Critical behavior of heterotic strings to all orders in string perturbation theory

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Using an explicit expression for the thermal soliton sector, we compute the would-be divergent terms of the free energy of heterotic strings when a nontrivial homology cycle in the Riemann surface is pinched. Modulo a plausible hypothesis, we find exactly the same critical temperature as in the lowest order. We also make some comments on the validity of our hypothesis. Our result is consistent with recent findings on the constant asymptotic form of the decay width for closed strings.

I. INTRODUCTION

It is well known that all string theories have mass spectra which grow exponentially. This means that the canonical equilibrium is not well defined—i.e., the canonical string partition function diverges—if the temperature exceeds a critical value T_c (cf. Ref. 1, and references therein).

The interpretation of what happens at $T=T_c$ is still somewhat unclear. The situation is formally analogous to a system of cosmic strings (at least in the noninteracting case). Numerical simulations (cf., for example, Ref. 2) have shown that, in this case, T_c represents the maximum possible temperature of the system, and when $T\to T_c^-$, the usual equivalence between microcanonical and canonical ensemble no longer holds.

What actually happens in this limit is that the mass of the Higgs boson goes to zero, so that the strings grow fat (which means that interactions are important). Numerical simulations show that most of the energy is concentrated in a single, very long string.

Another possible analogy is with the Hagedorn temperature in the context of the old dual hadronic models. With the advent of QCD, the Hagedorn temperature is interpreted as signaling a phase transition: from a confined (hadronic) phase to a deconfined phase of free quarks and gluons. This would have been very difficult to guess, though, knowing a model (such as the Hagedorn's original dual model) which is only valid for $T<T_c$. One could get an idea, for example, by noting that the divergences in the partition function disappear when one takes into account the fact that the total volume available for a hadronic bag to move is reduced by the volume occupied by the other hadrons already present (cf., for example, Ref. 3). But this calculation was done with the hindsight of QCD. One could argue that something similar is happening here, that superstrings are an effective theory valid only for $T<T_c$, which is part of a more complete one (the analogue of QCD) valid at all temperatures and existing in two different phases.

It is obvious that knowing the critical temperature for higher genus (that is, for interacting strings) would be an exceedingly useful piece of information.

It has been suggested actually (cf. Ref. 3) that interactions would cause the critical temperature to disappear. The proposed physical mechanism was quite appealing: If the high-mass states get a width which is itself a growing function of the mass (as it was the case in the model of Ref. 4),

$$\Gamma(m)\sim m^{cm} \tag{1.1}$$

(where c is a given constant), and we neglect in the computation of the canonical partition function all those states whose lifetime is smaller than the mean free time (which is itself of order β), that is, we include in the integrand of the free energy a factor $\theta(\Gamma(m)^{-1}-\beta)$, then this acts as an ultraviolet cutoff in the density of states, which means that the thermal partition function now makes sense for all values of T.

In this paper we will show this not to be the case, using an explicit expression for the thermal soliton sector (which contains all the β-dependent terms).

We find that there is a critical temperature for any genus-g contribution to the thermal free energy (in agreement with Ref. 5), but its numerical value is exactly the same as in the one-loop case. This same property for the bosonic string has recently been shown in Ref. 6.

We shall comment in the final section on how our result agrees with recent findings of other authors on the asymptotic behavior of the decay width for closed strings.

In a way, our conclusion is not surprising. After all, infrared divergences in string theory are due to tachyons propagating in handles which become very long and thin (this is what it is technically called the boundary of moduli space). It is natural that these divergences are insensitive to the rest of the Riemann surface—which is very far away from the pinched handle (in the constant curvature slice).

This result gives, however, no clues about the nature of the singularity at $T=T_c$.

II. GENUS-ONE CRITICAL BEHAVIOR REVISITED

We shall review in this section the lowest-order critical behavior for heterotic strings (cf. Ref. 1, and references therein). The result is, of course, well known, but the techniques (much fancier than needed for the particular purpose at hand) can be easily generalized to arbitrary genus.

The contribution of the thermal solitons, taking into account the phases due to quantum statistics, can be packed in a single θ function (cf. Ref. 7). This θ function has dimension $2g$ in the general case, and this means that the two-dimensional case is at genus one:

$$\theta\begin{bmatrix} 0 & 0 \\ s_2 & s_1 \end{bmatrix}(\mathbf{0}\,|\,\tilde{\Omega}) , \tag{2.1}$$

where

$$\tilde{\Omega}\equiv\Omega+\tfrac{1}{2}\mathcal{J} , \tag{2.2}$$

$$\mathcal{J}\equiv\begin{bmatrix} 0 & 1_g \\ 1_g & 0 \end{bmatrix} , \tag{2.3}$$

$$\Omega\equiv\frac{i\beta^2}{2\pi^2}\mathcal{T} , \tag{2.4}$$

$$\equiv\frac{i\beta^2}{2\pi^2}\begin{bmatrix} \tau_1\tau_2^{-1}\tau_1+\tau_2 & -\tau_1\tau_2^{-1} \\ -\tau_2^{-1}\tau_1 & \tau_2^{-1} \end{bmatrix} . \tag{2.5}$$

