy, and the distribution of spins is essentially random (Figure 4(a)). We say that the so-called *correlation length* ξ is small. It expresses the typical distance over which the behavior of a microscopic variable is correlated with (influenced by) the behavior of another. Close to T_c, ξ scales as $\xi \approx |\tau|^{-\nu}$, ν being another critical exponent. Below the critical point, the model is said to exhibit long-range order: the exchange interaction works over thermal perturbations (Figure 4(c)). Now let us move towards the critical point. If we observe this state (Figure 4(b)), we see that both types of spins are equally represented, but their distribution appears to be clustered in a non-trivial way. In fact, if a large enough system is seen, we can appreciate the existence of a self-similar pattern (i.e., a fractal object). There is no upper

Microscopic Model

Consider a two-dimensional square lattice of length side N (in one dimension, no PHT is obtained). The state of each lattice site, i, is indicated as $S_t(i)$ for each time step, t. Time is assumed discrete, and $\Sigma = \{-1, +1\}$. Each "spin," $S_t(i)$, interacts with its four nearest neighbors through an exchange interaction, J, which favors parallel alignment (the least-energy configuration).

The dynamics of the two-dimensional Ising model is easily simulated by computer. For each time step, we pick up at random a single spin, S_i, and its local field, h_i, defined as $h_i = J \Sigma S_i$ (the sum is carried out over nearest neighbors), is computed. Then the i-th spin will change with a probability, W_i, given by [6]:

$$W_i(S_i) = P(S_i \to -S_i) = (1/2)[1 - S_i \tanh(h_i/k_B T)]. \qquad (11)$$

FIGURE 4

cut-off length, and ordered structures exist on every length scale. In fact, this is the microscopic physics underlying a critical PHT: fluctuations on all scales of length are meaningful.

The correlation length, ξ, also represents the typical distance of connected cluster of spins. For a fixed ξ, fluctuations of blocks of all sizes up to size ξ can be found. When moving towards the critical point, ξ diverges to infinity, and so does the size of the maximum connected cluster. This means that there is a connected path on this cluster that spreads from one side of the system to the other. When this happens, the system is said to percolate. There are many different types of percolation [9], depending on the geometry of the system and on the existence of correlations in the formation of connected sites. The simplest example of percolation consists in randomly filling the sites of a square lattice with some probability, p. If p is increased from 0 to a critical value, p_c, the properties of a second-order PHT are recovered: one finds divergency of the correlation length and of the size of the clusters (until, at $p = p_c$, a percolation cluster that connects both sides of the system appears). At the critical point, the system displays fractal geometry (the size of the clusters scales following a power-law that helps to characterize the transition, too), and fluctuations of all sizes can be found. At or close to a critical point, the amount of information that might be transferred throughout the system is maximal.

Percolating systems are likely to be very robust against external perturbations, provided that each and every different size is already considered in the system, both in space and in time.

Entropy and Mutual Information Functions

Before continuing, we must define two additional measures that will be used in the next sections. Here we consider a given system formed by many subsystems (as in the Ising model). Let $\{x_i\}$ with $i = 1, 2, ..., m$, a given set of values $x_i \in \Sigma$, where $\#(\Sigma) = S$. Here $\{x_i\}$ can be a time series obtained from one of the subsystems, or the complete set of current states, or a combination of both. And let us assume that we can define two sets of probabilities $\{P_\alpha\}$ and $\{P_{\alpha\beta}\}$. Here P_α will be the probability of observing the $\alpha \in \Sigma$ state, and $\{P_{\alpha\beta}\}$, the probability of observing both $\alpha, \beta \in \Sigma$ states. These probabilities would be obtained by averaging over time; or averaging, at a given time step, over the elements of the system; or both.
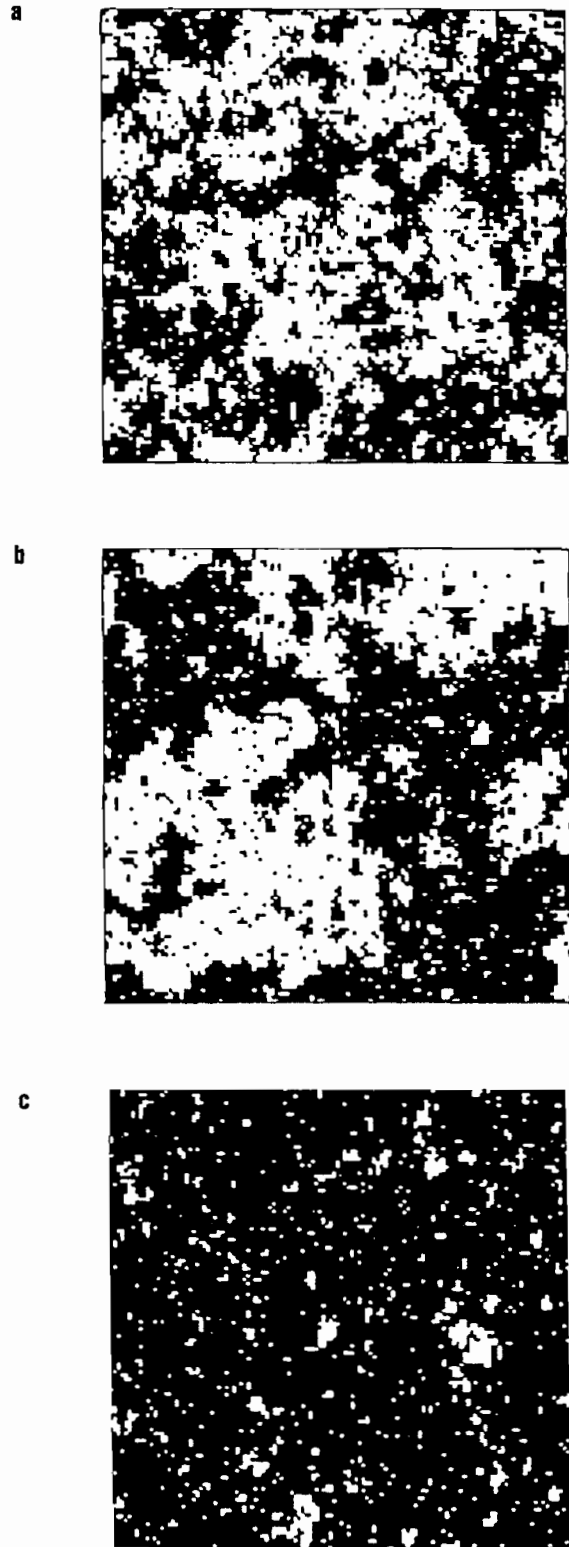
Then the Boltzmann (Shannon) entropy is given by

$$H = -\sum_{\alpha \in \Sigma} P_\alpha \log |P_\alpha| \tag{12}$$

and the mutual information by [10, 11]:

$$M = -\sum_{\alpha \in \Sigma} \left\{ \sum_{\alpha \in \Sigma} P_{\alpha\beta} \log \left[\frac{P_{\alpha\beta}}{P_\alpha P_\beta} \right] \right\}. \tag{13}$$

