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From loops to trees by-passing Feynman's theorem

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Abstract

We derive a duality relation between one-loop integrals and phase-space integrals emerging from them through single cuts. The duality relation is realized by a modification of the customary $+i0$ prescription of the Feynman propagators. The new prescription regularizing the propagators, which we write in a Lorentz covariant form, compensates for the absence of multiple-cut contributions that appear in the Feynman Tree Theorem. The duality relation can be applied to generic one-loop quantities in any relativistic, local and unitary field theories. It is suitable for applications to the analytical calculation of one-loop scattering amplitudes, and to the numerical evaluation of cross-sections at next-to-leading order.

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1 Introduction

The Feynman Tree Theorem (FTT) [1, 2] applies to any (local and unitary) quantum field theories in Minkowsky space with an arbitrary number d of space-time dimensions. It relates perturbative scattering amplitudes and Green's functions at the loop level with analogous quantities at the tree level. This relation follows from a basic and more elementary relation between loop integrals and phase-space integrals. Using this basic relation loop Feynman diagrams can be rewritten in terms of phase-space integrals of tree-level Feynman diagrams. The corresponding tree-level Feynman diagrams are then obtained by considering *multiple* cuts (single cuts, double cuts, triple cuts and so forth) of the original loop Feynman diagram.

We have recently proposed a method [3, 4, 5] to numerically compute multi-leg one-loop cross sections in perturbative field theories. The starting point of this method is a *duality* relation between one-loop integrals and phase-space integrals. Although the analogy with the FTT is quite close, there are important differences. The key difference is that the duality relation involves only *single* cuts of the one-loop Feynman diagrams. Both the FTT and the duality relation can be derived by using the residue theorem*.

In this paper, we illustrate and derive the duality relation. Since the FTT has recently attracted a renewed interest [6] in the context of twistor-inspired methods [7, 8] to evaluate one-loop scattering amplitudes [9], we also discuss its correspondence (including similarities and differences) with the duality relation.

The outline of the paper is as follows. In Section 2, we introduce our notation. In Section 3, we briefly recall how the FTT relates one-loop integrals with multiple-cut phase-space integrals. In Section 4, we present one of the main results of this publication: we derive and illustrate the duality relation between one-loop integrals and single-cut phase-space integrals. We also prove that the duality relation requires to properly regularize propagators by a complex Lorentz-covariant prescription, which is different from the customary $+i0$ prescription of the Feynman propagators. The duality is illustrated in Section 5 by considering the two-point function as the simplest example application. The correspondence between the FTT and the duality relation is formalized in Section 6. In Section 7, we explore the one-to-one correspondence between one-loop Feynman integrals and single-cut integrals on more mathematical grounds, and establish a generalized duality relation. The treatment of particle masses (including complex masses of unstable particles) when cutting loop integrals is discussed in Section 8. In Section 9, we analyze the effect of the gauge poles introduced by the propagators of the gauge fields in local gauge theories. In Section 10, we discuss the extension of the duality relation to one-loop Green's functions and scattering amplitudes. Some final remarks are presented in Section 11. Details about the derivation of the duality relation by using the residue theorem are discussed in Appendix A. The proof of an algebraic relation is presented in Appendix B.

*Within the context of loop integrals, the use of the residue theorem has been considered many times in textbooks and in the literature.

2 Notation

The FTT and the duality relation can be illustrated with no loss of generality by considering their application to the basic ingredient of any one-loop Feynman diagrams, namely a generic one-loop scalar integral $L^{(N)}$ with N ($N \geq 2$) external legs.

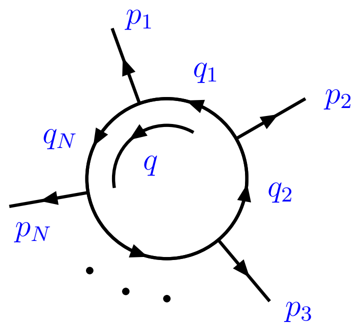

Figure 1: *Momentum configuration of the one-loop N-point scalar integral.*

The momenta of the external legs are denoted by $p_1^\mu, p_2^\mu, \dots, p_N^\mu$ and are clockwise ordered (Fig. 1). All are taken as outgoing. To simplify the notation and the presentation, we also limit ourselves in the beginning to considering massless internal lines only. Thus, the one-loop integral $L^{(N)}$ can in general be expressed as:

$$L^{(N)}(p_1, p_2, \dots, p_N) = -i \int \frac{d^d q}{(2\pi)^d} \prod_{i=1}^{N} \frac{1}{q_i^2 + i0} \quad , \tag{1}$$

where q^μ is the loop momentum (which flows anti-clockwise). The momenta of the internal lines are denoted by q_i^μ; they are given by

$$q_i = q + \sum_{k=1}^{i} p_k \quad , \tag{2}$$

and momentum conservation results in the constraint

$$\sum_{i=1}^{N} p_i = 0 \quad . \tag{3}$$

The value of the label i of the external momenta is defined modulo N, i.e. $p_{N+i} \equiv p_i$.

The number of space-time dimensions is denoted by d (the convention for the Lorentz-indices adopted here is $\mu = 0, 1, \dots, d-1$) with metric tensor $g^{\mu\nu} = \text{diag}(+1, -1, \dots, -1)$. Note that d does not necessarily have integer value, but it must satisfy $d \geq 1$ (as in the case of loop integrals in dimensional regularization). The space-time coordinates of any momentum k_μ are denoted as $k_\mu = (k_0, \mathbf{k})$, where k_0 is the energy (time component) of k_μ. It is also convenient to introduce light-cone coordinates $k_\mu = (k_+, \mathbf{k}_\perp, k_-)$, where $k_\pm = (k_0 \pm k_{d-1})/\sqrt{2}$.

Throughout the paper we use the following shorthand notation:

$$-i \int \frac{d^d q}{(2\pi)^d} \; \cdots \equiv \int_q \; \cdots \quad . \tag{4}$$

When we factorize off in a loop integral the integration over the momentum coordinate q_0 or q_+, we write

$$-i \int_{-\infty}^{+\infty} dq_0 \int \frac{d^{d-1}\mathbf{q}}{(2\pi)^d} \cdots \equiv \int dq_0 \int_{\mathbf{q}} \cdots \ , \tag{5}$$

and

$$-i \int_{-\infty}^{+\infty} dq_+ \int_{-\infty}^{+\infty} dq_- \int \frac{d^{d-2}\mathbf{q}_\perp}{(2\pi)^d} \cdots \equiv \int dq_+ \int_{(q_-,\mathbf{q}_\perp)} \cdots \ , \tag{6}$$

respectively. The customary phase-space integral of a physical massless particle with momentum q (i.e. an on-shell particle with positive-definite energy: $q^2 = 0$, $q_0 \geq 0$) reads

$$\int \frac{d^d q}{(2\pi)^{d-1}} \, \theta(q_0) \, \delta(q^2) \cdots \equiv \int_q \widetilde{\delta}(q) \cdots \ , \tag{7}$$

where we have defined

$$\widetilde{\delta}(q) \equiv 2\pi \, i \, \theta(q_0) \, \delta(q^2) = 2\pi \, i \, \delta_+(q^2) \ . \tag{8}$$

Using this shorthand notation, the one-loop integral $L^{(N)}$ in Eq. (1) can be cast into

$$L^{(N)}(p_1, p_2, \dots, p_N) = \int_q \prod_{i=1}^{N} G(q_i) \ , \tag{9}$$

where $G(q)$ denotes the customary Feynman propagator,

$$G(q) \equiv \frac{1}{q^2 + i0} \ . \tag{10}$$

We also introduce the advanced propagator $G_A(q)$,

$$G_A(q) \equiv \frac{1}{q^2 - i0 \, q_0} \ . \tag{11}$$

We recall that the Feynman and advanced propagators only differ in the position of the particle poles in the complex plane (Fig. 2). Using $q^2 = q_0^2 - \mathbf{q}^2 = 2q_+q_- - \mathbf{q}_\perp^2$, we therefore have

$$[G(q)]^{-1} = 0 \quad \Longrightarrow \quad q_0 = \pm\sqrt{\mathbf{q}^2 - i0} \ , \text{or} \ \ q_\pm = \frac{\mathbf{q}_\perp^2 - i0}{2q_\mp} \ , \tag{12}$$

and

$$[G_A(q)]^{-1} = 0 \quad \Longrightarrow \quad q_0 \simeq \pm\sqrt{\mathbf{q}^2} + i0 \ , \text{or} \ \ q_\pm \simeq \frac{\mathbf{q}_\perp^2}{2q_\mp} + i0 \ . \tag{13}$$

Thus, in the complex plane of the variable q_0 (or, equivalently[†], $q_\pm$), the pole with positive (negative) energy of the Feynman propagator is slightly displaced below (above) the real axis, while both poles (independently of the sign of the energy) of the advanced propagator are slightly displaced above the real axis.

[†]To be precise, each propagator leads to two poles in the plane q_0 and to only one pole in the plane q_+ (or q_-).

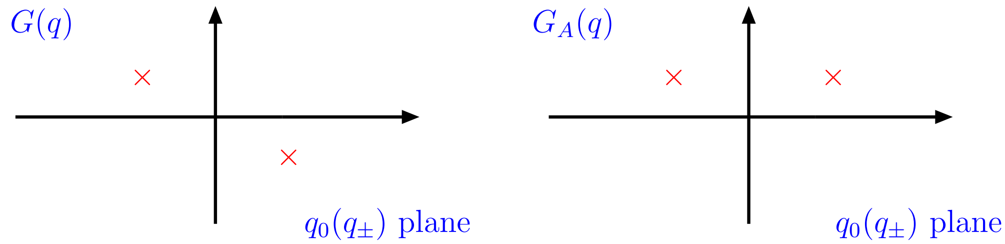

Figure 2: *Location of the particle poles of the Feynman (left) and advanced (right) propagators, $G(q)$ and $G_A(q)$, in the complex plane of the variable q_0 or $q_\pm$.*

3 The Feynman theorem

In this Section we briefly recall the FTT [1, 2].

To this end, we first introduce the advanced one-loop integral $L_A^{(N)}$, which is obtained from $L^{(N)}$ in Eq. (9) by replacing the Feynman propagators $G(q_i)$ with the corresponding advanced propagators $G_A(q_i)$:

$$L_A^{(N)}(p_1, p_2, \dots, p_N) = \int_q \prod_{i=1}^{N} G_A(q_i) \quad . \tag{14}$$

Then, we note that

$$L_A^{(N)}(p_1, p_2, \dots, p_N) = 0 \quad . \tag{15}$$

The proof of Eq. (15) can be carried out in an elementary way by using the Cauchy residue theorem and choosing a suitable integration path C_L. We have

$$L_A^{(N)}(p_1, p_2, \dots, p_N) = \int_{\mathbf{q}} \int dq_0 \prod_{i=1}^{N} G_A(q_i)$$
$$= \int_{\mathbf{q}} \int_{C_L} dq_0 \prod_{i=1}^{N} G_A(q_i) = -\,2\pi i \int_{\mathbf{q}} \sum \mathrm{Res}_{\{\mathrm{Im}\; q_0<0\}} \left[\prod_{i=1}^{N} G_A(q_i) \right] = 0 \quad . \tag{16}$$

The loop integral is evaluated by integrating first over the energy component q_0. Since the integrand is convergent when $q_0 \to \infty$, the q_0 integration can be performed along the contour C_L, which is closed at ∞ in the lower half-plane of the complex variable q_0 (Fig. 3–*left*). The only singularities of the integrand with respect to the variable q_0 are the poles of the advanced propagators $G_A(q_i)$, which are located in the upper half-plane. The integral along C_L is then equal to the sum of the residues at the poles in the lower half-plane and therefore vanishes.