We remind the reader that the general definition of θ functions with characteristics $(\mathbf{a},\mathbf{b})\in(\frac{1}{2}Z/Z)^{2g}$ is

$$\theta\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}=\sum_{n\in Z^g}\exp[\pi i(\mathbf{n}+\mathbf{a})^t\Omega(\mathbf{n}+\mathbf{a}) +2\pi i(\mathbf{n}+\mathbf{a})^t(\mathbf{z}+\mathbf{b})] , \tag{2.6}$$

where $\mathbf{z}\in C^g$. This means that if the cosmological constant reads

$$\Lambda_g=\int_{\mathcal{F}_g}d\mu(m)\sum_s\Lambda_s^{(g)}(\tau,\bar{\tau}) , \tag{2.7}$$

where the summation runs over all even spin structures $\binom{\mathbf{s}_1}{\mathbf{s}_2}$ [that is, such that $\exp(4\pi i\mathbf{s}_1^t\mathbf{s}_2)=1$]; then the free energy is given by

$$F_g(\beta)=-\int_{\mathcal{F}_g}d\mu(m)\sum_s\Lambda_s^{(g)}(\tau,\bar{\tau})\theta\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{s}_2 & \mathbf{s}_1 \end{bmatrix}(\mathbf{0}\,|\,\tilde{\Omega}) . \tag{2.8}$$

At genus one, in the ordinary torus, the period matrix reduces to the single complex parameter $\tau\equiv\tau_1+i\tau_2$, where τ_1 is the twist and $\tau_2\to\infty$ in the infrared domain (with the standard choice for the fundamental region of the modular group).

The behavior of the thermal solitons when $\tau_2\to\infty$ is easily seen to be

$$\theta\begin{bmatrix} 0 \\ s_1 \end{bmatrix}\left[0\,\middle|\,\frac{i\beta^2}{2\pi^2\tau_2}\right]+2\cos(2\pi s_2)$$

$$\times\exp\left[\frac{-\beta^2}{2\pi}\tau_2\right]\theta\begin{bmatrix} 0 \\ s_1\hat{+}\frac{1}{2} \end{bmatrix}\left[\frac{\beta^2\tau_1}{2\pi^2 i\tau_2}\,\middle|\,\frac{i\beta^2}{2\pi^2\tau_2}\right] \tag{2.9}$$

($\hat{+}$ indicates the sum modulo Z), where the $g=1$ θ functions with characteristics are simply related to the ordinary elliptic Jacobi θ functions:

$$\theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}=\theta_3 , \tag{2.10}$$

$$\theta\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}=\theta_2 , \tag{2.11}$$

$$\theta\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}=\theta_4 , \tag{2.12}$$

$$\theta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}=\theta_1 . \tag{2.13}$$

The plumbing fixture parameter t is related to τ_2 by the simple expression

$$|t|=e^{-2\pi\tau_2} . \tag{2.14}$$

We know that (2.9) cannot be the whole story, because it is a nondual expression, and we know (cf. Ref. 7) that duality must hold, even in this degenerate limit. In order to get an expression manifestly dual, we can use the inversion formulas for the θ functions (cf., for instance, Ref. 8): namely,

$$\theta\begin{bmatrix} s_1 \\ 0 \end{bmatrix}\left[0\,\middle|\,\frac{-2\pi^2\tau_2}{i\beta^2}\right]=\epsilon\left[\frac{-i\beta^2}{2\pi^2\tau_2}\right]^{1/2}\theta\begin{bmatrix} 0 \\ s_1 \end{bmatrix}\left[0\,\middle|\,-\frac{i\beta^2}{2\pi^2\tau_2}\right] , \tag{2.15}$$

$$\theta\begin{bmatrix} s_1\hat{+}\frac{1}{2} \\ 0 \end{bmatrix}\left[\tau_1\,\middle|\,-\frac{2\pi^2\tau_2}{i\beta^2}\right]=\epsilon\left[\frac{-i\beta^2}{2\pi^2\tau_2}\right]^{1/2}\theta\begin{bmatrix} 0 \\ s_1\hat{+}\frac{1}{2} \end{bmatrix}\left[\frac{\beta^2\tau_1}{2\pi^2 i\tau_2}\,\middle|\,-\frac{i\beta^2}{2\pi^2\tau_2}\right]\exp\left[\frac{-\beta^2\tau_1^2}{2\pi\tau_2}\right] \tag{2.16}$$