The first quantity gives us a measure of disorder, and it is fitted by $0 \le H \le \log(S)$, which corresponds to the simplest

Phase transitions and the Ising model. Three snapshots of a 200 × 200 lattice of spins are shown for different temperatures, T. (a) $T = 1.2\ T_c$; (b) $T = T_c$ (critical point, Curie temperature); (c) $T = 0.95\ T_c$.

state ($P_\alpha = 1$; $P_{\beta \neq \alpha} = 0$) and the uniform distribution ($P_\alpha = 1/S$, $\forall \alpha \in \Sigma$). If we consider the Ising model, this can be observed for the entropy per site. It is given by $H = -\{p_\star \log(p_\star) + (1 - p_\star) \log(1 - p_\star)\}$, and is $H = 0$ at low temperatures (when $m \rightarrow m_\star$) and $H = \log(2)$ at high temperatures. Using the same example, let us now consider two neighboring spins, and let us calculate the previous probabilities by averaging many steps over time. At low temperatures, the system is very ordered, and we have to expect $P_{\leftrightarrow} \rightarrow 1$ or $P_{\_\_} \rightarrow 1$, and so $M \rightarrow 0$. At the other extreme, both elements will behave randomly and nearly independently, and so we have to expect $P_{\alpha\beta} = P_\alpha P_\beta$, and so again $M \rightarrow 0$. At intermediate values, when correlations emerge close to the critical point, we expect a maximum in M,[5] as shown in Figure 5.

CELLULAR AUTOMATA AND PHASE TRANSITIONS

Though the Ising model is a very good representation of a wide set of equilibrium physical systems, we can ask ourselves if other (now *open*) open complex systems can be equally well represented, at least in their qualitative properties. Cellular automata (CA) models have been widely used as models of complex phenomena [12]. They also exhibit PHTs (see [12]).

Deterministic Cellular Automata (DCA)

Invented in 1948 by the Hungarian mathematicians Von Neumann and Ulam, these dynamical systems are defined in a general way by (2). These models exhibit an enormous variety of spatio-temporal patterns, but they can be classi-

Mutual information, M, (as defined in Section 2.3) for the Ising model. The information transfer between two nearest spins has been calculated using a 20 x 20 lattice, averaging over 500 time steps, and using 50 configurations. A maximum is obtained at the critical point.

fied into four main qualitative categories (Figure 6(a-d)): (a) Class I: evolution leads to a homogeneous state (steady attractor, fixed point); (b) Class II: evolution leads to a set of localized stable and/or periodic structures (periodic attractors); (c) Class III: evolution leads to chaotic patterns (chaotic attractor); and (d) Class IV: evolution leads to complex localized structures with very long transients. These automata are the only ones able to perform universal computation (Figure 6(d)).

This scheme was revised by means of Langton's λ parameter [4]. Following the notation of Section 1 and using $S = \#(\Sigma)$ and $\kappa = \#(\mathcal{N})$, we can define Langton's parameter for a given CA as

$$\lambda = \frac{S^\kappa - n_q}{S^\kappa}. \tag{14}$$

Here S^κ is the number of all the possible transitions for a specific rule-table, and n_q is the subset of such transitions (in the rule-table) leading to the so-called *quiescent state* (usually 0). This parameter gives a rough quantitative characterization of the CA complexity. Using λ, we can order the automata rules following the sequence *Class I $\rightarrow$ Class II $\rightarrow$ Class IV $\rightarrow$ Class III*, and so the complexity of CA rules is ordered. It has been shown that information grows up to a maximum at a "critical" λ_c, *where class-IV automata are observed*. This fact suggests that computation in natural systems should emerge close to PHT points [3, 4] (see also Haken [8, 14] for a detailed discussion of physical systems). The $H - M$ diagram shows us how complexity (M) and disorder (H) are related: information needs some degree of order to be stored (low entropy) but also some degree of disorder if transmission and manipulation are necessary [8, 15]. The critical point, where the maximum M is obtained, shows the compromise between both tendencies. This conjecture is known as *computation at the edge of chaos*, but the λ parameter, as first defined by Chris Langton, has been shown to be flawed. For example, the work of Mitchell, Hraber, and Crutchfield [13] has shown that symmetry-breaking processes can impede the evolution toward higher computational capability.

Stochastic Cellular Automata (SCA)

As happens with continuous, noise-driven systems (see the first section), CA models incorporating noise (the so-called stochastic cellular automata, [15]) can generate unexpected patterns of behavior. A study of the simplest one-dimensional DCA ($S = 2$, $\kappa = 3$) shows that they do not exhibit class-IV behavior.[6] But some SCA, which are classified as class-I if no randomness is used, belong to class-IV for a suitable noise level.