The advanced and Feynman propagators are related by

$$G_A(q) = G(q) + \widetilde{\delta}(q) \quad , \tag{17}$$

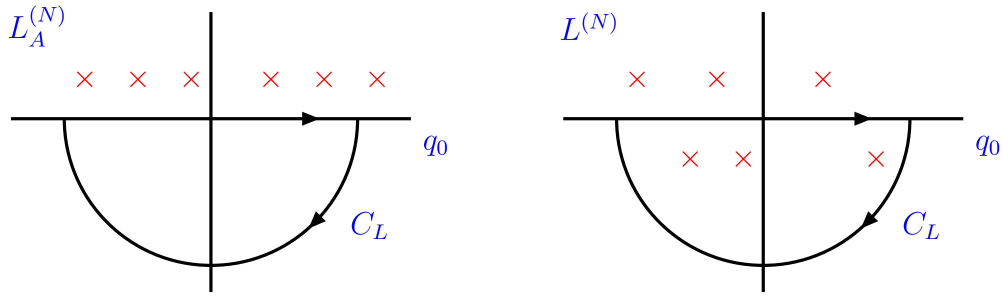

Figure 3: *Location of poles and integration contour C_L in the complex q_0-plane for the advanced (left) and Feynman (right) one-loop integrals, $L_A^{(N)}$ and $L^{(N)}$.*

which can straightforwardly be obtained by using the elementary identity

$$\frac{1}{x \pm i0} = \mathrm{PV}\left(\frac{1}{x}\right) \mp i\pi\, \delta(x) \quad , \tag{18}$$

where PV denotes the principal-value prescription. Inserting Eq. (17) into the right-hand side of Eq. (14) and collecting the contributions with an equal number of factors $G(q_i)$ and $\widetilde{\delta}(q_j)$, we obtain a relation between $L_A^{(N)}$ and the one-loop integral $L^{(N)}$:

$$\begin{aligned} L_A^{(N)}(p_1, p_2, \dots, p_N) &= \int_q \prod_{i=1}^{N} \left[G(q_i) + \widetilde{\delta}(q_i) \right] \\ &= L^{(N)}(p_1, p_2, \dots, p_N) + L_{\mathrm{1-cut}}^{(N)}(p_1, p_2, \dots, p_N) + \dots + L_{\mathrm{N-cut}}^{(N)}(p_1, p_2, \dots, p_N) \quad . \end{aligned} \tag{19}$$

Here, the single-cut contribution is given by

$$L_{\mathrm{1-cut}}^{(N)}(p_1, p_2, \dots, p_N) = \int_q \sum_{i=1}^{N} \widetilde{\delta}(q_i) \prod_{\substack{j=1 \\ j\neq i}}^{N} G(q_j) \quad . \tag{20}$$

In general, the m-cut terms $L_{\mathrm{m-cut}}^{(N)}$ $(m \leq N)$ are the contributions with precisely m delta functions $\widetilde{\delta}(q_i)$:

$$L_{\mathrm{m-cut}}^{(N)}(p_1, p_2, \dots, p_N) = \int_q \left\{ \widetilde{\delta}(q_1) \dots \widetilde{\delta}(q_m)\, G(q_{m+1}) \dots G(q_N) + \text{uneq. perms.} \right\} \quad , \tag{21}$$

where the sum in the curly bracket includes all the permutations of $q_1, \dots, q_N$ that give unequal terms in the integrand.

Recalling that $L_A^{(N)}$ vanishes, cf. Eq. (15), Eq. (19) results in:

$$L^{(N)}(p_1, p_2, \dots, p_N) = - \left[L_{\mathrm{1-cut}}^{(N)}(p_1, p_2, \dots, p_N) + \dots + L_{\mathrm{N-cut}}^{(N)}(p_1, p_2, \dots, p_N) \right] \quad . \tag{22}$$

This equation is the FTT in the specific case of the one-loop integral $L^{(N)}$. The FTT relates the one-loop integral $L^{(N)}$ to the multiple-cut[‡] integrals $L_{\mathrm{m-cut}}^{(N)}$. Each delta function $\widetilde{\delta}(q_i)$

[‡] If the number of space-time dimensions is d, the right-hand side of Eq. (22) receives contributions only from the terms with $m \leq d$; the terms with larger values of m vanish, since the corresponding number of delta functions in the integrand is larger than the number of integration variables.

in $L^{(N)}_{\rm m-cut}$ replaces the corresponding Feynman propagator in $L^{(N)}$ by cutting the internal line with momentum q_i. This is synonymous to setting the respective particle on shell. An m-particle cut decomposes the one-loop diagram in m tree diagrams: in this sense, the FTT allows us to calculate loop-level diagrams from tree-level diagrams.

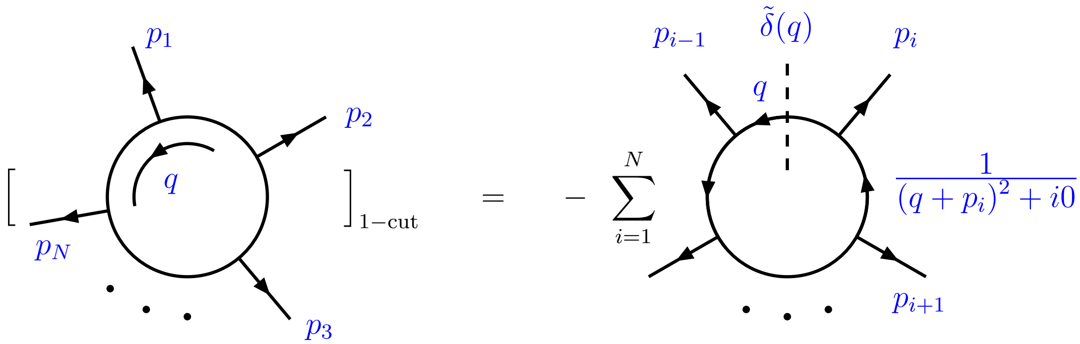

Figure 4: *The single-cut contribution of the Feynman Tree Theorem to the one-loop N-point scalar integral. Graphical representation as a sum of N basic single-cut phase-space integrals.*

In view of the discussion in the following sections, it is useful to consider the single-cut contribution $L^{(N)}_{\rm 1-cut}$ on the right-hand side of Eq. (22). In the case of single-cut contributions, the FTT replaces the one-loop integral with the customary one-particle phase-space integral, see Eqs. (7) and (20). Using the invariance of the loop-integration measure under translations of the loop momentum q, we can perform the momentum shift $q \to q - \sum_{k=1}^{i} p_k$ in the term proportional to $\widetilde{\delta}(q_i)$ on the right-hand side of Eq. (20). Thus, cf. Eq. (2), we have $q_i \to q$ and $q_j \to q + (p_{i+1} + p_{i+2} + \cdots + p_{i+j})$, with $i \neq j$. We can repeat the same shift for each of the terms ($i = 1, 2, \ldots, N$) in the sum on the right-hand side of Eq. (20), and we can rewrite $L^{(N)}_{\rm 1-cut}$ as a sum of N basic phase-space integrals (Fig. 4):

$$\begin{aligned} L^{(N)}_{\rm 1-cut}(p_1, p_2, \ldots, p_N) &= I^{(N-1)}_{\rm 1-cut}(p_1, p_1+p_2, \ldots, p_1+p_2+\cdots+p_{N-1}) + \text{cyclic perms.} \\ &= \sum_{i=1}^{N} I^{(N-1)}_{\rm 1-cut}(p_i, p_i+p_{i+1}, \ldots, p_i+p_{i+1}+\cdots+p_{i+N-2}) \ . \end{aligned} \quad (23)$$

We denote the basic one-particle phase-space integrals with n Feynman propagators by $I^{(n)}_{\rm 1-cut}$. They are defined as follows:

$$I^{(n)}_{\rm 1-cut}(k_1, k_2, \ldots, k_n) = \int_q \widetilde{\delta}(q) \prod_{j=1}^{n} G(q+k_j) = \int_q \widetilde{\delta}(q) \prod_{j=1}^{n} \frac{1}{2qk_j + k_j^2 + i0} \ . \quad (24)$$

The extension of the FTT from the one-loop integrals $L^{(N)}$ to one-loop scattering amplitudes $\mathcal{A}^{(1-\rm loop)}$ (or Green's functions) in perturbative field theories is straightforward, provided the corresponding field theory is *unitary* and *local*. The generalization of Eq. (22) to arbitrary scattering amplitudes is [1, 2]:

$$\mathcal{A}^{(1-\rm loop)} = -\left[\mathcal{A}^{(1-\rm loop)}_{\rm 1-cut} + \mathcal{A}^{(1-\rm loop)}_{\rm 2-cut} + \ldots \right] \ , \quad (25)$$

where $\mathcal{A}^{(1-\mathrm{loop})}_{\mathrm{m-cut}}$ is obtained in the same way as $L^{(N)}_{\mathrm{m-cut}}$, i.e. by starting from $\mathcal{A}^{(1-\mathrm{loop})}$ and considering all possible replacements of m Feynman propagators $G(q_i)$ of its loop internal lines with the 'cut propagators' $\widetilde{\delta}(q_i)$.

The proof of Eq. (25) directly follows from Eq. (22): $\mathcal{A}^{(1-\mathrm{loop})}$ is a linear combination of one-loop integrals that differ from $L^{(N)}$ only by the inclusion of interaction vertices and, eventually, particle masses. As briefly recalled below, these differences have harmless consequences on the derivation of the FTT.

Including particle masses in the advanced and Feynman propagators has an effect on the location of the poles produced by the internal lines in the loop. However, as long as the masses are *real*, as in the case of unitary theories, the position of the poles in the complex plane of the variable q_0 is affected only by a translation parallel to the real axis, with no effect on the imaginary part of the poles. This translation does not interfere with the proof of the FTT as given in Eqs. (14)–(22). Therefore, the effect of a particle mass M_i in a loop internal line with momentum q_i simply amounts to modifying the corresponding on-shell delta function $\widetilde{\delta}(q_i)$ when this line is cut to obtain $\mathcal{A}^{(1-\mathrm{loop})}_{\mathrm{m-cut}}$. This modification then leads to the obvious replacement:

$$\widetilde{\delta}(q_i) \to \widetilde{\delta}(q_i; M_i) = 2\pi\, i\, \theta(q_{i0})\, \delta(q_i^2 - M_i^2) = 2\pi\, i\, \delta_+(q_i^2 - M_i^2) \ . \qquad (26)$$

Including interaction vertices has the effect of introducing numerator factors in the integrand of the one-loop integrals. As long as the theory is local, these numerator factors are at worst polynomials of the integration momentum q [§]. In the complex plane of the variable q_0, this polynomial behavior does not lead to additional singularities at any finite values of q_0. The only danger, when using the Cauchy theorem as in Eq. (16) to prove the FTT, stems from polynomials of high degree that can spoil the convergence of the q_0-integration at infinity. Nonetheless, if the field theory is unitary, these singularities at infinity never occur since the degree of the polynomials in the various integrands is always sufficiently limited by the unitarity constraint.

4 A duality theorem

In this Section we derive and illustrate the duality relation between one-loop integrals and single-cut phase-space integrals. This relation is the main general result of the present work.

Rather than starting from $L^{(N)}_A$, we directly apply the residue theorem to the compu-

[§]This statement is not completely true in the case of gauge theories and, in particular, in the case of gauge-dependent quantities. The discussion of the additional issues that arise in gauge theories is postponed to Sect. 9.

tation of $L^{(N)}$. We proceed exactly as in Eq. (16), and obtain

$$L^{(N)}(p_1, p_2, \dots, p_N) = \int_{\mathbf{q}} \int dq_0 \prod_{i=1}^{N} G(q_i)$$
$$= \int_{\mathbf{q}} \int_{C_L} dq_0 \prod_{i=1}^{N} G(q_i) = -2\pi i \int_{\mathbf{q}} \sum \text{Res}_{\{\text{Im } q_0<0\}} \left[\prod_{i=1}^{N} G(q_i) \right] . \quad (27)$$

At variance with $G_A(q_i)$, each of the Feynman propagators $G(q_i)$ has single poles in both the upper and lower half-planes of the complex variable q_0 (see Fig. 3–*right*) and therefore the integral does not vanish as in the case of the advanced propagators. In contrast, here, the N poles in the lower half-plane contribute to the residues in Eq. (27).

The calculation of these residues is elementary, but it involves several subtleties. The detailed calculation, including a discussion of its subtle points, is presented in Appendix A. In the present Section we limit ourselves to sketching the derivation of the result of this computation.

The sum over residues in Eq. (27) receives contributions from N terms, namely the N residues at the poles with negative imaginary part of each of the propagators $G(q_i)$, with $i = 1, \dots, N$, see Eq. (12). Considering the residue at the i-th pole we write

$$\text{Res}_{\{i-\text{th pole}\}} \left[\prod_{j=1}^{N} G(q_j) \right] = \left[\text{Res}_{\{i-\text{th pole}\}} \, G(q_i) \right] \left[\prod_{\substack{j=1 \\ j\neq i}}^{N} G(q_j) \right]_{\{i-\text{th pole}\}} , \quad (28)$$

where we have used the fact that the propagators $G(q_j)$, with $j \neq i$, are not singular at the value of the pole of $G(q_i)$. Therefore, they can be directly evaluated at this value.

The calculation of the residue of $G(q_i)$ is straightforward and gives

$$\left[\text{Res}_{\{i-\text{th pole}\}} \, G(q_i) \right] = \left[\text{Res}_{\{i-\text{th pole}\}} \frac{1}{q_i^2 + i0} \right] = \int dq_0 \, \delta_+(q_i^2) \; . \quad (29)$$

This result shows that considering the residue of the Feynman propagator of the internal line with momentum q_i is equivalent to cutting that line by including the corresponding on-shell propagator $\delta_+(q_i^2)$. The subscript $+$ of δ_+ refers to the on-shell mode with positive definite energy, $q_{i0} = |\mathbf{q}_i|$: the positive-energy mode is selected by the Feynman $i0$ prescription of the propagator $G(q_i)$. The insertion of Eq. (29) in Eq. (27) directly leads to a representation of the one-loop integral as a linear combination of N single-cut phase-space integrals.