(with $\epsilon^8=1$). In this way we obtain the final expression for the limit of the thermal soliton sector:

$$\lim_{\tau_2\to\infty}\theta=\left[\frac{2\pi^2 i\tau_2}{\beta^2}\right]^{1/2}\left[\exp\left[\frac{-2\pi^3 s_1^2\tau_2}{\beta^2}\right]+\exp\left[\frac{-2\pi^3(s_1+1)^2\tau_2}{\beta^2}\right]\right.$$
$$+\exp\left[\frac{-2\pi^3(s_1-1)^2\tau_2}{\beta^2}\right]+2\cos(2\pi s_2)\exp\left[\frac{-\beta^2}{2\pi}\tau_2+\frac{\beta^2\tau_1^2}{2\pi\tau_2}\right]$$
$$\times\left\{\exp\left[\frac{-2\pi^3}{\beta^2}\left[s_1\hat{+}\frac{1}{2}\right]^2\tau_2+2\pi i\tau_1\left[s_1\hat{+}\frac{1}{2}\right]\right]\right.$$
$$+\exp\left\{\frac{-2\pi^3}{\beta^2}\left[1+\left[s_1\hat{+}\frac{1}{2}\right]\right]^2\tau_2+2\pi i\tau_1\left[1+\left[s_1\hat{+}\frac{1}{2}\right]\right]\right\}$$
$$\left.\left.+\exp\left\{\frac{-2\pi^3}{\beta^2}\left[-1+\left[s_1\hat{+}\frac{1}{2}\right]\right]^2\tau_2+2\pi i\tau_1\left[-1+\left[s_1\hat{+}\frac{1}{2}\right]\right]\right\}\right\}\right] . \tag{2.17}$$

On the other hand, it is very easy to get the limit of the integrand of the cosmological constant $\Lambda_s(\tau,\bar{\tau})$:

$$\lim_{\tau_2\to\infty}\Lambda_s(\tau,\bar{\tau})=\tau_2^{-4}\exp(i\pi\tau_1+3\pi\tau_2)\sum_{\text{even } s}\exp[2\pi i(s_1+s_2+\tfrac{1}{2})]\exp(4\pi i\tau s_1^2)$$
$$\times\{1+8e^{\pi i\tau}\cos[2\pi(s_1\tau+s_2)]+24e^{2\pi i\tau}\cos^2[2\pi(s_1\tau+s_2)]$$
$$+32e^{3\pi i\tau}\cos^3[2\pi(s_1\tau+s_2)]+16e^{4\pi i\tau}\cos^4[2\pi(s_1\tau+s_2)]\} . \tag{2.18}$$

All the contributions from $s_1=\frac{1}{2}$ cancel out, owing to the integral over the twist, except one, which has a divergence of the type

$$\exp\left[2\pi\tau_2-\frac{\beta^2}{2\pi}\tau_2-\frac{2\pi^3}{\beta^2}\tau_2\right] . \tag{2.19}$$

This term has bosonic duality (that is, it is invariant when $\beta\to\beta^*\equiv 2\pi^2/\beta$). It converges if $2\pi\le\beta^2/2\pi+2\pi^3/\beta^2$, a condition which is satisfied at all temperatures.

The contributions from $s_1=0$ are more complicated. There are nine nontrivial terms to consider. Three of them give zero after integrating over the twist. Another five cancel out because of the one-loop Gliozzi-Scherk-Olive (GSO) projection factor $\exp[2\pi(s_1+s_2+\frac{1}{2})]$, and finally, there is a single term left which diverges as

$$\exp\left[\left[3\pi-\frac{\pi^3}{2\beta^2}-\frac{\beta^2}{2\pi}\right]\tau_2\right] , \tag{2.20}$$

which has heterotic duality $\beta\to\beta^*\equiv\pi^2/\beta$ and converges if $\beta>\beta_c\equiv\pi(\sqrt{2}+1)$ (or $\beta<\beta_c^*$). This is the standard result for the critical behavior of the heterotic string.

III. BEHAVIOR OF THE THERMAL SOLITONS AT ARBITRARY GENUS

The $2g\times 2g$ matrix $\tilde{\Omega}$ appearing in the contribution to the free energy coming from genus g is given by

$$\tilde{\Omega}=\frac{i\beta^2}{2\pi^2}\mathcal{T}+\frac{1}{2}\mathcal{I} . \tag{3.1}$$

The first factor $\Omega\equiv(i\beta^2/2\pi^2)\mathcal{T}$ has been already studied with some detail in Ref. 6. The period matrix of the Riemann surface degenerates (cf. Ref. 9) in the form

$$2\pi\tau=\begin{bmatrix} a-i(\alpha+\ln|t|) & c^t+i\gamma^t \\ c+i\gamma & d+i\delta\end{bmatrix} , \tag{3.2}$$

where $a,\alpha\in R$; $c,\gamma\in R^{g-1}$; and δ and d are both real square matrices of $g-1$ dimensions. All of them, moreover, are of order 1 with respect to the plumbing fixture coordinate t $(\to 0)$.