Stochasticity can be introduced in several ways. For example, if a one-dimensional CA is used, with $\Sigma = \{0,1\}$ and coupling with nearest neighbors, the i-th automaton takes the value $\mu \in \Sigma$ according to a conditional probability: $P\{S_{t+1}(i) = \mu | S_t(i - 1), S_t(i), S_t(i + 1)\}$. It can be shown that by means of a percolation phenomenon (the so-called directed percola-

FIGURE 6
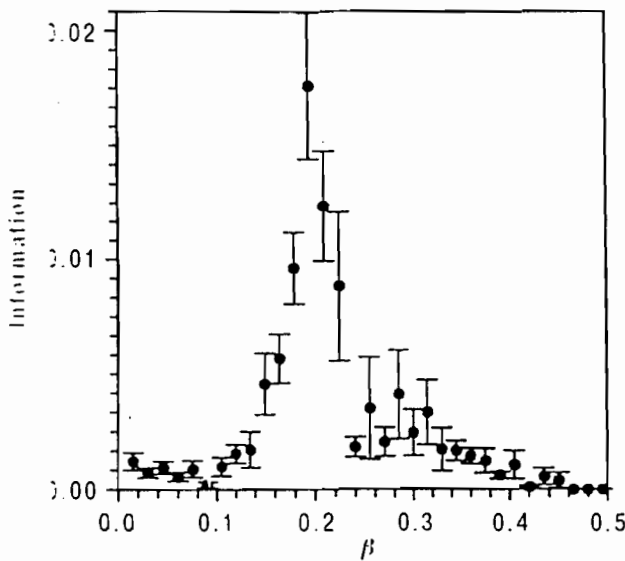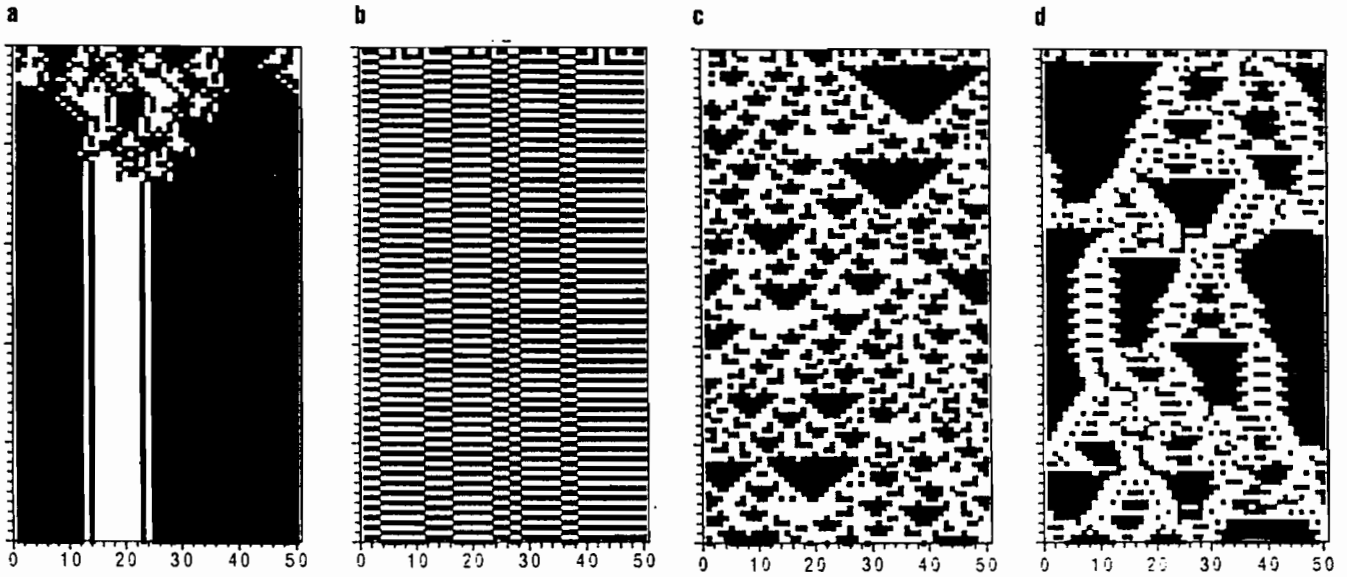
Patterns of behavior of one-dimensional cellular automaton models, as defined by Wolfram's classification: (a) class I; (b) class II; (c) class III; and (d) class IV.

tion [9]), complex patterns emerge at critical points, with statistical properties reminiscent of class-IV DCA. In other cases, a small amount of noise (something unavoidable in real systems) can modify the long-term behavior of a class-II CA in such a way that the transient structures (which can look like class-IV propagating structures) are stabilized by means of noise.

SELF-ORGANIZED CRITICALITY

As we have seen, complex patterns involving the appearance of fractal structures are observed at critical points. Fractal forms are widespread in nature, from mountain landscapes and river networks to clusters of galaxies. Other self-similar patterns are observed in the time evolution of many systems which exhibit the so-called $1/f$ noise. But the origin of such patterns has been a matter of debate for a long time. Free-scale behavior seems to be linked with complexity. How does it emerge?

The Sandpile Automaton

In 1987, Per Bak, Chen Tang, and Kurt Wiesenfield [16] suggested a new theory which they called "self-organized criticality" (SOC). They suggested that a wide variety of natural dissipative systems spontaneously evolve toward the critical point. No tuning of some external parameter is necessary, and in this sense these phenomena should be "parameter-free." The best known example of a physical system exhibiting SOC (at least in some cases) is the sandpile (Figure 7(a)). When we build the pile of sand by adding a few grains at a time, the slope grows until a maximum value is reached. Once this situation is obtained, the addition of new grains very often generates small avalanches. Larger avalanches are less common but also occur. From time to time, a very large avalanche takes place. These events generate a $1/f$ dynamics (i.e., self-similar

fluctuations in time). resulting in a critical state: the distribution of avalanches is. in fact, a power law.

Bak et al. [16] suggested that a simple cellular automata model (the sandpile automaton. SA) is able to reproduce the observed behavior. Let us consider a two-dimensional grid where the "state" of each point is given by the local "slope." $z(i,j)$. There are two essential rules defining the automaton: (i) Addition of a unit: we randomly choose a grid point and add a "grain": $z(i,j) \rightarrow z(i,j) + 1$; and (ii) Toppling: if the local slope is larger than a critical value, K, (usually $K = 4$ with Neumann neighborhood), then some units are transferred to nearest positions: $z(i,j) \rightarrow z(i,j) - 4$ and $z(i \pm 1, j \pm 1) \rightarrow z(i \pm 1, j \pm 1) + 1$.

Power laws are a consequence of the self-organization process: once the pile reaches the critical slope, the model exhibits the general features of the sandpile. In particular, if we study how many avalanches of each size occur after the addition of a single "grain" to the pile, we find that the distribution of events where a total of s sites topple obeys a power law (Figure 7(b)): $N(s) \approx s^{-\tau}$. Thus, if one waits long enough, one will certainly see events that are as large as one has the patience to wait for. As Bak has pointed out, *avalanches are an unavoidable and intrinsic part of sandpile dynamics*. On the other hand, the spatial distribution of critical slope K also shows fractal behavior. This result strongly suggests a nontrivial connection between temporal and spatial self-similarity [17].