The calculation of the residue prefactor on the r.h.s. of Eq. (28) is more subtle (see Appendix A) and yields

$$\left[\prod_{j\neq i} G(q_j) \right]_{\{i-\text{th pole}\}} = \left[\prod_{j\neq i} \frac{1}{q_j^2 + i0} \right]_{\{i-\text{th pole}\}} = \prod_{j\neq i} \frac{1}{q_j^2 - i0 \, \eta(q_j - q_i)} , \quad (30)$$

where η is a *future-like* vector,

$$\eta_\mu = (\eta_0, \eta) \; , \qquad \eta_0 \geq 0, \; \eta^2 = \eta_\mu \eta^\mu \geq 0 \; , \quad (31)$$

i.e. a d-dimensional vector that can be either light-like ($\eta^2 = 0$) or time-like ($\eta^2 > 0$) with positive definite energy η_0. Note that the calculation of the residue at the pole of the internal line with momentum q_i changes the propagators of the other lines in the loop integral. Although the propagator of the j-th internal line still has the customary form $1/q_j^2$, its singularity at $q_j^2 = 0$ is regularized by a different $i0$ prescription: the original Feynman prescription $q_j^2 + i0$ is modified in the new prescription $q_j^2 - i0\,\eta(q_j - q_i)$, which we name the 'dual' $i0$ prescription or, briefly, the η prescription. The dual $i0$ prescription arises from the fact that the original Feynman propagator $1/(q_j^2 + i0)$ is evaluated at the *complex* value of the loop momentum q, which is determined by the location of the pole at $q_i^2 + i0 = 0$. The $i0$ dependence from the pole has to be combined with the $i0$ dependence in the Feynman propagator to obtain the total dependence as given by the dual $i0$ prescription. The presence of the vector η_μ is a consequence of using the residue theorem. To apply it to the calculation of the d dimensional loop integral, we have to specify a system of coordinates (e.g. space-time or light-cone coordinates) and select one of them to be integrated over at fixed values of the remaining $d-1$ coordinates. Introducing the auxiliary vector η_μ with space-time coordinates $\eta_\mu = (\eta_0, \mathbf{0}_\perp, \eta_{d-1})$, the selected system of coordinates can be denoted in a Lorentz-invariant form. Applying the residue theorem in the complex plane of the variable q_0 at fixed (and *real*) values of the coordinates $\mathbf{q}_\perp$ and $q'_{d-1} = q_{d-1} - q_0\eta_{d-1}/\eta_0$ (to be precise, in Eq. (27) we actually used $\eta_\mu = (1, \mathbf{0})$), we obtain the result in Eq. (30).

The η dependence of the ensuing $i0$ prescription is thus a consequence of the fact that the residues at each of the poles are not Lorentz-invariant quantities. The Lorentz-invariance of the loop integral is recovered only after summing over all the residues.

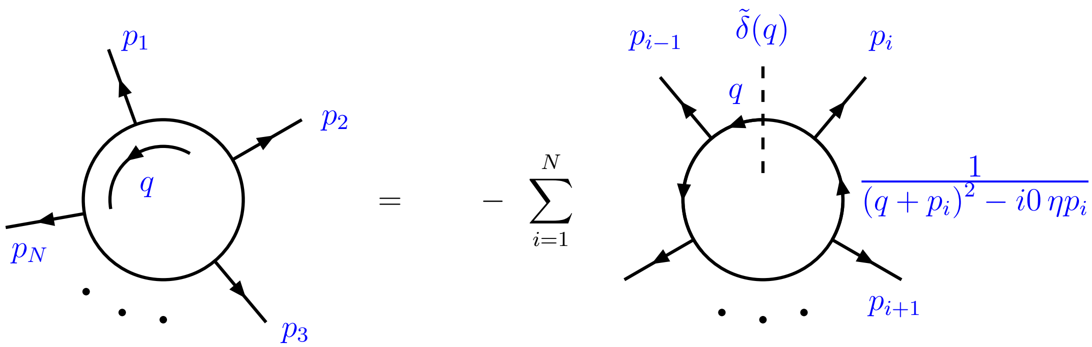

Figure 5: *The duality relation for the one-loop N-point scalar integral. Graphical representation as a sum of N basic dual integrals.*

Inserting the results of Eq. (28)–(30) in Eq. (27) we directly obtain the duality relation between one-loop integrals and phase-space integrals:

$$L^{(N)}(p_1, p_2, \dots, p_N) = -\,\widetilde{L}^{(N)}(p_1, p_2, \dots, p_N) \ , \tag{32}$$

where the explicit expression of the phase-space integral $\widetilde{L}^{(N)}$ is (Fig. 5)

$$\widetilde{L}^{(N)}(p_1, p_2, \dots, p_N) = \int_q \sum_{i=1}^{N} \widetilde{\delta}(q_i) \prod_{\substack{j=1 \\ j \neq i}}^{N} \frac{1}{q_j^2 - i0\,\eta(q_j - q_i)} \ , \tag{33}$$

and η is the auxiliary vector defined in Eq. (31). Each of the $N-1$ propagators in the integrand is regularized by the dual $i0$ prescription and, thus, it is named 'dual' propagator. Note that the momentum difference $q_i - q_j$ is independent of the integration momentum q: it only depends on the momenta of the external legs of the loop (see Eq. (2)).

Using the invariance of the integration measure under translations of the momentum q, we can perform the same momentum shifts as described in Sect. 3. In analogy to Eq. (23), we can rewrite Eq. (33) as a sum of N basic phase-space integrals (Fig. 5):

$$\begin{aligned}\widetilde{L}^{(N)}(p_1, p_2, \dots, p_N) &= I^{(N-1)}(p_1, p_1+p_2, \dots, p_1+p_2+\dots+p_{N-1}) + \text{cyclic perms.} \\ &= \sum_{i=1}^{N} I^{(N-1)}(p_i, p_i+p_{i+1}, \dots, p_i+p_{i+1}+\dots+p_{i+N-2}) \ . \qquad (34)\end{aligned}$$

The basic one-particle phase-space integrals with n dual propagators are denoted by $I^{(n)}$, and are defined as follows:

$$I^{(n)}(k_1, k_2, \dots, k_n) = \int_q \widetilde{\delta}(q) \ \mathcal{I}^{(n)}(q; k_1, k_2, \dots, k_n) = \int_q \widetilde{\delta}(q) \prod_{j=1}^{n} \frac{1}{2qk_j + k_j^2 - i0\,\eta k_j} \ . \qquad (35)$$

We now comment on the comparison between the FTT (Eqs. (20)–(24)) and the duality relation (Eqs. (32)–(35)). The multiple-cut contributions $L^{(N)}_{\text{m-cut}}$, with $m \geq 2$, of the FTT are completely absent from the duality relation, which only involves single-cut contributions similar to those in $L^{(N)}_{\text{1-cut}}$. However, the Feynman propagators present in $L^{(N)}_{\text{1-cut}}$ are replaced by dual propagators in $\widetilde{L}^{(N)}$. This compensates for the absence of multiple-cut contributions in the duality relation.

The $i0$ prescription of the dual propagator depends on the auxiliary vector η. The basic dual integrals $I^{(n)}$ are well defined for arbitrary values of η. However, when computing $\widetilde{L}^{(N)}$, the future-like vector η has to be the *same* in all its contributing dual integrals (propagators): only then $\widetilde{L}^{(N)}$ does not depend on η.

In our derivation of the duality relation, the auxiliary vector η originates from the use of the residue theorem. Independently of its origin, we can comment on the role of η in the duality relation. The one-loop integral $L^{(N)}(p_1, p_2, \dots, p_N)$ is a function of the Lorentz-invariants $(p_i p_j)$. This function has a complicated analytic structure, with pole and branch-cut singularities (scattering singularities), in the multidimensional space of the complex variables $(p_i p_j)$. The $i0$ prescription of the Feynman propagators selects a Riemann sheet in this multidimensional space and, thus, it unambiguously defines $L^{(N)}(p_1, p_2, \dots, p_N)$ as a single-valued function. Each single-cut contribution to $\widetilde{L}^{(N)}$ has additional (unphysical) singularities in the multidimensional complex space. The dual $i0$ prescription fixes the position of these singularities. The auxiliary vector η *correlates* the various single-cut contributions in $\widetilde{L}^{(N)}$, so that they are evaluated on the same Riemann sheet: this leads to the cancellation of the unphysical single-cut singularities. In contrast, in the FTT, this cancellation is produced by the introduction of the multiple-cut contributions $L^{(N)}_{\text{m-cut}}$.

We remark that the expression (34) of $\widetilde{L}^{(N)}$ as a sum of basic dual integrals is just a matter of notation: for massless internal particles $\widetilde{L}^{(N)}$ is actually a *single* phase-space integral whose integrand is the sum of the terms obtained by cutting each of the internal lines of the loop. In explicit form, we can write:

$$\widetilde{L}^{(N)}(p_1,p_2,\dots,p_N) = \int_q \widetilde{\delta}(q) \sum_{i=1}^{N} \mathcal{I}^{(N-1)}(q;p_i,p_i+p_{i+1},\dots,p_i+p_{i+1}+\dots+p_{i+N-2}) \ , \quad (36)$$

where the function $\mathcal{I}^{(n)}$ is the integrand of the dual integral in Eq. (35). Therefore, the duality relation (32) directly expresses the one-loop integral as the phase-space integral of a tree-level quantity. To name Eq. (32), we have introduced the term 'duality' precisely to point out this direct relation* between the d-dimensional integral over the loop momentum and the $(d-1)$-dimensional integral over the one-particle phase-space. For the FTT, the relation between loop-level and tree-level quantities is more involved, since the multiple-cut contributions $L^{(N)}_{\rm m-cut}$ (with $m \geq 2$) contain integrals of expressions that correspond to the product of m tree-level diagrams over the phase-space for different number of particles.

The simpler correspondence between loops and trees in the context of the duality relation is further exploited in Sect. 10, where we discuss Green's functions and scattering amplitudes.

5 Example: The scalar two-point function

In this Section we illustrate the application of the FTT and of the duality relation to the evaluation of the one-loop two-point function $L^{(2)}$. A detailed discussion (including detailed results in analytic form and numerical results) of higher-point functions will be presented elsewhere [5] (see also Refs. [3, 4]).

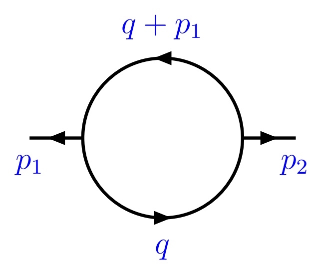

Figure 6: *The one-loop two-point scalar integral* $L^{(2)}(p_1,p_2)$.

The two-point function (Fig. 6), also known as bubble function Bub, is the simplest non-trivial one-loop integral with massless internal lines:

$$\mathrm{Bub}(p_1^2) \equiv L^{(2)}(p_1,p_2) = -i \int \frac{d^dq}{(2\pi)^d} \frac{1}{[q^2+i0]\,[(q+p_1)^2+i0]} \ . \quad (37)$$

Here, we have visibly implemented momentum conservation $(p_1+p_2=0)$ and exploited Lorentz invariance ($L^{(2)}(p_1,p_2)$ can only depend on p_1^2, which is the sole available invariant).

*The word duality also suggests a stronger (possibly one-to-one) correspondence between dual integrals and loop integrals, which is further discussed in Sect. 7.

Since most of the one-loop calculations have been carried out in four-dimensional field theories (or in their dimensionally-regularized versions), we set $d = 4 - 2\epsilon$. Note, however, that we present results for arbitrary values of ϵ or, equivalently, for any value d of space-time dimensions.

The result of the one-loop integral in Eq. (37) is well known:

$$\text{Bub}(p^2) = c_\Gamma \, \frac{1}{\epsilon(1-2\epsilon)} \, \left(-p^2 - i0\right)^{-\epsilon} \quad , \tag{38}$$

where c_Γ is the customary d-dimensional volume factor that appears from the calculation of one-loop integrals:

$$c_\Gamma \equiv \frac{\Gamma(1+\epsilon)\,\Gamma^2(1-\epsilon)}{(4\pi)^{2-\epsilon}\,\Gamma(1-2\epsilon)} \quad . \tag{39}$$

We recall that the $i0$ prescription in Eq. (38) follows from the corresponding prescription of the Feynman propagators in the integrand of Eq. (37). The $i0$ prescription defines $\text{Bub}(p^2)$ as a single-value function of the real variable p^2. In particular, it gives $\text{Bub}(p^2)$ an imaginary part with an unambiguous value when $p^2 > 0$:

$$\text{Bub}(p^2) = c_\Gamma \, \frac{1}{\epsilon(1-2\epsilon)} \, \left(|p^2|\right)^{-\epsilon} \left[\theta(-p^2) + \theta(p^2)\, e^{i\pi\epsilon}\right] \quad . \tag{40}$$

5.1 General form of single-cut integrals

To apply the FTT and the duality relation, we have to compute the single-cut integrals $I^{(1)}_{1-\text{cut}}$ and $I^{(1)}$, respectively. Since these integrals only differ because of their $i0$ prescription, we introduce a more general regularized version, $I^{(1)}_{\text{reg}}$, of the single-cut integral. We define:

$$I^{(1)}_{\text{reg}}(k; c(k)) = \int_q \widetilde{\delta}(q) \, \frac{1}{2qk + k^2 + i0\, c(k)} = \int \frac{d^d q}{(2\pi)^{d-1}} \, \delta_+(q^2) \, \frac{1}{2qk + k^2 + i0\, c(k)} \quad . \tag{41}$$

Although $c(k)$ is an arbitrary function of k, $I^{(1)}_{\text{reg}}$ only depends on the sign of the $i0$ prescription, i.e. on the sign of the function $c(k)$: setting $c(k) = +1$ we recover $I^{(1)}_{1-\text{cut}}$, cf. Eq. (24), while setting $c(k) = -\eta k$ we recover $I^{(1)}$ (see Eq. (35)).