When computing $\mathcal{T}$, there is a single divergent element: namely,

$$\mathcal{T}_{11}=-\frac{\ln|t|}{2\pi} . \tag{3.3}$$

This means that the limit of the big (order $2g$) θ function is given by

$$\lim_{t\to 0}\theta\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{s}_2 & \mathbf{s}_1\end{bmatrix}(\mathbf{0}|\,\tilde{\Omega})$$
$$=\theta\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{s}_2' & \mathbf{s}_1\end{bmatrix}(\mathbf{0}|\,\tilde{\Omega}')+|t|^{\beta^2/4\pi^2}2\cos(2\pi s_2^{(1)})$$
$$\times\theta\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{s}_2' & \mathbf{s}_1\hat{+}\frac{1}{2}\end{bmatrix}(\mathbf{z}|\,\tilde{\Omega}') , \tag{3.4}$$

where $\tilde{\Omega}'$ means the $(2g-1)\times(2g-1)$ matrix obtained after deleting the divergent $\tilde{\Omega}_{11}$ element and $s_1\hat{+}\frac{1}{2}$ means $(s_1^{(1)},\ldots,s_1^{(g)})\hat{+}(\frac{1}{2},0,\ldots,0)$, where, as usual, $\hat{+}$ is the sum modulo Z. The primes on the spin structures mean that the first component has been deleted; i.e., $\mathbf{s}'\equiv(s^{(2)},\ldots,s^{(g)})$.

The argument $\mathbf{z}$ appearing in the second term of the second member of (3.1) is most interesting. As we shall see in detail in the sequel, it will eventually lead to the twist, which will in turn lead after integration to the cancellation of several terms that would otherwise be divergent. Its actual value is

$$\mathbf{z}\equiv\frac{i\beta^2}{2\pi^2}[\mathcal{T}_{1i}(i>1)]\in C^{2g-1}\ . \tag{3.5}$$

Our formula in (3.1) has no manifest heterotic duality. This means that we should perform an inversion on the θ's before getting this symmetry in an explicit form. The simplest inversion formula (cf., for example, Ref. 8) to be applied to the first term in the sum is (3.1) gives

$$\theta\begin{bmatrix}-\mathbf{s}_2' & \mathbf{s}_1\\ 0 & 0\end{bmatrix}(0|\,|-\tilde{\Omega}'^{-1})=\epsilon(\det\tilde{\Omega}')^{1/2}\theta\begin{bmatrix}0 & 0\\ \mathbf{s}_2' & \mathbf{s}_1\end{bmatrix}(0|\,|\tilde{\Omega}')\ , \tag{3.6}$$

where $\epsilon^8=1$. We shall ignore systematically in the sequel all these phases, which must at any state cancel among themselves if the total result is to be free of global anomalies.

When considering the contributions from $\tilde{\Omega}'^{-1}$, it is necessary to realize that the imaginary part of the period matrix has the structure

$$\tau_2^{-1}=\begin{bmatrix} O\begin{bmatrix}\dfrac{1}{\ln|t|}\end{bmatrix} & O\begin{bmatrix}\dfrac{1}{\ln|t|}\end{bmatrix}\\ O\begin{bmatrix}\dfrac{1}{\ln|t|}\end{bmatrix} & \delta^{-1}\end{bmatrix}\ . \tag{3.7}$$

This fact will determine, when inverting $\mathcal{T}$ by boxes, the appearance of a single divergent element in the matrix, namely,

$$(\mathcal{T}'^{-1})_{gg}=-\frac{\ln|t|}{2\pi}\ . \tag{3.8}$$

It can be checked easily that the modifications caused by the addition of the matrix $\frac{1}{2}\mathcal{I}$ do not change this leading, divergence, meaning that

$$(\tilde{\Omega}'^{-1})_{gg}=\frac{\pi i\ln|t|}{\beta^2}\ . \tag{3.9}$$

This implies that the $t\to 0$ limit of the first term, given by (3.1), will be

$$(\det\tilde{\Omega}')^{-1/2}\left[\theta\begin{bmatrix}-\mathbf{s}_2' & \mathbf{s}_1'\\ 0 & 0\end{bmatrix}(0|\,|-\tilde{\Omega}''^{-1})|t|^{(\pi^2/\beta^2)(s_1^{(1)})^2}+\theta\begin{bmatrix}-\mathbf{s}_2' & \mathbf{s}_1'\\ 0 & 0\end{bmatrix}(\mathbf{z}_+|\,|-\tilde{\Omega}''^{-1})|t|^{(\pi^2/\beta^2)(1-s_1^{(1)})^2}\right.$$
$$\left.+\theta\begin{bmatrix}-\mathbf{s}_2' & \mathbf{s}_1'\\ 0 & 0\end{bmatrix}(\mathbf{z}_-|\,|-\tilde{\Omega}''^{-1})|t|^{(\pi^2/\beta^2)(1+s_1^{(1)})^2}\right]\ , \tag{3.10}$$