SOC and Universality

The sandpile model (and other models of real systems which exhibit SOC) is only a metaphor of the real sandpile but a powerful one. The key word for this consistence is *universality*. What this means (and it is particularly true for critical sys-

FIGURE 7
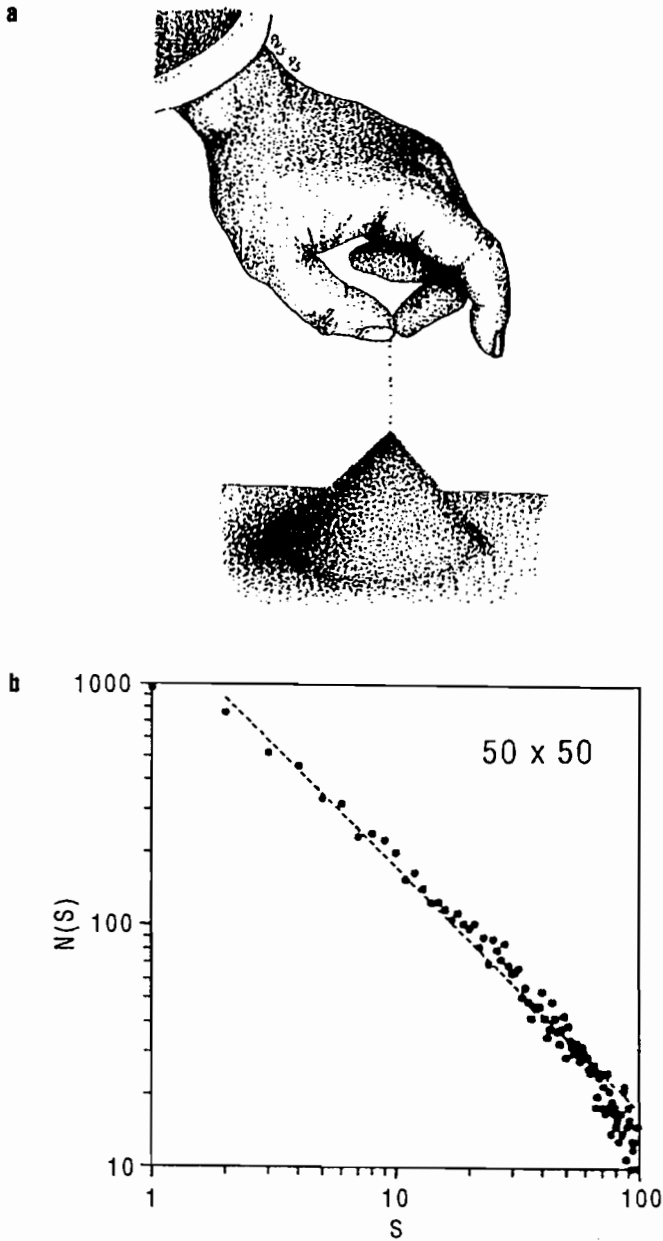
(a) The real sandpile; (b) power-law distribution obtained from a computer simulation of a two-dimensional sandpile cellular automaton, as defined in The Sandpile Automation section

TABLE I

Properties of Kauffman nets for different values of K

CONNECTIVITY	CYCLE LENGTH	N ATTRACTORS	STABILITY
$K = N$	$0.5 \times 2^{N/2}$	N/e	low
$K = 1$	$\exp(1/8 \log^2 N)$	$(2/\sqrt{e})^N$	low
$K = 2$	$\sqrt{N}$	$\sqrt{N}$	high

tems) is that important features of large-scale phenomena are roughly insensitive to the particular details of the models and are shared by very (apparently) different systems. Thus, it has been suggested that earthquakes or even turbulence would be associated with SOC phenomena (see [17]) belonging to different universality classes. Actually, this is the only known mechanism by means of which an explanation for self-organized complexity is obtained. Whether or not SOC is a universal mechanism of "complexity generation" will be a matter of future research.

EXAMPLES: MODELS AND REALITY

We now present several examples of complex systems exhibiting critical properties and introduce some theoretical approaches to phase transitions.

Kauffman Networks and Genome Stability

Random Boolean networks (also known as Kauffman networks, KN) were first proposed by Stuart Kauffman as simple (yet reasonable) models of genetic systems [18]. Since their definition in 1962, they have become one of the most celebrated models of complex systems. In these networks, a set of N binary elements is used. The state of each element at a given time step, t, is given by $S_i(t) \in \{0, 1\}$, $(i = 1,...,N)$. The dynamical state of $S_i(t)$ is updated (synchronously) by means of a Boolean function, Λ_i. Each element receives inputs from exactly K elements, resulting in a dynamical system defined from

$$S_i(t + 1) = \Lambda_i [S_{i_1}(t), S_{i_2}(t),..., S_{i_K}(t)] \qquad (15)$$

$(i = 1,2,...,N)$. The critical point, K_c was analytically determined by Derrida and his colleagues [19] in a set of remarkable theoretical studies using the so-called *annealed* approximation. The dynamical and structural properties of the critical point can be well understood by means of a very interesting example: the problem of cell differentiation from the global activity of a complex genomic system.

All cells in a multicellular organism share the same genome formed by N genes (and some amount of selfish DNA). If we assume (reasonably) that genes are essentially on-off elements, a given differentiated cell (a neuron, a kidney cell, etc.) is characterized by a subset of active genes. The number of different cell types is not arbitrary and much less than N. Why? The existence of a critical point in KN might be the answer. The key assumption of Kauffman's theory is that *a cell type is an attractor*, more precisely, a state cycle attractor. Starting from different initial conditions (IC), a KN evolves toward a more or less complex periodic pattern, the system's attractor. Several IC can reach identical attractors, forming a *basin of attraction*. The global properties of these attractors—i.e., their length, number (n_a) and stability against small perturbations—are controlled by the connectivity, K, and are summarized in Table I.