The calculation of the integral in Eq. (41) is elementary, and the result is

$$I^{(1)}_{\text{reg}}(k; c(k)) = -\frac{c_\Gamma}{2\cos(\pi\epsilon)} \, \frac{1}{\epsilon(1-2\epsilon)} \, \left[\frac{k^2}{k_0} - i0\, k^2\, c(k)\right]^{-\epsilon} \, \left[k_0 - i0\, k^2\, c(k)\right]^{-\epsilon} \quad . \tag{42}$$

Note that the typical volume factor, $\widetilde{c}_\Gamma$, of the d-dimensional phase-space integral is

$$\widetilde{c}_\Gamma = \frac{\Gamma(1-\epsilon)\,\Gamma(1+2\epsilon)}{(4\pi)^{2-\epsilon}} \quad . \tag{43}$$

The factor $\cos(\pi\epsilon)$ in Eq. (42) originates from the difference between $\widetilde{c}_\Gamma$ and the volume factor c_Γ of the loop integral:

$$\frac{\widetilde{c}_\Gamma}{c_\Gamma} = \frac{\Gamma(1+2\epsilon)\,\Gamma(1-2\epsilon)}{\Gamma(1+\epsilon)\,\Gamma(1-\epsilon)} = \frac{1}{\cos(\pi\epsilon)} \quad . \tag{44}$$

We also note that the result in Eq. (42) depends on the sign of the energy k_0. This follows from the fact that the integration measure in Eq. (41) has support on the future light-cone, which is selected by the positive-energy requirement of the on-shell constraint $\delta_+(q^2)$.

The denominator contribution $(2qk+k^2)$ in the integrand of Eq. (41) is positive definite in the kinematical region where $k^2 > 0$ and $k_0 > 0$. In this region the $i0$ prescription is inconsequential, and $I_{\rm reg}^{(1)}$ has no imaginary part. Outside this kinematical region, $(2qk+k^2)$ can vanish, leading to a singularity of the integrand. The singularity is regularized by the $i0$ prescription, which also produces a non-vanishing imaginary part. The result in Eq. (42) explicitly shows these expected features, since it can be rewritten as

$$\begin{aligned} I_{\rm reg}^{(1)}(k;c(k)) &= -\frac{c_\Gamma}{2\cos(\pi\epsilon)} \frac{(|k^2|)^{-\epsilon}}{\epsilon(1-2\epsilon)} \left\{ \theta(-k^2) \left[\cos(\pi\epsilon) - i \sin(\pi\epsilon)\, {\rm sign}(c(k)) \right] \right. \\ &+ \left. \theta(k^2) \left[\theta(k_0) + \theta(-k_0) \left(\cos(2\pi\epsilon) + i \sin(2\pi\epsilon)\, {\rm sign}(c(k)) \right) \right] \right\} \quad . \qquad (45) \end{aligned}$$

We note that the functions ${\rm Bub}(k^2)$ and $I_{\rm reg}^{(1)}(k;c(k))$ have different analyticity properties in the complex k^2 plane. The bubble function has a branch-cut singularity along the positive real axis, $k^2 > 0$. The phase-space integral $I_{\rm reg}^{(1)}(k;c(k))$ has a branch-cut singularity along the entire real axis if $k_0 < 0$, while the branch-cut singularity is placed along the negative real axis if $k_0 > 0$.

5.2 Duality relation for the two-point function

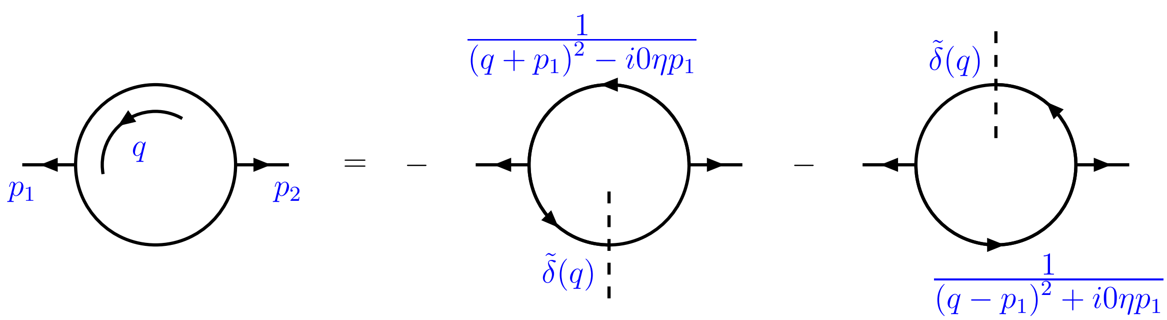

Figure 7: *One-loop two-point function: the duality relation.*

We now consider the duality relation (Fig. 7) in the context of this example. The dual representation of the one-loop two-point function is given by

$$\widetilde{L}^{(2)}(p_1,p_2) = I^{(1)}(p_1) + \Big(p_1 \leftrightarrow -p_1\Big) \quad , \qquad (46)$$

cf. Eqs. (34) and (35). The basic dual integral $I^{(1)}(k)$ is obtained by setting $c(k) = -\eta k$ in Eq. (42). Since η^μ is a future-like vector, $c(k)$ has the following important property:

$${\rm sign}(\eta k) = {\rm sign}(k_0) \quad , \quad {\rm if} \;\; k^2 \geq 0 \quad . \qquad (47)$$

Using this property, the result in Eq. (42) can be written as

$$I^{(1)}(k) = -\,\frac{c_\Gamma}{2}\, \frac{(-k^2-i0)^{-\epsilon}}{\epsilon(1-2\epsilon)} \left[1 - i\, \frac{\sin(\pi\epsilon)}{\cos(\pi\epsilon)}\, {\rm sign}(k^2 \eta k) \right] \quad . \qquad (48)$$

Comparing this expression with Eq. (38), we see that the imaginary contribution in the square bracket is responsible for the difference with the two-point function. However, since $\mathrm{sign}(-\eta k) = -\mathrm{sign}(\eta k)$, this contribution is odd under the exchange $k \to -k$ and, therefore, it cancels when Eq. (48) is inserted in Eq. (46). Taken together,

$$\widetilde{L}^{(2)}(p_1, p_2) = I^{(1)}(p_1) + \Big(p_1 \leftrightarrow -p_1\Big) = -\, c_\Gamma \, \frac{(-p_1^2 - i0)^{-\epsilon}}{\epsilon(1-2\epsilon)} \quad , \tag{49}$$

which fully agrees with the duality relation $\widetilde{L}^{(2)}(p_1, p_2) = -\,\mathrm{Bub}(p_1^2)$.

5.3 FTT for the two-point function

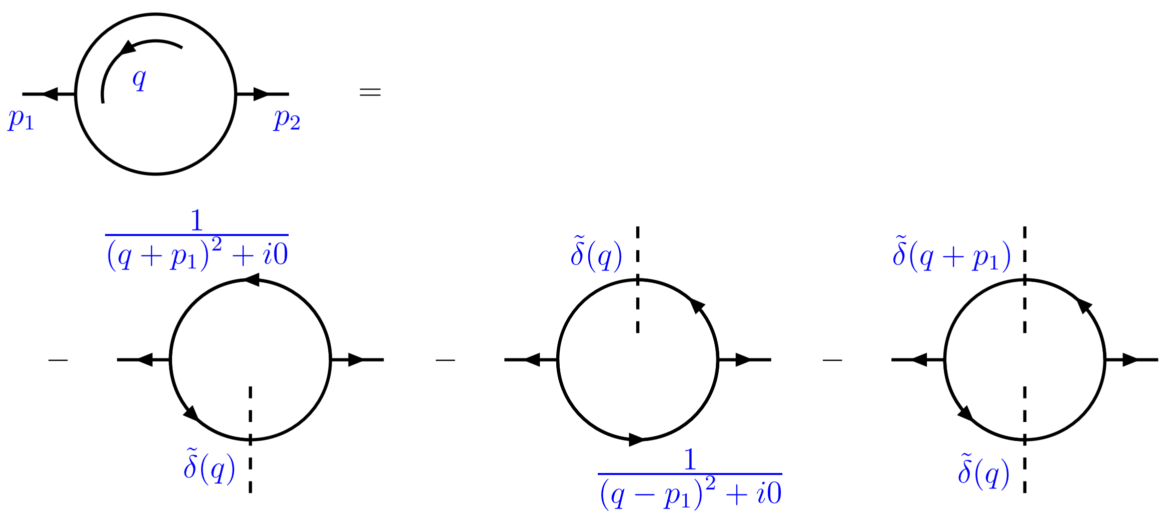

Figure 8: *One-loop two-point function: the Feynman Tree Theorem*

We now would like to discuss the FTT (Fig. 8) in the case of the two-point function. To this end, we want to check the relations of Eqs. (21)–(24). For the FTT, the two-point function is cast into the form

$$L^{(2)}(p_1, p_2, \ldots, p_N) = -\left[\, L^{(2)}_{1-\mathrm{cut}}(p_1, p_2) + L^{(2)}_{2-\mathrm{cut}}(p_1, p_2) \,\right] \quad , \tag{50}$$

where the single-cut and double-cut contributions are

$$L^{(2)}_{1-\mathrm{cut}}(p_1, p_2) = I^{(1)}_{1-\mathrm{cut}}(p_1) + \Big(p_1 \leftrightarrow -p_1\Big) \quad , \tag{51}$$

and

$$L^{(2)}_{2-\mathrm{cut}}(p_1, p_2) = \int_q \; \widetilde{\delta}(q) \; \widetilde{\delta}(q+p_1) = i \int \frac{d^d q}{(2\pi)^{d-2}} \, \theta(q_0) \, \delta(q^2) \; \theta(q_0+p_{10}) \; \delta((q+p_1)^2) \quad , \tag{52}$$

respectively. The basic single-cut integral $I^{(1)}_{1-\mathrm{cut}}(k)$ of Eq. (51) is obtained by setting $c(k) = +1$ in Eq. (42); we then have

$$I^{(1)}_{1-\mathrm{cut}}(k) = -\,\frac{c_\Gamma}{2} \, \frac{(-k^2 - i0)^{-\epsilon}}{\epsilon(1-2\epsilon)} \left[1 - i \, \frac{\sin(\pi\epsilon)}{\cos(\pi\epsilon)} \left[\theta(-k^2) + \theta(k^2)\, \mathrm{sign}(k_0) \right] \right] \quad . \tag{53}$$

Comparing the individual single-cut results, Eqs. (48) and (53), we see that the imaginary contributions in the square brackets are different. Inserting Eq. (53) into Eq. (51), the part of the imaginary contribution that is proportional to $\text{sign}(k_0)$ cancels (this part is odd under the exchange $k \to -k$), while the remaining part does not:

$$L^{(2)}_{1-\text{cut}}(p_1, p_2) = I^{(1)}_{1-\text{cut}}(p_1) + \Big(p_1 \leftrightarrow -p_1\Big) = -\, c_\Gamma \, \frac{(-p_1^2 - i0)^{-\epsilon}}{\epsilon(1-2\epsilon)} \left[1 - i\, \frac{\sin(\pi\epsilon)}{\cos(\pi\epsilon)}\, \theta(-p_1^2)\right] \ . \tag{54}$$

We see that also the sum of the two single-cut contributions of Eqs. (49) and (54) are different: the difference is due to the replacement of the dual $i0$ prescription with the Feynman $i0$ prescription. In particular, the difference is a purely imaginary term with support on the space-like region $p_1^2 < 0$, whereas the two-point function is purely real in the same region. In the FTT, this difference is compensated by the double-cut contribution $L^{(2)}_{2-\text{cut}}$.

The calculation of the double-cut contribution in Eq. (52) results in

$$L^{(2)}_{2-\text{cut}}(p_1, p_2) = -\, i\, c_\Gamma \, \frac{(|p_1^2|)^{-\epsilon}}{\epsilon(1-2\epsilon)} \, \frac{\sin(\pi\epsilon)}{\cos(\pi\epsilon)} \, \theta(-p_1^2) \ . \tag{55}$$

Inserting Eqs. (54) and (55) into the right-hand side of the FTT expression of Eq. (50), we find agreement with the result from the direct one-loop computation of the two-point function.