where $\tilde{\Omega}''$ means that we have deleted both the $\tilde{\Omega}_{11}$ and $\tilde{\Omega}_{gg}$ elements. On the other hand, the arguments $\mathbf{z}_\pm\in C^{2g-2}$ are given by

$$\begin{aligned}\mathbf{z}_+&=\{-\tfrac{1}{2}(1-s_1^{(1)})(\Omega'^{-1})_{gi}\quad(i\neq 1,g)\}\ ,\\ \mathbf{z}_-&=\{\tfrac{1}{2}(1+s_1^{(1)})(\Omega'^{-1})_{gi}\quad(i\neq 1,g)\}\ .\end{aligned} \tag{3.11}$$

In the second term, in the sum on the second member of (3.1), we shall use the formula

$$\theta\begin{bmatrix}0 & 0\\ \mathbf{s}_2' & \mathbf{s}_1\hat{\mp}\frac{1}{2}\end{bmatrix}(\pm\mathbf{z}|\,|\tilde{\Omega}')=(\det\tilde{\Omega}')^{-1/2}e^{-i\pi\mathbf{z}^t\tilde{\Omega}'^{-1}\mathbf{z}}\theta\begin{bmatrix}-\mathbf{s}_2' & -\mathbf{s}_1\hat{\mp}\frac{1}{2}\\ 0 & 0\end{bmatrix}(\pm\tilde{\Omega}'^{-1}\mathbf{z}|\,|-\tilde{\Omega}'^{-1})\ . \tag{3.12}$$

Actually, we know explicitly what $\mathbf{z}$ is; its components are given by

$$\mathbf{z}=\frac{i\beta^2}{2\pi^2}\left[\frac{1}{2\pi}(c^t\delta^{-1}d+\gamma^t)_{1i},\frac{1}{\ln|t|}[a-(c^t\delta^{-1}\gamma)_{11}],-(c^t\delta^{-1})_{1j}\right]\ , \tag{3.13}$$

where $i,j\neq 1$.

Using this expression, one can easily check that the component number g of $\tilde{\Omega}'^{-1}\mathbf{z}$ is given by

$$2\pi i(\tilde{\Omega}'^{-1}\mathbf{z})_g=-i[a-(c^t\delta^{-1}\gamma)_{11}]\equiv-2\pi i\varphi\ , \tag{3.14}$$

where φ is the twist in the plumbing fixture, the quantity which plays the role of τ_1 in one-loop calculations. On the other hand, the minus signs in the characteristics and in the argument of the θ function are irrelevant, because it can be easily shown that

$$\theta\begin{bmatrix}\mathbf{r}_1\\ \mathbf{r}_2\end{bmatrix}(\mathbf{z}|\,|\Gamma)=\theta\begin{bmatrix}\mathbf{r}_1\\ \mathbf{r}_2\end{bmatrix}(-\mathbf{z}|\,|\Gamma)\ , \tag{3.15}$$

for even characteristics, as is always the case here ($\mathbf{r}_2=0$ always in the thermal θ functions), and

$$\theta\begin{bmatrix}-\mathbf{r}_1\\ \mathbf{r}_2\end{bmatrix}(\mathbf{z}|\,|\Gamma)=\theta\begin{bmatrix}\mathbf{r}_1\\ -\mathbf{r}_2\end{bmatrix}(-\mathbf{z}|\,|\Gamma)\ . \tag{3.16}$$

Performing now the $t \to 0$ limit, and grouping again together the different terms, we get