At $K_c = 2$, nets *crystallize spontaneous order*. KN with critical connectivity exhibit unexpected and powerful collective order. Both the expected length of state cycles and the number of attractors are $n_A = O(\sqrt{N})$. This implies that in a genome of 10,000 elements with $2^{10,000} \approx 10^{3,000}$ combinations of activities, only 100 asymptotic patterns (different cell types) are observed. Another property of critical KN is that (as cells) each attractor is stable to most minimal perturbations. We have, then, an example of PHT where the properties of real genomes and cells are found to appear. Even if a distribution of input connectivities is used instead of a fixed K, the previous results are obtained [20]. A very important consequence of these results is that the observed properties involved in genetic regulatory systems would be the result of intrinsic dynamical processes and not just the action of natural selection.

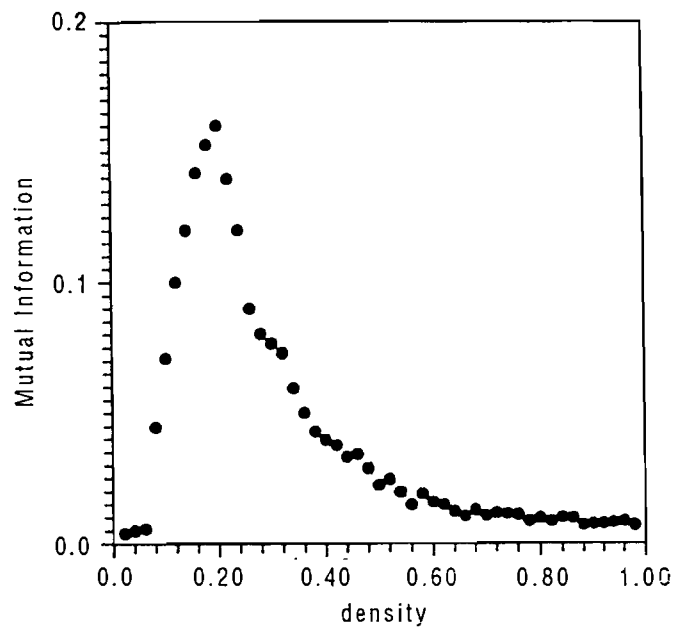## Social Insects and Swarms

Social insects are a paradigmatic example of the collective properties that might emerge from a set of simple individual entities. The appearance of social behavior is a quantum leap in evolution: the insect societies have found their place in all ecosystems [21]. In spite of the fact that social insects in isolation typically have a rather simple behavior, the colony as a whole is able to perform very complex tasks, including computation. *Insect societies share common features with neural structures like the brain:* (i) They are formed by a more or less large number of elements (neurons, say, or ants) which interact locally; (ii) Single elements display a reduced number of behavioral patterns; (iii) When we look at the system as a whole, macroscopic patterns of activity emerge as a result of local interactions among elements that are not reducible to the behavior of single units; and (iv) The dynamics of these systems is robust against noise, failure, or even the removal of single units.

The remarkable studies of Deneubourg and his colleages of the Bruxelles school [22] early showed that several patterns of activity in social insects were linked with self-organization processes. These phenomena are sometimes the result of a symmetry-breaking mechanism [22, 23] like those analyzed in the section Second-Order Transitions.

As an example of computation at the edge of chaos in ant colonies, let us consider the colony oscillations of the *Leptothorax* ants [24, 25]. These colonies show a striking pattern of global short-time periodic oscillations of activity. The number of active individuals moves up and down in a roughly periodical way. It is also known that individual ants are not periodic but chaotic [26] indicating that global behavior is an emergent property. As the density of individuals is increased (this can be done experimentally) from low to high, the coherence of these oscillations becomes more and more clear. In natural conditions, the nest density appears to be at an intermediate value where neither total order nor total disorder is allowed (see [27]).

It has been conjectured [28] that this phenomenon enables

Mutual information in a fluid neural network (see section Social Insects and Swarms), calculated for different densities of automata on a 10 x 10 lattice, and following two given individuals over 10^4 time steps. A maximum is obtained at a critical density.

ants to perform mutual exclusion (ME). ME is an important general problem of concurrent programming, an instance of which is resource allocation. We say that activity A_1 of process P_1 and activity A_2 of process P_2 must exclude each other if the execution of A_1 may not overlap the execution of A_2. In our case, mutual exclusion is attained with the combined effects of autosynchronization and the spatial arrangement of brood in the nests of *L. acervorum*, which appears to limit the number of workers that can access the brood simultaneously [28].

On the other hand, we have introduced clear evidence that the phenomena of autosynchronization is due to a state dependent external noise in the form of both a spontaneous activation and a density dependent activity propagation, giving rise to a noise induced transition [29]. Using the formal approach of fluid neural networks [27], we have shown that, if the entropy and mutual information are calculated (by averaging over time the states of pairs of individuals), maximum information is reached at a given critical density, ρ_c (Figure 8). For a biologically plausible set of parameters, we have the transition at the same $\rho_c = 0.2$; this has been observed experimentally. These facts point to the possibility of a noise-induced computational capability in the colony, for which we have coined the term *noise-induced computation* [29].

Macroevolution and Extinction

In the neo-Darwinist view of evolution, the interest in individuals as the units for selection has been shifted towards genes. In this framework, evolution is then understood as the temporal changes in gene frequencies driven by natural selection. In the

FIGURE 9
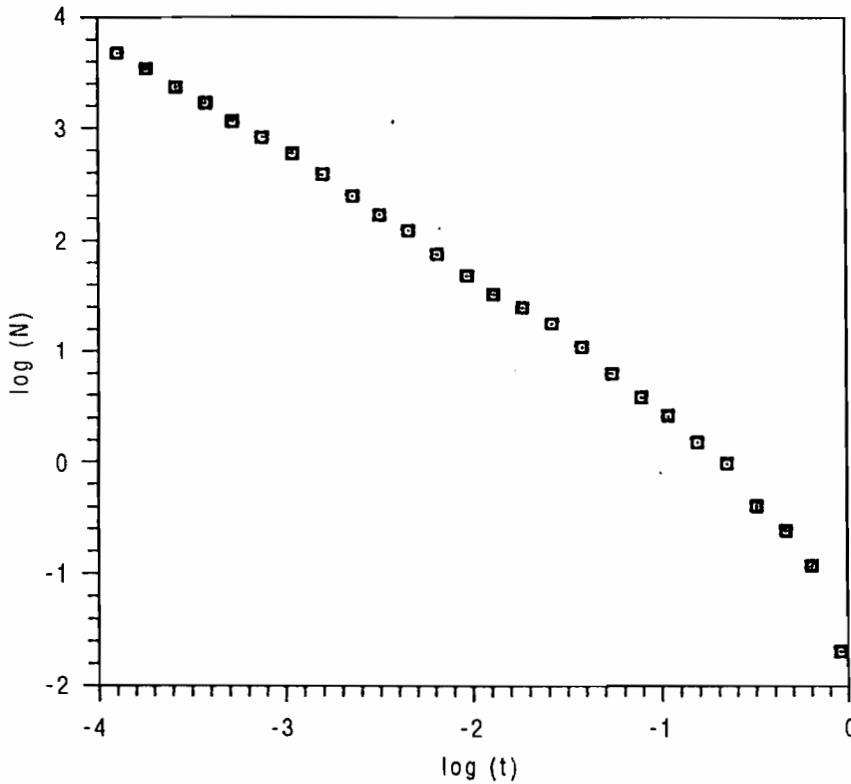
Power-law behavior of the Bak-Sneppen model of large-scale evolution. The lifetime of "evolutionary avalanches" is shown. It is conjectured that this quantity is related to the life span of species over time.