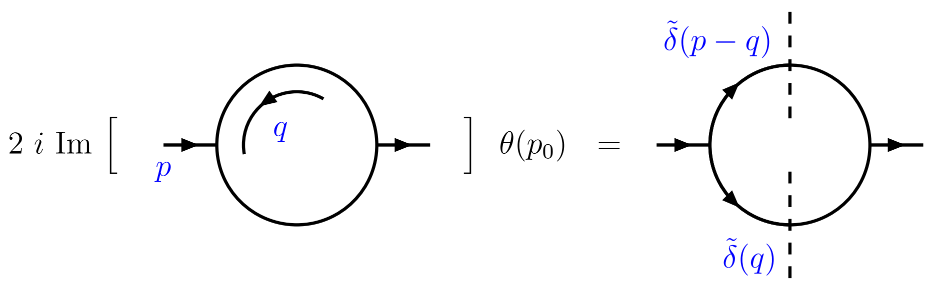

Figure 9: *One-loop two-point function: the imaginary part.*

To conclude this illustration of the FTT, we add a remark. The double-cut contribution $L^{(2)}_{2-\text{cut}}$ is different from the unitarity-cut contribution that gives the imaginary part of the bubble function (or, equivalently, the discontinuity of $\text{Bub}(p^2)$ across its branch-cut singularity). The imaginary part of the two-point function can be obtained by applying the Cutkosky rules (Fig. 9):

$$2\, i\ \text{Im}\left[\text{Bub}(p^2)\right]\ \theta(p_0) = \int_q \ \widetilde{\delta}(q)\ \widetilde{\delta}(p-q) = i \int \frac{d^d q}{(2\pi)^{d-2}}\ \theta(q_0)\, \delta(q^2)\ \theta(p_0 - q_0)\ \delta((q-p)^2) \ . \tag{56}$$

We see that the double-cut contributions in Eq. (52) and (56) are different due to the determination of the positive-energy flow in the internal lines. Once the energy of the line with momentum q is fixed to be positive, the on-shell line with momentum $q+k$ has positive energy in Eq. (52) and negative energy in Eq. (56). The computation of the double-cut integral in Eq. (56) yields

$$\int_q \widetilde{\delta}(q)\, \widetilde{\delta}(p-q) = +i\, c_\Gamma \, \frac{(|p^2|)^{-\epsilon}}{\epsilon(1-2\epsilon)} \, 2\sin(\pi\epsilon)\, \theta(p^2)\, \theta(p_0) \ , \tag{57}$$

which indeed differs from the expression in Eq. (55). Inserting Eq. (57) in Eq. (56), we obtain the imaginary part of $\text{Bub}(p^2)$, in complete agreement with the result (40) of the one-loop integral.

We also note that the Cutkosky rules in Eq. (56) can be derived in a direct way (i.e., without the explicit computation of any integrals) from the duality relation. The derivation is as follows. Applying the identity (18) to the dual propagator, we have

$$\text{Im}\,[I^{(1)}(p)] = \pi \int_q \widetilde{\delta}(q)\, \delta((q+p)^2) \ \ \text{sign}(\eta p) \ . \tag{58}$$

We now use the duality relation to compute the imaginary part of the two-point function, which is given by

$$2\,i\ \text{Im}\left[\text{Bub}(p^2)\right]\, \theta(p_0) = -2\,i\, \theta(p_0) \left[\, \text{Im}\, I^{(1)}(p) + (p \leftrightarrow -p) \,\right] \ . \tag{59}$$

Inserting Eq. (58) in Eq. (59), we obtain

$$\begin{aligned} 2\,i \ \ \text{Im}\left[\text{Bub}(p^2)\right]\, \theta(p_0) &= -2\pi\, i\, \text{sign}(\eta p)\, \theta(p_0) \int_q \widetilde{\delta}(q) \Big[\, \delta((q+p)^2) - \delta((q-p)^2) \,\Big] \\ = \ -\, (2\pi\, i)^2\, \text{sign}(\eta p)\, \theta(p_0) &\int_q \ \delta(q^2)\, \delta((q-p)^2) \left\{ \theta(q_0-p_0) - \theta(q_0) \right\} \ , \end{aligned} \tag{60}$$

where the first term in the square bracket has been rewritten by performing the shift $q \to q-p$ of the integration variable q. The energy constraints in Eq. (60) result in

$$\theta(p_0) \left\{ \theta(q_0-p_0) - \theta(q_0) \right\} = -\, \theta(q_0)\, \theta(p_0-q_0) \ . \tag{61}$$

This can be inserted in Eq. (60) to obtain

$$2\,i\ \text{Im}\left[\text{Bub}(p^2)\right]\, \theta(p_0) = \text{sign}(\eta p) \int_q \widetilde{\delta}(q)\, \widetilde{\delta}(p-q) \ . \tag{62}$$

We observe that the constraints $q^2 = (p-q)^2 = 0$ and $q_0 > 0$, $p_0 - q_0 > 0$ imply $\text{sign}(\eta q) = \text{sign}(\eta(p-q)) = +1$ (see Eq. (47)) and, hence, $\text{sign}(\eta p) = +1$. Therefore Eq. (62) becomes identical to Eq. (56).

6 Relating Feynman's theorem and the duality theorem

The one-loop integral $L^{(N)}$ can be expressed by using either the FTT or the duality relation. Comparing Eq. (22) with Eq. (32), we thus derive

$$\widetilde{L}^{(N)}(p_1, p_2, \dots, p_N) = L^{(N)}_{\text{1-cut}}(p_1, p_2, \dots, p_N) + \cdots + L^{(N)}_{\text{N-cut}}(p_1, p_2, \dots, p_N) \ . \tag{63}$$

This expression relates single-cut dual integrals with multiple-cut Feynman integrals. It has been derived in an indirect way, by applying the residue theorem to the evaluation of one-loop integrals.

In this Section we present another proof of Eq. (63). The proof is direct and purely algebraic. It further clarifies the connection between the FTT and the duality relation.

Our starting point is a basic identity between dual and Feynman propagators. The identity applies to the dual propagators when they are inserted in a single-cut integral. Then

$$\widetilde{\delta}(q) \, \frac{1}{2qk + k^2 - i0\,\eta k} = \widetilde{\delta}(q) \left[G(q+k) + \theta(\eta k) \, 2\pi i \, \delta((q+k)^2) \right] = \widetilde{\delta}(q) \left[G(q+k) + \theta(\eta k) \, \widetilde{\delta}(q+k) \right] \; . \tag{64}$$

The equality on the first line of Eq. (64) directly follows from Eq. (18). The equality on the second line is obtained as follows. Using the constraint $\widetilde{\delta}(q)$, we have $q^2 = 0$ and $q_0 > 0$. Therefore, from Eq. (47) we thus have $\eta q > 0$. Using $\eta q > 0$ and the constraint $\theta(\eta k)$, we have $\eta(q+k) > 0$. Combining $\eta(q+k) > 0$ with $(q+k)^2 = 0$, from Eq. (47) we thus have $q_0 + k_0 > 0$. This enables the replacement $\delta((q+k)^2) \to \delta_+((q+k)^2)$, which finally yields Eq. (64).

6.1 Two-point function

The relation (64) can be used to prove Eq. (63). We first consider the case $N = 2$. Inserting Eq. (64) in Eq. (35) and comparing with Eqs. (24) and (52), we obtain

$$I^{(1)}(p_1) = I^{(1)}_{1-\mathrm{cut}}(p_1) + \theta(\eta p_1) \int_q \widetilde{\delta}(q) \, \widetilde{\delta}(q+p_1) = I^{(1)}_{1-\mathrm{cut}}(p_1) + \theta(\eta p_1) \, L^{(2)}_{2-\mathrm{cut}}(p_1, p_2) \; . \tag{65}$$

We can now use this equation to compute $\widetilde{L}^{(2)}$:

$$\widetilde{L}^{(2)}(p_1, p_2) = I^{(1)}(p_1) + I^{(1)}(p_2) = L^{(2)}_{1-\mathrm{cut}}(p_1, p_2) + \left[\theta(\eta p_1) + \theta(\eta p_2) \right] L^{(2)}_{2-\mathrm{cut}}(p_1, p_2) \; . \tag{66}$$

This relation is equivalent to Eq. (63), since by merely using momentum conservation, $p_1 + p_2 = 0$, we find

$$\theta(\eta p_1) + \theta(\eta p_2) = \theta(\eta p_1) + \theta(-\eta p_1) = 1 \; . \tag{67}$$

6.2 General N-point function

More generally, the identity (64) relates the basic dual integrals $I^{(n)}$ with multiple-cut Feynman integrals. Inserting Eq. (64) in Eq. (35) and using Eq. (24), we obtain

$$I^{(n)}(k_1, k_2, \ldots, k_n) = I^{(n)}_{1-\mathrm{cut}}(k_1, k_2, \ldots, k_n) + I^{(n)}_{\eta}(k_1, k_2, \ldots, k_n) = I^{(n)}_{1-\mathrm{cut}}(k_1, k_2, \ldots, k_n) + \sum_{m=1}^{n} I^{(n)}_{m,\eta}(k_1, k_2, \ldots, k_n) \; , \tag{68}$$

where

$$I^{(n)}_{m,\eta}(k_1,k_2,\ldots,k_n) = \int_q \widetilde{\delta}(q) \left\{ \widetilde{\delta}(q+k_1)\ldots\widetilde{\delta}(q+k_m)\, G(q+k_{m+1})\ldots G(q+k_n) \right.$$
$$\left. \times\, \theta(\eta k_1)\ldots\theta(\eta k_m) + \text{ uneq. perms.} \right\} \ . \quad (69)$$

Note that the key difference between $I^{(n)}_{m,\eta}$ and the multiple-cut contributions of the FTT (see Eq. (21)) is the presence of the momentum constraints, $\theta(\eta k_i)$, in Eq. (69).

For a proof in the case of the N-point function, we employ the following relation:

$$I^{(N-1)}_{m-1,\eta}(p_1, p_1+p_2, \ldots, p_1+p_2+\cdots+p_{N-1}) + \text{cyclic perms.} = L^{(N)}_{\text{m-cut}}(p_1,p_2,\ldots,p_N) \ . \quad (70)$$

Summing over the cyclic permutations of $I^{(N-1)}$ as in Eq. (34), and using Eqs. (68), (23) and (70), we straightforwardly obtain the relation in Eq. (63).

We note that the proof of Eq. (70) is mainly a matter of combinatorics, and it does not require the explicit evaluation of any m-cut integral. Eventually, the main ingredient of the proof is the following algebraic identity

$$\theta(\eta p_1)\,\theta(\eta(p_1+p_2))\ \ldots\ \theta(\eta(p_1+p_2+\cdots+p_{N-1})) + \text{cyclic perms.} = 1 \ . \quad (71)$$

It is a direct consequence of momentum conservation, namely $\sum_{i=1}^{N} p_i = 0$. The derivation of Eq. (71) is presented in Appendix B.

To simplify the combinatorics in the proof of Eq. (70), we first rewrite $I^{(n)}_{m,\eta}$ in Eq. (69) as

$$I^{(n)}_{m,\eta}(k_1,k_2,\ldots,k_n) = I^{(n)}_{m,F}(k_1,k_2,\ldots,k_n) + \delta I^{(n)}_{m,\eta}(k_1,k_2,\ldots,k_n) \ , \quad (72)$$

where

$$I^{(n)}_{m,F}(k_1,k_2,\ldots,k_n) = \frac{1}{m+1} \int_q \widetilde{\delta}(q) \left\{ \widetilde{\delta}(q+k_1)\ldots\widetilde{\delta}(q+k_m)\, G(q+k_{m+1})\ldots G(q+k_n) \right.$$
$$\left. + \text{ uneq. perms.} \right\} \ , \quad (73)$$

and

$$\delta I^{(n)}_{m,\eta}(k_1,k_2,\ldots,k_n) = \int_q \widetilde{\delta}(q) \left\{ \widetilde{\delta}(q+k_1)\ldots\widetilde{\delta}(q+k_m)\, G(q+k_{m+1})\ldots G(q+k_n) \right.$$
$$\left. \times \left[\theta(\eta k_1)\ldots\theta(\eta k_m) - \frac{1}{m+1} \right] + \text{ uneq. perms.} \right\} . \quad (74)$$

This leaves us with the task to prove the relations

$$I^{(N-1)}_{m-1,F}(p_1, p_1+p_2, \ldots, p_1+p_2+\cdots+p_{N-1}) + \text{cyclic perms.} = L^{(N)}_{\text{m-cut}}(p_1,p_2,\ldots,p_N) \ , \quad (75)$$

and

$$\delta I^{(N-1)}_{m-1,\eta}(p_1, p_1+p_2, \ldots, p_1+p_2+\cdots+p_{N-1}) + \text{ cyclic perms.} = 0 \ . \quad (76)$$

Obviously, Eqs. (72), (75) and (76) imply Eq. (70).