$$\begin{aligned}
\lim_{t\to 0} \theta_{2g} \times (\det\tilde{\Omega}')^{1/2} = {} & |t|^{(\pi^2/\beta^2)(s_1^{(1)})^2} \theta \begin{bmatrix} -\mathbf{s}_2' & \mathbf{s}_1' \\ 0 & 0 \end{bmatrix} (\mathbf{0} \,|\, -\underline{\tilde{\Omega}'^{-1}}) + |t|^{(\pi^2/\beta^2)(1-s_1^{(1)})^2} \theta \begin{bmatrix} -\mathbf{s}_2' & \mathbf{s}_1' \\ 0 & 0 \end{bmatrix} (\mathbf{z}_+ \,|\, -\underline{\tilde{\Omega}'^{-1}}) \\
& + |t|^{(\pi^2/\beta^2)(1+s_1^{(1)})^2} \theta \begin{bmatrix} -\mathbf{s}_2' & \mathbf{s}_1' \\ 0 & 0 \end{bmatrix} (\mathbf{z}_- \,|\, -\underline{\tilde{\Omega}'^{-1}}) + |t|^{(\beta^2/4\pi^2)} 2\cos(2\pi s_2^{(1)}) e^{i\pi \mathbf{z}^t \tilde{\Omega}'^{-1}\mathbf{z}} \\
& \times \Bigg[|t|^{(\pi^2/\beta^2)(1-s_1^{(1)})^2} e^{2\pi(1/2-s_1^{(1)})i\varphi} \theta \begin{bmatrix} -\mathbf{s}_2' & -\mathbf{s}_1' \\ 0 & 0 \end{bmatrix} (\underline{\tilde{\Omega}'^{-1}\mathbf{z}} \,|\, -\underline{\tilde{\Omega}'^{-1}}) \\
& + |t|^{(\pi^2/\beta^2)(1+s_1^{(1)})^2} e^{-2\pi(1/2+s_1^{(1)})i\varphi} \theta \begin{bmatrix} -\mathbf{s}_2' & -\mathbf{s}_1' \\ 0 & 0 \end{bmatrix} (\underline{\tilde{\Omega}'^{-1}\mathbf{z}} + \Delta_+\mathbf{z} \,|\, -\underline{\tilde{\Omega}'^{-1}}) \\
& + |t|^{(\pi^2/\beta^2)(3/2-s_1^{(1)})^2} e^{2\pi(3/2-s_1^{(1)})i\varphi} \theta \begin{bmatrix} -\mathbf{s}_2' & -\mathbf{s}_1' \\ 0 & 0 \end{bmatrix} (\underline{\tilde{\Omega}'^{-1}\mathbf{z}} + \Delta_-\mathbf{z} \,|\, -\underline{\tilde{\Omega}'^{-1}}) \Bigg] , \qquad (3.17)
\end{aligned}$$

where a bar below a vector means that the g component has been deleted, and below a matrix means that the correspondent element (gg) has been deleted as well. On the other hand,

$$\begin{aligned}
\Delta_+\mathbf{z} &\equiv \{ \tfrac{1}{2}(\tfrac{1}{2}+s_1^{(1)})\tilde{\Omega}'^{-1}_{ig}, \quad i=2,\ldots,2g,; i\neq g \} , \\
\Delta_-\mathbf{z} &\equiv \{ \tfrac{1}{2}(\tfrac{3}{2}-s_1^{(1)})\tilde{\Omega}'^{-1}_{ig}, \quad i=2,\ldots,2g,; i\neq g \} .
\end{aligned} \qquad (3.18)$$

It can be checked that this behavior is very similar to the one we got at genus one in (2.17), once the identification $|t| = e^{-2\pi\tau_2}$ is made.

IV. BEHAVIOR OF THE INTEGRAND OF THE COSMOLOGICAL CONSTANT AT ARBITRARY GENUS

Before we can impose convergence on the total free energy given in (2.7) in the limit $t \to 0$, we need to know exactly what tachyonic divergences the temperature is supposed to tame. For the bosonic string, there is a general theorem, due to Belavin and Knizhnik,[10] which states that the integrand has a quartic pole:

$$\lim_{t\to 0} \Lambda_B \frac{d^2\tau}{\tau_2^2} = \frac{dt \wedge \overline{dt}}{|t|^4} . \qquad (4.1)$$

Unfortunately, no general theorem of this sort is available for heterotic strings (cf., for example, Ref. 11 for a general review). The main difficulty stems from the existence of different spin structures and the fact that there are cancellations among them which are controlled by the genus-g GSO projection.

What we have in (2.8) for the free energy is in fact an expression defined on the spin covering of the moduli space of ordinary Riemann surfaces. This finite spin covering is achieved by integrating out the anticommuting supermoduli. The GSO projection or, what is the same, the sum over different spin structures is only achieved after integrating over the odd coordinates. The problem is that this integration is not free of ambiguities (cf. Ref. 12). In other words, we do not know whether or not the supermanifold is split. It is interesting to remark that even if the supermanifold were split there may be problems for the compactified supermoduli which actually would be the interesting space for our purposes.

Perhaps the appropriate procedure to address the problem with the GSO projection could be to sum over the spin structures before integrating out the odd variables. This would mean that instead of (2.8) we would have to know the expression for the free energy as an integral over the supermoduli space. The operator formalism[13] could be a useful tool for this purpose.

In the absence of exact results, we can make a plausible hypothesis on the behavior of $\Lambda_{\text{heterotic}}$. It should be kept in mind, however, that this is only a hypothesis: It would not be the first time where a plausible assumption has been disproved later on by a detailed calculation in string theory.

Actually, pinching a handle is equivalent (in the constant curvature slice) to letting it become very long and thin. It would be only natural to assume, then, that the integrand degenerates into the one-loop part [i.e., Eq. (2.18)] times some other terms which depend only on the period matrix $\tau_{g-1} \equiv d + i\delta$ of the genus $g-1$ Riemann surface with two points identified between them which is the degeneration limit of the genus-g Riemann surface.