Some theoretical models have tried to shed light on this problem. One is the Kauffman-Johnsen (KJ) model based on random Boolean nets [18]. The other, simpler one is the Bak-Sneppen (BS) [33] model defined by the following set of rules: We consider a string of N random numbers. $x_i \in (0, 1)$, which are the "species" of our ecosystem. The steps are (a) we choose the minimum x_j and change it by a new random number, $x_j \in (0, 1)$; (b) the two nearest neighbors are also changed: $x_{j+1} \in (0,1), x_{j-1} \in (0, 1)$ (periodic boundaries are assumed); and (c) the previous steps are repeated.

In this model the quantities $\{x_i\}$ play the role of a "barrier height," separating the local fitness maximum from other better maxima. In other words, the barrier height is a measure of how far the i-th species is from its maximum fitness. Then if the fitness, x_j, is low, it is more likely to find nearby better states. In the BS model, the barriers are the measure of stability: smaller barriers are more unstable and easily changed. In spite of the oversimplified nature of this model, it is able to show punctuated equilibrium [34] in terms of bursts

general case where mutations and selection are considered, the evolution of gene frequencies follows a nonlinear stochastic differential equation for the gene frequency x:

$$\frac{\partial f(x,t)}{\partial t} = -\Phi_\mu(x)\frac{\partial^2 f(x,t)}{\partial^2 x} + \Psi_\beta(x)\frac{\partial f(x,t)}{\partial x}. \qquad (16)$$

Here $\Phi_\mu(x)$, $\Psi_\beta(x)$ are suitable continuous functions incorporating the specific type of mutation, selection pressures, etc. We also see diffusion and drift terms (the partial derivatives over x). Once the whole set of parameters is known, the steady-state solution (i.e., $f_s(x)$ such that $\partial_t f = 0$) gives us the final frequency distribution. Implicitly, it is assumed that these theoretical approaches can be translated to the large-scale events and so explain extinction patterns and species diversification.

Some data from the fossil record, in fact, suggest that a critical, emergent phenomenon would be behind the overall pattern of macroevolution. First, there is the frequency distribution of extinction events, which fits a power law [30]. Second, there is the existence of fractal scaling in the structure of taxonomic data [31]. More recently, the study of time fluctuations of some fossil families has shown the existence of 1/f behavior [32].

of change into the ecosystem. It also shows power law distributions of several quantities (Figure 9). These properties are, in fact, shared by such other complex systems as economics, and it has been conjectured that punctuated equilibrium is a common property of all complex systems [30].[7]
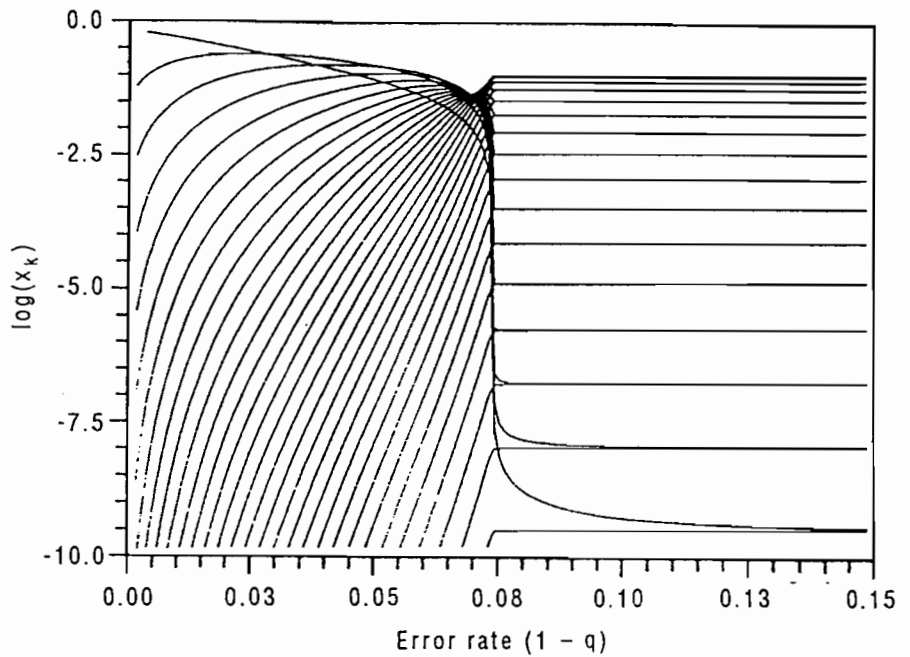
Quasi-Species and the Error Catastrophe

Consider a set of molecules (RNA chains, viruses, etc.). We indicate by I_i a given sequence (assumed to be a string of bits of length v). We can imagine the phase space of our system as a v-dimensional hypercube. Up to 2^v different molecules are possible. We assume that replication of molecules takes place together with mutations (i.e., errors). Selective pressures can be introduced in such a way that some sequences survive and others do not. The mutation rates between a given pair $\{I_i, I_k\}$ are indicated as W_{ik} (where $i, k = 1, 2,....2^v$). It can be shown [36] that the population $n_i(t)$ of the i-th sequence, I_i, will evolve in time following the dynamical system

$$\frac{dn_i(t)}{dt} = W_{ii}n_i(t) + \sum_{k \neq i} W_{ik}n_k(t) - \frac{n_i(t)}{\Sigma_k n_k(t)}\phi(t). \qquad (17)$$

where W_{ik} are the mutation rates from I_k, to I_i, and W_{ii} is the net production of I_i through (correct) self-copying. It is de-

FIGURE 10

fined as $W_{ii} = A_iQ_{ii} - D_i$, where A_i and D_i give the rate constants of total (correct and erroneous) replication and degradation, respectively. The factor Q_{ii} is the frequency of correct copying, the so-called quality factor. It can be indicated as $Q_{ii} = q^v$, q being the mutation rate per unit. Finally, the last term on the right stands for the dilution flux that removes molecules in proportion to the amount produced.