The relation (75) can be proven as follows. According to Eq. (21), $L^{(N)}_{\rm m-cut}$ is a sum of m-cut contributions with a fully symmetric dependence on the momenta q_i of the internal lines of the loop integral. The expression on the left-hand side of Eq. (75) is also a fully symmetric linear combination of m-cut contributions: the symmetrization follows from the sum over the permutations in Eqs. (73) and (75). Hence, owing to their symmetry, the left-hand side and the right-hand side of Eq. (75) are necessarily proportional, and the proportionality coefficient is just unity. To show this, we can give weight unity to each m-cut contribution and simply count the number of m-cut contributions on both sides of Eq. (75). The number of terms in $L^{(N)}_{\rm m-cut}$ equals the total number of permutations in the curly bracket of Eq. (21), namely

$$\binom{N}{m} = \frac{N!}{m!\,(N-m)!} \quad . \tag{77}$$

The number of terms on the left-hand side of Eq. (75) is

$$\frac{1}{m}\binom{N-1}{m-1} N = \frac{1}{m}\,\frac{(N-1)!}{(m-1)!\,(N-m)!}\, N \quad , \tag{78}$$

where the factor $1/m$ is the weight of each contribution to $I^{(N-1)}_{m-1,F}$, the factor $\binom{N-1}{m-1}$ is the number of permutations that contribute to $I^{(N-1)}_{m-1,F}$ (see Eq. (73)), and the factor N is the number of cyclic permutations in Eq. (75). As we can see, the numbers given by Eqs. (77) and (78) coincide, thus yielding the equality in Eq. (75).

The relation (76) can be proven as follows. The left-hand side is a sum of m-cut contributions of the loop integral $L^{(N)}$. We can organize these contributions in a sum of $\binom{N}{m}$ diagrams as on the right-hand side of Eq. (21): each diagram has m fixed internal lines that have been cut. The coefficient of each diagram is computed according to the expression on the left-hand side of Eq. (76). As discussed below, this coefficient vanishes algebraically, thus yielding the result in Eq. (76).

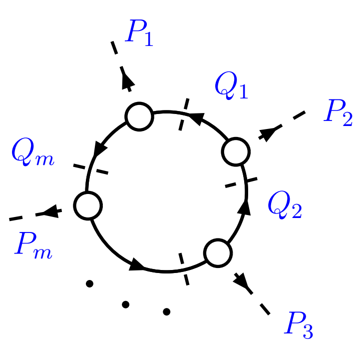

Figure 10: *A one-loop diagram with m cut lines. Each blob denotes a set of internal lines that are not cut.*

We consider one of the diagram with m cut lines, and we denote the momenta of these internal lines as $Q_1, Q_2, \ldots, Q_m$ (Fig. 10). We define $P_i = Q_i - Q_{i-1}$, so that P_i is the total external momentum between the cut lines with momenta Q_i and Q_{i-1}. The computation of the diagram involves the factor

$$\widetilde{\delta}(Q_1)\,\widetilde{\delta}(Q_2)\,\ldots\,\widetilde{\delta}(Q_m) \quad , \tag{79}$$

and two other factors. One factor stems from the product of the Feynman propagators of the uncut internal lines and it is inconsequential to the present discussion. The other factor arises from the term in the square bracket on the right-hand side of Eq. (74). We note that $\delta I_{m-1,\eta}^{(N-1)}$ involves the product $\widetilde{\delta}(q)\,\widetilde{\delta}(q+k_1)\,\ldots\,\widetilde{\delta}(q+k_{m-1})$ of m delta functions, but the term in the square bracket is symmetric only with respect to the argument of $m-1$ delta functions. Therefore, inserting Eq. (74) into Eq. (76) and performing the sum over the permutations, the term in the square bracket leads to m different contributions: each contribution corresponds to one of the assignments $\widetilde{\delta}(q) \to \widetilde{\delta}(Q_i)$ with $i = 1, 2, \ldots, m$. In conclusion, the square-bracket term contributes to multiply the left-hand side of Eq. (79) by a factor proportional to the following expression:

$$\left[\theta(\eta P_1)\,\theta(\eta(P_1+P_2))\,\ldots\,\theta(\eta(P_1+P_2+\cdots+P_{m-1})) - \frac{1}{m}\right] + \text{cyclic perms.}$$
$$= \left\{\theta(\eta P_1)\,\theta(\eta(P_1+P_2))\,\ldots\,\theta(\eta(P_1+P_2+\cdots+P_{m-1})) + \text{cyclic perms.}\right\} - 1 \ . \quad (80)$$

This expression vanishes, because of Eq. (71) and the momentum conservation constraint $\sum_{i=1}^{m} P_i = 0$. Therefore, each m-cut diagram of the left-hand side of Eq. (76) has a vanishing coefficient.

7 Dual bases and generalized duality

One-loop Feynman integrals and single-cut dual integrals are not in a *one-to-one* correspondence. Starting from this observation we discuss in more general terms the nature of the correspondence between one-loop and single-cut integrals in this section.

Using the duality relation, any one-loop Feynman integral $L^{(N)}$ can be expressed as a linear combination of the basic dual integrals $I^{(N-1)}$, but the opposite statement is not true. Therefore, the dual integrals $I^{(n)}$ form a linear basis of the functional space generated by the loop integrals, but overall they generate a larger space containing that of the one-loop Feynman integrals.

To express $I^{(N-1)}$ as a linear combination of loop integrals, we have to introduce generalized one-loop integrals, whose integrands contain both Feynman and advanced propagators. We define them through

$$L^{(N)}(p_1, \alpha_1, p_2, \alpha_2, \ldots, p_N, \alpha_N) = \int_q \prod_{i=1}^{N} G_{\alpha_i}(q_i) \ , \quad (81)$$

where the label α_i can take two values, $\alpha_i = F, A$, and $G_F(q_i) = G(q_i)$ is the Feynman propagator, while $G_A(q_i)$ is the advanced propagator. In particular, when $\alpha_1 = \alpha_2 = \cdots = \alpha_N = F$ we recover the one-loop Feynman integral in Eq. (9), while we obtain the one-loop advanced integral in Eqs. (14) and (15) for the case $\alpha_1 = \alpha_2 = \cdots = \alpha_N = A$.

The relation between $I^{(N-1)}$ and the generalized one-loop integrals in Eq. (81) is obtained by rewritten the dual propagators as a linear combination of G and G_A. Using

Eqs. (17) and (64) we have:

$$\widetilde{\delta}(q) \, \frac{1}{2qk + k^2 - i0\,\eta k} = \widetilde{\delta}(q) \, \Big[G(q+k) + \theta(\eta k) \Big(G_A(q+k) - G(q+k) \Big) \Big] = \widetilde{\delta}(q) \, \Big[\theta(-\eta k) \, G(q+k) + \theta(\eta k) \, G_A(q+k) \Big] \;, \tag{82}$$

which can be inserted in Eq. (35). We thus obtain

$$I^{(n)}(k_1, k_2, \dots, k_n) = \int_q \widetilde{\delta}(q) \, \prod_{j=1}^{n} \Big[\theta(-\eta k_j) \, G(q+k_j) + \theta(\eta k_j) \, G_A(q+k_j) \Big] = \int_q \Big(G_A(q) - G(q) \Big) \, \prod_{j=1}^{n} \Big[\theta(-\eta k_j) \, G(q+k_j) + \theta(\eta k_j) \, G_A(q+k_j) \Big] \,, \tag{83}$$

where again we have used Eq. (17) to express $\widetilde{\delta}(q)$ as a linear combination of $G(q)$ and $G_A(q)$. The right-hand side of Eq. (83) is a sum of generalized one-loop integrals. Note that the η dependence of $I^{(n)}$ appears only in the coefficients $\theta(\pm\eta k_j)$.

In the simplest case, with $n = 1$, Eq. (83) reads:

$$\begin{aligned} I^{(1)}(p_1) &= - \, \theta(-\eta p_1) \int_q \, G(q) \, G(q+p_1) \\ &+ \Big[\theta(-\eta p_1) \int_q \, G_A(q) \, G(q+p_1) - \theta(\eta p_1) \int_q \, G(q) \, G_A(q+p_1) \Big] \\ &= - \, \theta(-\eta p_1) \, L^{(2)}(p_1, -p_1) + \Big[\theta(-\eta p_1) \, L^{(2)}(p_1, F, -p_1, A) - (p_1 \leftrightarrow -p_1) \Big] \;, \end{aligned} \tag{84}$$

where we have used Eq. (15). Note that the term in the square bracket is odd under the exchange $p_1 \leftrightarrow -p_1$. Therefore the sum $I^{(1)}(p_1) + I^{(1)}(-p_1)$ consistently reproduces the duality relation (i.e., equivalently, it reproduces the two-point function $L^{(2)}(p_1, -p_1)$).

More generally, the linear relation in Eq. (83) implies that the dual integrals $I^{(N-1)}$ belong to the functional space, which is generated by the generalized one-loop integrals of Eq. (81)

Nonetheless, we have not yet established a one-to-one correspondence between single-cut and one-loop integrals. In fact, the correspondence in Eq. (83) is not invertible. The generalized one-loop integrals can be expressed in terms of single-cut integrals by a proper generalization of the duality relation in Eqs. (32) and (33). However, the single-cut integrals of this generalized relation involve the integration of both dual and advanced propagators.

The generalized duality relation is:

$$L^{(N)}(p_1, \alpha_1, p_2, \alpha_2, \dots, p_N, \alpha_N) = - \int_q \, \sum_{i=1}^{N} \, \widetilde{\delta}(q_i) \, \delta_{\alpha_i, F} \times \prod_{\substack{j=1 \\ j \neq i}}^{N} \left[\delta_{\alpha_j, F} \, \frac{1}{q_j^2 - i0\,\eta(q_j - q_i)} + \delta_{\alpha_j, A} \, G_A(q_j) \right] \,. \tag{85}$$

This result can be derived by applying the residue theorem (see Appendix A).

Alternatively, Eq. (85) can also be derived by applying an algebraic procedure similar to the one used in Sect. 6 to prove Eq. (63). This procedure consists of rewriting the right-hand side of Eqs. (81) and (85) as multiple-cut integrals of expressions involving only *advanced* propagators. The resulting expressions can be shown to agree with each other. The rewrite of Eqs. (81) and (85) is achieved by using Eq. (17) to replace Feynman and dual propagators with advanced propagators. More precisely, in the case of the dual propagators, Eqs. (17) and (82) give:

$$\widetilde{\delta}(q) \, \frac{1}{2qk + k^2 - i0 \, \eta k} = \widetilde{\delta}(q) \, \Big[G_A(q+k) - \theta(-\eta k) \, \widetilde{\delta}(q+k) \, \Big] \; . \tag{86}$$

To exemplify this algebraic procedure, we can explicitly show its application to the simple, though non-trivial, case of the one-loop integral $L^{(3)}(p_1, F, p_2, F, p_3, A)$. The right-hand side of Eq. (81) yields

$$\int_q G_A(q) \, G(q+p_1) \, G(q+p_1+p_2) = - \int_q G_A(q)$$
$$\times \Big[\widetilde{\delta}(q+p_1) \, G_A(q+p_1+p_2) + \widetilde{\delta}(q+p_1+p_2) \, G_A(q+p_1) - \widetilde{\delta}(q+p_1) \, \widetilde{\delta}(q+p_1+p_2) \Big] \, , \tag{87}$$

where we have also used Eq. (15). After using Eq. (86), the right-hand side of Eq. (85) reads

$$- \int_q G_A(q) \left[\widetilde{\delta}(q+p_1) \, \frac{1}{(q+p_1+p_2)^2 - i0 \, \eta p_2} + \widetilde{\delta}(q+p_1+p_2) \, \frac{1}{(q+p_1)^2 + i0 \, \eta p_2} \right]$$
$$= \; - \int_q G_A(q) \, \Big[\widetilde{\delta}(q+p_1) \Big(G_A(q+p_1+p_2) - \theta(-\eta p_2) \, \widetilde{\delta}(q+p_1+p_2) \Big)$$
$$+ \, \widetilde{\delta}(q+p_1+p_2) \Big(G_A(q+p_1) - \theta(\eta p_2) \, \widetilde{\delta}(q+p_1) \Big) \Big] \; . \tag{88}$$

By simple inspection, we see that the expressions in Eqs. (87) and (88) coincide.

The generalized duality in Eq. (85) relates one-loop integrals to single-cut phase-space integrals. Note that only the Feynman propagators of the loop integral are cut; the uncut Feynman propagators are instead replaced by dual propagators. The advanced propagators of the loop integral are not cut, and they appear unchanged in the integrand of the phase-space integral.

Moreover, the correspondence in Eq. (85) between one-loop and single-cut integrals is invertible. Using the same algebraic steps as in Eqs. (82) and (83), we indeed obtain:

$$\int_q \widetilde{\delta}(q) \left(\prod_{j=1}^{m} \frac{1}{2qk_j + k_j^2 - i0 \, \eta k_j} \right) \prod_{i=1}^{k} G_A(q+k_i)$$
$$= \int_q \Big(G_A(q) - G(q) \Big) \, \prod_{j=1}^{m} \, \Big[\theta(-\eta k_j) \, G(q+k_j) + \theta(\eta k_j) \, G_A(q+k_j) \Big] \prod_{i=1}^{k} G_A(q+k_i) \; . \tag{89}$$

The functional space generated by the generalized one-loop integrals is thus equivalent to the space generated by the single-cut integrals on the left-hand side of Eq. (89). The one-loop integrals of Feynman and advanced propagators and the single-cut integrals of dual and advanced propagators can therefore be regarded as equivalent dual basis of the same functional space.