To be explicit, this assumption would be

$$\lim_{t\to 0}\Lambda^{(\mathbf{s}_1,\mathbf{s}_2)}_{\text{heterotic}}=|t|^{-3/2}e^{i\pi\varphi[1+4(s_1^{(1)})^2]}|t|^{2(s_1^{(1)})^2}e^{2\pi i(s_1^{(1)}+s_2^{(2)}+1/2)}\{1+8|t|^{1/2}e^{i\pi\varphi}\cos[2\pi(s_1^{(1)}\tau_{11}+s_2^{(1)})]$$
$$+24|t|e^{2\pi i\varphi}\cos^2[2\pi(s_1^{(1)}\tau_{11}+s_2^{(1)})]+32|t|^{3/2}e^{3\pi i\varphi}\cos^3[2\pi(s_1^{(1)}\tau_{11}+s_2^{(1)})]$$
$$+16|t|^2e^{4\pi i\varphi}\cos^4[2\pi(s_1^{(1)}\tau_{11}+s_2^{(1)})]\}\times\Lambda^{\mathbf{s}_1,\mathbf{s}_2}_{s_1^{(1)}}(\tau_{g-1})\ . \tag{4.2}$$

Note that we assume, first of all, that the signs of the t-dependent terms are the same as they are at one loop (including the one-loop GSO projection factor $e^{2\pi i(s_1^{(1)}+s_2^{(1)}+1/2)}$), and moreover, we assume that the contribution of the rest of the Riemann surface does not depend on $s_2^{(1)}$; that is, it depends only on whether the states propagating along the pinched handle are Neveu-Schwarz or Ramond.

Now let us check our hypothesis in relation to the class of gauges in which the cancellation of the cosmological constant (at least for the matter current part) comes from a Riemann identity ($\mathcal{RJ}$) related to a given odd spin structure at genus g, $({\alpha_0 \atop \beta_0})$: namely,

$$\mathcal{RJ}_g\begin{bmatrix}\alpha_0\\ \beta_0\end{bmatrix}\equiv\sum_{({\alpha \atop \beta})\text{ even}}e^{4\pi i(\alpha^t\beta_0-\beta^t\alpha_0)}\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}^4(0|\,|\tau)$$
$$=0\ . \tag{4.3}$$

These sorts of gauges can be seen as incorporating the GSO projection in the same way as in the one-loop case in which Eq. (4.3) reduces to the Jacobi equation. Nevertheless, contrary to the one-loop case in which there is only one odd spin structure which is modular invariant, at genus g there are $2^{g-1}(2^g-1)$ different odd spin structures which transform between them under the action of the modular group. Then the price we have to pay is the loss of modular invariance (we only have invariance under the subgroup of the modular group which preserves our choice of odd spin structure).

It has been seen in Ref. 14 that an explicit calculation at genus two of the critical temperature using this gauge gives different results depending on which handle is shrunk. This can be understood if we realize that the odd spin structure that we choose gives different "one-loop GSO projections" for each handle (there is at least one handle in which we obtain the correct one-loop GSO projection). This result could not be true if we had full modular invariance because the order of the leading term in this limit must be modular invariant.

These gauges are closely related to the picture-changing formalism and the mentioned ambiguity.[12,13] The picture-changing-operator (PCO) insertions represent different elections of odd coordinates, and depending on our election, the odd part of the integrand of the cosmological constant is projected in different ways after integrating over the odd moduli.

We could also try to do finite-temperature calculations at arbitrary genus using the light-cone gauge (cf. Ref. 15, and references therein). Following Ref. 16, we can obtain the transverse partition function from a $g-1$ Mandelstam diagram with only one incoming and one outgoing string, and periodic boundary conditions on the Euclidean time. This leads us with a genus-g vacuum-to-vacuum amplitude and only a solitonic sector in the temporal direction. The advantage is that the PCO insertions as well as the GSO projection are completely determined, paying now the price of a harder loss of modular invariance (perhaps it could be recovered using the techniques developed in Ref. 17). To obtain the limit in another handle, we would have to identify it with Euclidean time and calculate a "different" diagram changing the PCO insertions and GSO projection at the same time.

We would like to note that our hypothesis includes a concrete assumption on how the unknown GSO projection would degenerate in this boundary of the spin moduli space.

Let us now find the consequences of our hypothesis. Accepting the assumed behavior of the integrand of the cosmological constant, it is now very easy to check the values of β which make the leading term of the free energy convergent.