Now one of the most important consequences of this model is the appearance of a final distribution of sequences (not just one). This distribution is known as a molecular quasi-species. It is formed by a given (dominant) sequence I_m, also known as the master sequence, together with a set of mutants deriving from it. These mutants are grouped around the master sequence in such a way that often their average sequence equals that of the master. Retroviral populations are typically formed by a highly dynamical quasi-species.

The interesting point is that q

The error catastrophe. Using the model defined by eq. (17), we plot the stationary concentrations of the concentrations $x_k = n_k / \sum n_j$ $(k = 1,...,50)$. More precisely, here n_k is the frequency of those sequences which differ in k bits from the master sequence. As the error rate is increased, a critical boundary is reached where a phase transition takes place. It has been shown that, typically, RNA viruses replicate close to such error threshold.

plays the role of a temperature.[8] The nonequilibrium Darwinian organizing process has a clear parallel in the existence of order-disorder PHTs. In our case, very small q values (high fidelity, $q \to 0$) lead to a single copy (a uniform virus population), and very high error values ($q \to 1$) lead to a totally random set of molecules without any biological identity.

It can be shown that a PHT point exists for a critical fidelity ε_c (or error threshold q_c) given by [36–38]

$$\varepsilon_c = 1 - q_c = Ln(\sigma)/v. \tag{18}$$

In Figure 10, we see how ε operates on the stationary distribution of sequences for $v = 50$. We observe a sharp transition, known as the "error catastrophe," at $\varepsilon_c \approx 0.08$. At the left-hand side, the master sequence dominates, but once reached (when $\varepsilon > \varepsilon_c$), it becomes a small fraction inside a random population.

How is ε_c related to viruses? For a retrovirus, very small ε means to be very sensitive to the immune response. The cells of the immune system recognize very specific foreign molecules. Once detected, the molecules can be removed. Viruses will survive only if new mutants are generated fast enough. But obviously, ε should be less than ε_c, or the new sequences will not be functional. Where are real viruses organized? Experimental evidence shows clearly that retroviruses are typically self-organized very close to the error catastrophe [39]. In this sense,

RNA viruses explore their "RNA world" [38] by moving toward the edge of disorder. In this way, the broad spectrum of mutants makes evolutionary optimization faster. The virus population is less likely to be caught in local fitness optima.

Diversity and Rainforest Dynamics

Complex systems find one of their brightest examples in the tropical rainforest. The great variety of species, even in local areas, is legendary. Since the early days of ecological theory, the causal explanation for this phenomenon was that the species composition of such ecosystems is maintained near an equilibrium state. But other analyses soon suggested a different explanation: that high diversity is maintained only in a nonequilibrium state [40], in a state of "intermediate disturbance." Here "disturbance" means an external one. However, the system can be maintained far from equilibrium by means of its internal dynamics. A recent study [41–42] has shown that the spatial distribution of trees in a rainforest (the Barro Colorado Rainforest, in Panama) shows (multi-) fractal properties. Specifically, we analyzed the spatial distribution of those locations where the size of the canopy is less than 10 meters (usually, the canopy reaches 50–60 meters high). These low canopy points are the result of recent treefall. Treefall and gap formation are known to contribute, very importantly, to the generation and maintenance of diversity in rainforests. The frequency distribution of gap sizes,

$N(G)$, was also shown to be a power law $N(G) \approx G^{-\psi}$ with $\psi = 2.01 \pm 0.24$ (Figure 11). These results strongly suggest that the Barro Colorado (and maybe other rainforests) might be an ecosystem poised at a critical state.

With only these quantities at hand, the evidence of criticality is appealing but may not be conclusive enough. Again, theoretical models come to our aid. A CA model of forest dynamics, the *Forest Game (FG)* [41, 42] can be used to analyze the previous conjecture. The model is defined on a $L \times L$ lattice where trees of size $S_i \in [0, S_c]$ are defined. A simple set of rules is (asynchronously) applied. We take a lattice point at random and allow the following possibilities:

$$\text{(i) Birth: } 0 \xrightarrow{P_b} S_0 \quad \text{(ii) Death: } S \xrightarrow{P_d} 0 \text{ and } S_c \to 0;$$
$$\text{and (iii) Growth: } S \xrightarrow{\Phi} S + \Delta S.$$

Here P_d and P_b indicate the death and birth probabilities. When a new tree is born, it has a minimum size, $S_0 = 0.1$. The third transition introduces a critical (maximum) size, S_c. Finally, the fourth transition enables trees to grow, provided that a given condition is fulfilled. Specifically, the growth term is defined by

$$\Delta S = \Phi \left[1 - \frac{\gamma}{8} \sum_q S_q \right]$$

where $\Phi(z) = z$ if $z > 0$ and zero otherwise. γ is the strength of the interaction, and the sum is extended over the eight neighbors. An additional rule of gap creation is included: as a tree falls, a gap in the canopy is created by removing all neighbors in such a way that the total biomass removed cannot be larger than the size of the falling tree. Then we study how the dynamics is modified as γ is changed. When very small γ are used, the FG shows coherent behavior: local parts of the forest become highly synchronized. At high γ, the forest becomes frozen with a random pattern of disperse gap points. For large p_d, the forest becomes disordered. But for a wide set of intermediate interaction values, a *complex forest* is observed where *all* the observed properties of the Barro Colorado plot are reproduced, particularly, those properties linked with a system at a critical state: (a) Fractal behavior of gap points (with multi-fractal spectrum); (b) Power-law distribution of gap sizes (Figure 11) with $\psi \approx 2.0$; and (c) $1/f$-fluctuations of biomass.