8 Massive integrals, complex masses and unstable particles

As discussed at the end of Sect. 3, the introduction of particle masses and massive propagators does not lead to difficulties in the generalization of the FTT from the massless case. The same discussion and the same conclusions apply to the duality relation, since this relation can be derived by applying the residue theorem in close analogy with the derivation of the FTT. Therefore, as long as the mass is *real*, the effect of a particle mass M_i in the Feynman propagator of a loop internal line with momentum q_i amounts to modifying (according to the replacement in Eq. (26)) the corresponding on-shell delta function $\widetilde{\delta}(q_i)$ when this line is cut to obtain the dual representation $\widetilde{L}^{(N)}$ (see Eqs. (33) and (85)) of the loop integral $L^{(N)}$. Note also that the $i0$ prescription of the dual propagators is not affected by the masses. More precisely, if the Feynman propagator of the j-th internal line has mass M_j, the corresponding dual propagator is

$$\frac{1}{q_j^2 - M_j^2 - i0\,\eta(q_j - q_i)} \quad , \tag{90}$$

independently of the value M_i of the mass in the i-th line – the cut line.

In any unitary quantum field theory, the masses of the basic fields are real. If some of these fields describe unstable particles, a proper (physical) treatment of the corresponding propagators in perturbation theory requires a Dyson summation of self-energy insertions, which produces finite-width effects introducing *finite* imaginary contributions in the propagators. A typical form of the ensuing propagator G_C (such as the propagator used in the complex-mass scheme[†] [10]) is

$$G_C(q;s) = \frac{1}{q^2 - s} \ , \tag{91}$$

where s denotes the *complex* mass of the unstable particle:

$$s = \mathrm{Re}\, s + i\, \mathrm{Im}\, s \ , \quad \text{with} \quad \mathrm{Re}\, s > 0 > \mathrm{Im}\, s \quad . \tag{92}$$

These complex masses, together with complex couplings, are introduced in both tree-level and one-loop Feynman diagrams. A natural question that arises in the context of the present paper is whether the duality relation between one-loop and phase-space integrals (and the FTT, as well) can deal with complex-mass propagators or, more generally, with propagators of unstable particles. The answer to this question is positive, as we discuss below.

We consider a one-loop N-point scalar integral (see Eq. (9)) where one or more of the Feynman propagators of the internal lines are replaced by complex-mass propagators

[†]In the complex-mass scheme, unitarity can be perturbatively recovered (modulo higher-order terms) order by order.

$G_C(q_i; s_i)$. To derive a representation of this one-loop integral in terms of single-cut phase space integrals, we then apply the same procedure as in Sect. 4. The only difference is the presence of the complex-mass propagators. In the complex plane of the loop integration variable q_0, the complex-mass propagators produce poles that are located far off the real axis, the displacement being controlled by the finite imaginary part of the complex masses. Using the Cauchy theorem as in Eq. (27), we derive a duality relation that is analogous to Eq. (32). The only difference is that the the right-hand side of Eq. (32) has to be modified:

$$\widetilde{L}^{(N)}(p_1, p_2, \dots, p_N) \to \; \widetilde{L}^{(N)}(p_1, p_2, \dots, p_N) + \widetilde{L}_C^{(N)}(p_1, p_2, \dots, p_N) \; . \tag{93}$$

Here, $\widetilde{L}^{(N)}$ denotes the terms that correspond to the residues at the poles of the Feynman propagators of the loop integral, while $\widetilde{L}_C^{(N)}$ denotes those from the poles of the complex-mass propagators.

$\widetilde{L}^{(N)}$ is thus expressed as

$$\widetilde{L}^{(N)}(p_1, p_2, \dots, p_N) = \int_q \sum_{i \in F} \widetilde{\delta}(q_i; M_i) \left[\prod_{j \neq i} \; \cdots \right] \; , \tag{94}$$

where the sum refers to the internal lines i of the loop with a Feynman propagator (we use the notation $i \in F$ to denote these cut lines). The term in the square bracket denotes the product of the propagators of the uncut lines. The Feynman propagators of the loop are replaced by the corresponding dual propagators (as in Eq. (33)), while the complex-mass propagators are *unchanged*[‡].

The expression of $\widetilde{L}_C^{(N)}$ is similar to Eq. (94), but the cut lines i are those with complex-mass propagators (we use the notation $i \in C$ to denote these cut lines). Taken together

$$\begin{aligned} \widetilde{L}_C^{(N)}(p_1, p_2, \dots, p_N) &= \int_q \sum_{i \in C} \widetilde{\delta}(q_i; s_i) \left[\prod_{j \neq i} \; \cdots \right] \\ &= \int \frac{d^{d-1}\mathbf{q}}{(2\pi)^{d-1}} \sum_{i \in C} \frac{1}{2\sqrt{\mathbf{q}_i^2 + s_i}} \left[\prod_{j \neq i} \; \cdots \right]_{q_{i0} = \sqrt{\mathbf{q}_i^2 + s_i}} \; , \end{aligned} \tag{95}$$

where the term in the square bracket contains the propagators of the uncut lines. Note that in the integral representation on the first line of Eq. (95) the 'on-shell' delta function $\widetilde{\delta}(q_i; s_i)$ of the cut propagator has a formal meaning, since it singles out the residue at the complex-mass pole, $q_{i0} = q_{i0}^{(C,+)} = \sqrt{\mathbf{q}_i^2 + s_i}$, which has a *finite* (and negative) imaginary part. The explicit expression of $\widetilde{L}_C^{(N)}$ is thus given in the second line of Eq. (95). Owing to the finite imaginary component of $q_{i0}^{(C,+)}$, we can remove the $i0$ prescription in any of the Feynman propagators inside the square bracket.

The outcome of our discussion of the duality relation can also be used to explain how the FTT can be generalized to deal with complex-mass propagators of the internal lines.

[‡]The dual propagators arise from the infinitesimal $i0$ displacement produced by the residue at the pole of the Feynman propagator, see Sect. 4 and Appendix A. This infinitesimal imaginary displacement has no effect on the complex-mass propagators, owing to the finite imaginary part of the complex mass.

Following the derivation of the FTT in Sect. 3, we can replace the advanced one-loop integral $L_A^{(N)}$ of Eq. (14) with a one-loop integral that contains both advanced propagators and complex-mass propagators. This one-loop integral can be rewritten in two different ways. First (exploiting Eq. (17)), it can be expressed, as in the right-hand side of Eq. (19), in terms of a linear combination of the required one-loop integral (i.e. the integral with Feynman and complex-mass propagators) and of multiple-cut phase-space integrals $L^{(N)}_{\rm m-cut}$. Alternatively, it can be evaluated directly by applying the Cauchy theorem as in Eq. (16). This direct evaluation leads to the computation of the residues at the poles of the complex-mass propagators (the poles of the advanced propagators do not contribute, since they are placed outside the integration contour): the computation gives exactly the contribution in Eq. (95). Comparing the expressions obtained in these two ways, we conclude that the generalization of the FTT to include complex-mass propagators is realized by the following replacement in the right-hand side of Eq. (22):

$$L^{(N)}_{\rm 1-cut}(p_1,p_2,\dots,p_N) \to \; L^{(N)}_{\rm 1-cut}(p_1,p_2,\dots,p_N) + \widetilde{L}^{(N)}_C(p_1,p_2,\dots,p_N) \;\; . \tag{96}$$

Here, $L^{(N)}_{\rm 1-cut}$ is the usual contribution (see Eq. (20)) emerging from the single cuts of the sole Feynman propagators of the internal lines (the complex-mass propagators are not cut), while $\widetilde{L}^{(N)}_C$ is given by Eq. (95). Note, in particular, that the complex-mass propagators do *not* produce further m-cut contributions ($m \geq 2$) to the FTT in addition to the real-mass terms $L^{(N)}_{\rm m-cut}$ in Eq. (21).

We add a final comment on one-loop integrals with unstable internal particles. The propagator of an unstable particle can have a form that differs from the complex-mass propagator in Eq. (91). We can introduce, for instance, a complex mass, $s(q^2)$, that depends on the momentum q of the propagator. We can also include a non-resonant component, in addition to the resonant contribution of the complex-mass pole. Independently of its specific form, the propagator of the unstable particle produces singularities that are located at a *finite* imaginary distance from the real axis in the complex plane of the loop integration variable q_0. Such contributions can be included in the duality relation and in the FTT by performing the replacements in Eq. (93) and in Eq. (96), respectively. In general, the term $\widetilde{L}^{(N)}_C$ has a form that differs from Eq. (95) and depends on the actual expression of the propagator and, in particular, on the singularity structure (poles, branch cuts, ...) of the propagator in the complex plane.

9 Gauge theories and gauge poles

The quantization of gauge theories requires the introduction of a gauge-fixing procedure, which specifies the spin polarization vectors of the gauge bosons and the ensuing content of (possible) compensating fictitious particles (e.g. the Faddeev–Popov ghosts in unbroken non-Abelian gauge theories, or the would-be Goldstone bosons in spontaneously broken gauge theories).

The fictitious particles have their own Feynman propagators, which have to be taken into account when applying either the FTT or the duality relation. This is done in a straightforward manner: if some internal lines in a one-loop integral correspond to fictitious particles, they have to be cut exactly in the same way as for physical particles. The multiple-cut phase-space integrals of the FTT and the single-cut phase-space integral of the duality relation will include the contributions from the cuts of the Feynman propagators of these fictitious particles.

The impact of the propagators of the gauge particles is more delicate, since they introduce 'gauge poles'. This point is discussed below.

The propagator of the (spin 1) gauge boson with momentum q is obtained by multiplying the customary Feynman propagator $G(q)$ with the tensor $d^{\mu\nu}(q)$, which arises from the sum of the spin polarizations. The general form of the polarization tensor is

$$d^{\mu\nu}(q) = -g^{\mu\nu} + (\zeta - 1)\, \ell^{\mu\nu}(q)\, G_G(q) \quad . \tag{97}$$

The second term on the right-hand side is absent only in the 't Hooft–Feynman gauge ($\zeta = 1$). In any other gauge, this term is present and the tensor $\ell^{\mu\nu}(q)$ propagates longitudinal polarizations, which are proportional to q^μ, or q^ν, or $q^\mu q^\nu$. On the one hand, the specific form of $\ell^{\mu\nu}(q)$ is not relevant in the context of the following discussion; the only relevant point is that $\ell^{\mu\nu}(q)$ has a *polynomial* dependence on the momentum q. On the other hand, the factor $G_G(q)$ (we call it 'gauge-mode' propagator) has a potentially dangerous, non-polynomial dependence on q and, in particular, it produces poles with respect to the momentum variable q.

When considering one-loop quantities in gauge theories, we deal with one-loop integrals containing gauge boson propagators as internal lines of the loop. Therefore, to derive the FTT or the duality relation, we have to consider the effect produced by the gauge polarization tensors. In the *'t Hooft–Feynman gauge* the effect is *harmless*: the polarization tensor is simply $-g^{\mu\nu}$ and factorizes off the loop integration. When applying the Cauchy residue theorem as in Sects. 3 and 4 in any other gauge, we have to take into account the possible additional contributions that arise from the presence of the poles of the gauge-mode propagator $G_G(q)$ (the presence of polynomial terms from $\ell^{\mu\nu}(q)$ does not interfere with the residue theorem).

We first discuss the case of spontaneously broken gauge theories. Here, the gauge boson has a finite mass M, and the form of the gauge-mode propagator $G_G(q)$ is

$$G_G(q) = \frac{1}{\zeta(q^2 + i0) - M^2} \quad . \tag{98}$$

Considering the unitary gauge ($\zeta = 0$), the gauge-mode propagator does not depend on q and factorizes off the loop integration in any of the one-loop integrals. Therefore, the *unitary gauge* has only inconsequential implications on the use of the FTT and the duality relation for one-loop calculations in gauge theories. If we instead consider a generic renormalizable gauge (or R_ζ gauge) with $\zeta \neq 0$, we see that the gauge-mode propagator introduces a pole when $q^2 = M^2/\zeta - i0$. This is an additional pole with respect to the physical pole (when $q^2 = M^2 - i0$) from the associated Feynman propagator. For the extension of the FTT and the duality relation of Sects. 3 and 4 to one-loop computations in the R_ζ gauge, one has to properly consider the introduction of additional single-cut and multiple-cut contributions from gauge-mode propagators. We will not pursue this issue any further in the present paper.

We now discuss the case of unbroken gauge theories, where the gauge boson is massless. We separately consider two classes of gauges: covariant gauges and physical gauges.

In covariant gauges, we have

$$G_G(q) = \frac{1}{q^2 + i0} \quad . \tag{99}$$

Since the gauge-mode propagator $G_G(q)$ is equal to the Feynman propagator, the two propagators together generate a *second-order* pole when $q^2 = -i0$. The extension of the FTT and the duality relation of Sects. 3 and 4 to hold for one-loop computations in covariant gauges requires a proper treatment of the contributions from this type of second-order poles[§]. This issue is not pursued any further in the present paper.