The Ramond sector does not contribute because of the integral over the twist. The Neveu-Schwarz sector leads to a bunch of different terms. Three of them disappear because of the twist; four others because of the one-loop GSO projection, and, finally, there are two temperature-dependent conditions: one with the bosonic duality

$$2\pi\le\frac{\beta^2}{2\pi}+\frac{2\pi^3}{\beta^2}\ , \tag{4.4}$$

which does not impose any restriction on the allowed temperatures, and the other with the heterotic duality

$$3\pi\le\frac{\beta^2}{2\pi}+\frac{\pi^3}{2\beta^2}\ , \tag{4.5}$$

which is exactly the same we got for genus one.

V. CONCLUSIONS

We have obtained (modulo a plausible hypothesis) the result that string interactions do not modify the critical temperature for heterotic strings. In a way, this result is not surprising. After all, the divergence is a local effect in the boundary of moduli space. The physical implication of it is, reversing the argument given in the Introduction, that the high mass does not get enough width as to be cut off from the ones contributing to the free energy.

The fact that there exists a critical temperature at an arbitrary genus has been previously realized in Refs. 5 and 18.

Let us now reexamine carefully the argument we gave in the Introduction on the step functions on the width of the states. Actually, we should smear out somewhat the Heaviside function, because there is a certain probability, given by $e^{-\beta\Gamma(m)}$, for a state with decay width $\Gamma(m)$ to live longer than the mean free time β. This means that if we assume $\Gamma(m)=g^2m^\gamma$, the figure $\gamma=1$ is critical: For $\gamma>1$ there is no critical temperature; for $\gamma<1$ the critical temperature is not modified by interactions; and $\gamma=1$ is marginal: The critical temperature is modified by numerical factors.

Recent numerical calculations[19] of the decay width for closed strings show that, actually, the asymptotic behavior of $\Gamma(m)$ corresponds to $\gamma=-1$, in agreement with our results. Incidentally, the same calculation for open strings[20] shows that $\gamma=1$, implying that one expects modifications in the numerical values of the critical temperature in this case.

We would like to stress that our result is also consistent with the general belief on the "softness" of string interactions at very high energies (cf., for example, Ref. 21).

On the other hand, it has been recently remarked (cf. Ref. 22) that if the integrand of the free energy is properly regularized, and then an analytic continuation is performed, then the divergency for $\beta<\beta_c$ disappears and, instead, $\mathrm{Im}F(\beta)\neq 0$.

The interest of these calculations stems from the fact that it has been claimed in Ref. 23 that they reproduce the imaginary parts consistent with unitarity, a result which, if true, would be most interesting.

The process of analytic continuation is not without problems, however: When the calculation is performed for noncritical dimensions, the Liouville mode is not taken into account; the other procedure of inserting external momenta, in addition to being nonmodular invariant, does not guarantee decoupling of unwanted spins. More than that, the statement equivalent to unitarity for the free energy is that it should be real for β real. Were this not the case, the only possible interpretation is that the Hamiltonian is not self-adjoint, perhaps because in this region it is only an approximation to the true Hamiltonian.

It is worth remarking that we did not find any signal of Jeans instability, which manifests itself in quantum gravity at finite temperature (cf. Ref. 24 and also Ref. 25) as a negative contribution to the mass squared for the graviton proportional to GT^4.

Actually, we did not find this contribution for the bosonic string either (cf. Ref. 6), in which case the behavior of the integrand is given exactly by the Belavin-Knizhnik theorem, and there is no need of introducing our "plausible hypothesis," so that our calculation is an exact one.

What happens is that it is not easy to find Jeans instability by examining the free energy only. If we had computed instead the thermal graviton propagator, we should have the corresponding negative contribution, as in ordinary quantum gravity (with the caveat that the analytic continuation to Minkowski space is somewhat problematic; cf. the work of Moore in Ref. 26, where he tried to find directly the widths as imaginary parts of the mass corrections from the pinching of a nontrivial cycle).

But the instability appears in the free energy only after resummation of the "ring diagrams," in the form of an imaginary part for the logarithms which appear as a byproduct of the resummation itself (cf. Ref. 17).

It is true, nevertheless, that any amplitude at order g in string perturbation theory contains information on all the other amplitudes at order $g-1$, by using the degeneracy limit on a nontrivial cycle. In this way, the thermal graviton propagator at $g-1$ loops can be computed from the knowledge of the free energy at order g by taking the corresponding limit, making an expansion in powers of the "plumbing fixture" parameter t, and taking the appropriate projection on a t-independent part. It is this t-independent part in the degeneracy limit which should contain the Jeans instability. Our techniques only allow for an elimination of the dominant t-dependent divergences. The issue of the finiteness of the coefficients of the expansion in t, after integration over the moduli space, although possible in principle, is outside the scope of the present work.

Note added in proof. One of us (T.O.) has been trying to find an expression for $\Lambda^{s_1 s_2}_{\text{heterotic}}$ satisfying our hypothesis in an explicit modular-invariant way. An expression with these properties exists and, furthermore, it vanishes identically before integration over the moduli space. All that will be explained in more detail in a forthcoming paper.

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