The agreement of theoretical and field results suggests that rainforests are certainly in a nonequilibrium state; in fact, they are on or close to a PHT point. A direct implication of this result is that high diversity is a consequence of criticality. Again, a very simple model (we use only one "species" of tree instead of the 186 known species of trees in Barro Colorado) leads to a good description of the rainforest, which is expected when universality is involved.

SUMMARY AND PROSPECTS

We have reviewed several theoretical problems involving the existence of PHTs and critical phenomena in natural complex systems. Other studied, for example the large-scale dynamics of brain activity as an external stimulus is slowly changed [43], show clearly those properties expected in PHT phenomena. A simple

$$\psi_{BC} = 2.01 \pm 0.24$$
$$\psi_{FG} = 2.00 \pm 0.10$$
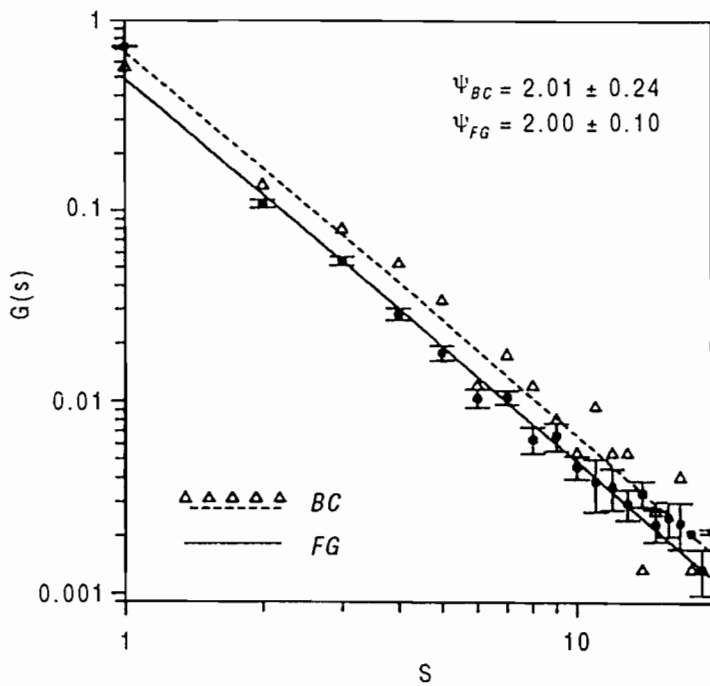
Power-law distributions of gap sizes for the Barro Colorado rainforest (dashed line) and for the forest game cellular automaton (continuous line). The last curve has been calculated at the so-called "complex forest" domain of parameter space. The agreement between these sets of data seems to suggest that the Barro Colorado rainforest would be poised at the critical state.

TABLE II

Phase transitions and complex systems (SOC: self-organized criticality)

SYSTEM	MODEL(S)	PARAMETER	PROPERTIES
Ferromagnet	Ising Model	Temperature	Fractals/Power laws
Sand pile	Sand pile CA	SOC	$1/f$-noise
Macroevolution	KJ/BS models	SOC	Extinction patterns
Rainforests	Forest Game	Interaction strength	Fractal patterns
RNA viruses	Eigen Model	Mutation rate	Error catastrophe
Ant colonies	Fluid Neural Nets	Density	Maximum I transfer
Brain Dynamics	JFHK model	Behavioral	Critical slowing down
DNA Genome	Kauffman nets	Connectivity	Antichaos

theoretical model [44] in which the main experimental results were well reproduced.

These systems, their nature, parameter dependence (if any), and implications are summarized in Table II. In some cases, the existence of a critical boundary is important as an organizing principle: the genome (and the underlying patterns of cell differentiation) or the existence of computational processes in ant colonies are only two examples. Sometimes they are linked with a mechanism of diversity generation (as in RNA viruses and rainforests), and sometimes they can introduce sudden changes in the complexity of ecosystems as when landscapes are fragmented below a critical threshold [45].

In modeling real systems, two general comments are in order. First, if complexity is linked with critical states, then a simplified model is justified, as far as it is able to reproduce relevant properties of the real system. Then there is the matter of constraints: biological systems are often limited in their structural and dynamical properties [46]. Thus, it can happen that not all points in the parameter space need to be available, and sometimes *exact* critical states might be not observed. Second, if SOC is a robust phenomenon able to drive complex systems toward the critical point, it is assumed that the final state is reached through a parameter-free process. This makes it apparently necessary that the underlying model be explicitly parameter-free. Though this can be done, a parameter-dependent model (as it is the forest game) where different types of dynamical states are observed, can nevertheless be used to detect criticality in the real world. This is achieved if the parameter-space of the theoretical model has a critical domain where a PHT takes place. If the predicted properties at this point match those observed in the real system, then the next step will be to look for a mechanism of evolution towards such boundary.

These results are only some examples of the rich variety of PHT phenomena occuring in complex systems. They show how simple, nonlinear models are able to capture the essential properties of real systems close to critical points. How general this situation is will be a matter for future research, but clearly, as Prigogine and Stengers [47] write, "the more complex a system is, the more numerous are the types of fluctuations that threaten its stability." There is little doubt that the interplay between fluctuations—amplified through nonlinearities—and stabilizing phenomena (as diffusion) find their most creative boundary at the edge of chaos.

ACKNOWLEDGMENTS

FOOTNOTES

1. Other important approaches where phase transitions can also be explored are, for example, reaction-diffusion models and coupled map lattices.
2. These are known as mean field theories.
3. Two possible $G(x)$ are used, the so-called (i) Ito ($G(x) = f(x)$) and (ii) Stratonovich ($G(x) = f(x) + [\partial_x \sigma^2(x)]/4$)
4. It can be shown that, as in the section Second-Order Transitions, a *probabilistic potential* $V(x)$ can be defined in such a way that we can write $f_s(x) = K\exp[-2V(x)/\sigma^2]$ (see [7]).
5. When $\#\Sigma > 2$, this situation can be more complicated, as shown by W. Li [11].
6. Except perhaps rule 193.
7. This observation was first reported by B. Huberman et al. in relation with economics and the so-called "computational ecologies" [35].
8. It has been shown by Leuthausser that Eigen's evolution model is equivalent to a two-dimensional Ising model [37].

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