In physical gauges, the typical form of the gauge-mode propagator is

$$G_G(q) = \frac{1}{(n \cdot q)^k} \quad , \qquad k = 1 \ \text{ or } \ 2 \ , \tag{100}$$

where n^μ denotes an auxiliary gauge vector. We see that $G_G(q)$ leads to a (first- or second-order) pole when $n \cdot q = 0$. In Coulomb gauge we have $n_\mu = (0, \mathbf{q})$, where $\mathbf{q}$ is the space component of the gauge boson momentum $q_\mu = (q_0, \mathbf{q})$. In axial ($n \cdot A = 0$) or planar gauges, n^μ is a fixed external vector and the pole has to be regularized according to a proper prescription (the precise position of the pole has to be specified by some imaginary displacement from the real axis), which we do not specify here, since its specific form has no effect on the discussion that follows.

We now consider a generic one-loop integral, whose integrand contains gauge-mode propagators in addition to Feynman propagators. To derive a duality relation by using the residue theorem in the complex plane of the variable q_0 (as in Sect. 4), we have to take into account the possible contributions from the poles of the gauge-mode propagators.

In Coulomb gauge, the pole of $G_G(q)$ is located at $\mathbf{q}^2 = 0$. Applying the residue theorem in the q_0 plane at fixed values of $\mathbf{q}$ (see Sect. 4 and Appendix A), the gauge pole does not contribute. We conclude that the gauge-mode propagator remains *untouched* in going from the one-loop integral to its representation as a single-cut dual integral. Note, however, that this conclusion follows from having kept $\mathbf{q}$ fixed while performing the integration over q_0. Therefore, the auxiliary future-like vector η^μ of the duality relation is necessarily *fixed* (see Appendix A) to be $\eta_\mu = (\eta_0, \mathbf{0})$, i.e. aligned along the time direction.

In axial or planar gauges, the pole of $G_G(q)$ is located at $nq = n_0 q_0 - n_{d-1} q_{d-1} = 0$. Without loosing generality, we can assume $n_\mu = (n_0, \mathbf{0}_\perp, n_{d-1})$ and apply (see Sect. 4) the residue theorem in the complex plane q_0 at fixed values of the coordinates $\mathbf{q}_\perp$ and $q'_{d-1} = q_{d-1} - q_0 \eta_{d-1}/\eta_0$. Setting $\eta_{d-1}/\eta_0 = n_0/n_{d-1}$, we have $nq = -n_{d-1} q'_{d-1}$. Hence, $G_G(q)$ does not depend on the integration variable q_0. We conclude that the gauge-mode propagator, *including* the regularization prescription of its gauge pole, is *untouched* in going from the one-loop integral to its representation as a single-cut dual integral. Note, however, that we have set $\eta_{d-1}/\eta_0 = n_0/n_{d-1}$. Therefore, since the vector η^μ specifying the dual prescription is future-like, the above conclusion is valid only if the gauge vector

[§]Of course, this does not apply to the 't Hooft–Feynman gauge, where $G_G(q)$ is absent.

n^μ is either *space-like* or *light-like* ($n^2 \leq 0$) and, moreover, the dual vector is fixed to be *orthogonal* to the gauge vector, $n \cdot \eta = 0$. These requirements are not fulfilled if n^μ is time-like¶. The derivation of the duality relation in time-like gauges requires to properly include contributions from cuts of the gauge-polarization tensors (these contributions depend on the specific regularization of the gauge poles): this derivation is beyond the scope of this paper.

Our discussion and conclusions regarding the duality relation in physical gauges can straightforwardly be used to draw similar conclusions on the validity of the FTT. The only difference is that in the latter case there is no auxiliary dual vector η^μ. To be precise, in Coulomb gauge and in space-like or light-like gauges, the FTT is valid in its customary form, without introducing any multiple-cut contributions stemming from the gauge-polarization tensors. In time-like gauges, the poles of the gauge-polarization tensors can play a role, and their effect has to be taken into account when applying the FTT.

10 Loop-tree duality at the amplitude level

In the final part of Sect. 3, we have discussed how the FTT can be extended to evaluate not only basic one-loop integrals $L^{(N)}$ but also complete one-loop quantities (such as Green's functions and scattering amplitudes). The same reasoning (see also Sects. 8 and 9) applies to the extension of the duality relation to the amplitude level.

The analogue of Eq. (25) is the following duality relation:

$$\mathcal{A}^{(1-\mathrm{loop})} = - \, \widetilde{\mathcal{A}}^{(1-\mathrm{loop})} \quad , \tag{101}$$

where $\mathcal{A}^{(1-\mathrm{loop})}$ generically denotes a one-loop quantity. The expression $\widetilde{\mathcal{A}}^{(1-\mathrm{loop})}$ on the right-hand side of Eq. (101) is obtained in the same way as $\widetilde{L}^{(N)}$ in Eqs. (32) and (33). We start from any Feynman diagram in $\mathcal{A}^{(1-\mathrm{loop})}$ and consider all possible replacements of each Feynman propagator $G(q_i)$ of its loop internal lines with the cut propagator $\widetilde{\delta}(q_i; M_i)$; the uncut Feynman propagators in the loop are then replaced by the corresponding dual propagators.

The duality relation (101) is valid in any field theory that is unitary and local. Some words of caution are, however, needed (see the conclusions of Sect. 9) about its applicability to theories with local gauge symmetries. In spontaneously broken gauge theories, the duality relation is valid in the 't Hooft–Feynman gauge and in the unitary gauge. In unbroken gauge theories, the duality relation is valid in the 't Hooft–Feynman gauge; it is also valid in physical gauges specified by a gauge vector n^ν, provided the auxiliary duality vector η^μ is chosen such that $n \cdot \eta = 0$ (this excludes gauges where n^ν is time-like).

Equation (101) establishes a correspondence between one-loop Feynman diagrams and the phase-space integral of tree-level Feynman diagrams. The right-hand side of Eq. (101)

¶For example, in the axial gauge $A_0 = 0$, we have $nq = n_0 q_0$, and the pole of the gauge-mode propagator does not decouple from the integration over q_0.

can be written in the following sketchy form:

$$\mathcal{A}^{(1-\text{loop})} \sim \int_q \sum_P \widetilde{\delta}(q; M_P) \sum_{\text{d.o.f.}(P)} \mathcal{A}_P^{(\text{tree})} \ , \tag{102}$$

where $\sum_P$ denotes the sum over the particles that can propagate in the loop internal lines that are cut, and $\sum_{\text{d.o.f.}(P)}$ denotes the sum over the degrees of freedom (such as spin, colors, ..) of the particle P. The integrand $\mathcal{A}_P^{(\text{tree})}$ is given by the sum of the tree-level Feynman diagrams that are obtained by cutting the one-loop Feynman diagrams on the left-hand side.

The structure of Eq. (102) implies a natural question[‖]. If $\mathcal{A}^{(1-\text{loop})}$ is the one-loop expression of a specific quantity $\mathcal{A}$, how is $\mathcal{A}_P^{(\text{tree})}$ related to the tree-level expression of the same quantity $\mathcal{A}$? In the next subsections, we show how the duality relation can be formulated directly at the amplitude level, when the quantity $\mathcal{A}$ is a Green's function. We also discuss the case of on-shell scattering amplitudes.

10.1 Green's functions

In the following, $\mathcal{A}_N(p_1, \dots, p_N)$ denotes a generic off-shell Green's function with N external lines (the outgoing momentum of the i-th line is p_i). To be precise, we consider Green's functions that are *connected* and *amputated* of the free propagators of the external lines. The tree-level and one-loop expressions of $\mathcal{A}$ are $\mathcal{A}^{(\text{tree})}$ and $\mathcal{A}^{(1-\text{loop})}$, respectively. The tree-level scattering amplitude for a given physical process is obtained from $\mathcal{A}^{(\text{tree})}(p_1, \dots, p_N)$ by setting the external momenta on their physical mass shell ($p_i^2 = M_i^2$, $p_{i0} \geq 0$ for an outgoing particle, $-p_{i0} \geq 0$ for an incoming particle) and including the appropriate wave-function factors of the external particles. The one-loop scattering amplitude is obtained from $\mathcal{A}^{(1-\text{loop})}$ by specifying the renormalization procedure.

To simplify the illustration of the duality relation, we first consider the case with only one type of massive scalar particles. We thus refer to a theory with a single real scalar field ϕ ($\phi^* = \phi$) of mass M. The particles are self-interacting through polynomial interactions (e.g. ϕ^3 or ϕ^4). In this case, the duality relation (102) has the following explicit form:

$$\mathcal{A}_N^{(1-\text{loop})}(p_1, \dots, p_N) = +\frac{1}{2} \int \frac{d^d q}{(2\pi)^{d-1}} \, \delta_+(q^2 - M^2) \, \widetilde{\mathcal{A}}_{N+2}^{(\text{tree})}(q, -q, p_1, \dots, p_N) \ , \tag{103}$$

where the integrand factor $\mathcal{A}^{(\text{tree})}$ on the right-hand side is exactly the tree-level counterpart of the one-loop quantity $\mathcal{A}_N^{(1-\text{loop})}$ on the left-hand side. The tree-level counterpart $\mathcal{A}_{N+2}^{(\text{tree})}$ involves two additional external lines with outgoing momenta q and $-q$.

The tilde superscript in $\widetilde{\mathcal{A}}^{(\text{tree})}$ denotes the replacement of Feynman propagators with dual propagators. More precisely, to obtain $\widetilde{\mathcal{A}}^{(\text{tree})}(q, -q, \dots)$ from $\mathcal{A}^{(\text{tree})}(q, -q, \dots)$, we assign a dual propagator (rather than a Feynman propagator) to each internal line with momentum $q + k_j$ (k_j is a linear combination of the external momenta p_i). We note that this

[‖]Issues related to similar questions were discussed by Feynman [2] in the context of the FTT.

step can also be performed by using a short-cut recipe, namely by applying the momentum shift $q^\mu \to q^\mu - i0\,\eta^\mu/(2\eta q)$ in the Feynman propagators of $\mathcal{A}^{(\text{tree})}(q,-q,\dots)$.

The momenta q and $-q$ of the two additional external lines of $\mathcal{A}^{(\text{tree})}_{N+2}(q,-q,\dots)$ in Eq. (103) are on their physical mass-shell: in this respect, $\mathcal{A}^{(\text{tree})}_{N+2}$ is a scattering amplitude (there are no wave-function factors for scalar particles). More precisely, $\widetilde{\mathcal{A}}^{(\text{tree})}_{N+2}(q,-q,\dots)$ is the tree-level physical amplitude that corresponds to the *forward-scattering process* of a particle with momentum q in the external field produced by N self-interacting sources (the N external legs).

In a theory with different types of particles and antiparticles, the generalization of Eq. (103) is obtained by including a sum over the particle types P. We find:

$$\mathcal{A}^{(1-\text{loop})}_N(\dots) = +\frac{1}{2}\int \frac{d^dq}{(2\pi)^{d-1}} \sum_P \delta_+(q^2-M_P^2)\;\sigma(P)\;\widetilde{\mathcal{A}}^{(\text{tree})}_{N+2}(P(q)\leftarrow P(q),\dots)\;\;, \tag{104}$$

where the momenta p_i of N external legs are denoted by 'dots', since they play no active role on both sides of the equation. Note that $\sum_P$ includes the sum over *both* particles and antiparticles (if $P \neq \overline{P}$). The coefficient $\sigma(P)$ on the right-hand side of Eq. (104) is a Bose–Fermi statistics factor: $\sigma(P) = +1$ if P is a bosonic particle (e.g. spin 0 Higgs boson, spin 1 gauge boson), and $\sigma(P) = -1$ if P is a fermionic particle (e.g. spin 1/2 fermion, Faddeev–Popov ghost).

The tree-level expression $\mathcal{A}^{(\text{tree})}_{N+2}(P(q)\leftarrow P(q),\dots)$ is the amplitude for the forward-scattering process $P(q) \to P(q)$ in the field of the N external legs. It can be written as

$$\mathcal{A}^{(\text{tree})}_{N+2}(P(q)\leftarrow P(q),\dots) = \sum_{\text{spin, color},..} \langle P(q)\,|\,\mathcal{A}^{(\text{tree})}_{N+2}(P(q),\overline{P}(-q),\dots)\,|\,P(q)\,\rangle\;, \tag{105}$$

where the ('ket' and 'bra') vectors $|\,P(q)\,\rangle$ and $\langle P(q)\,|$ generically denote the (spin-dependent, color-dependent, ...) wave-function factors of the forward-scattered particle P.

We illustrate the general notation in Eq. (105) with a few explicit examples:

- $P=$ gluon (λ labels the spin-polarization or helicity states; μ,ν are Lorentz indices; a,b are color indices) yields

$$\begin{aligned}\mathcal{A}^{(\text{tree})}_{N+2}(g(q)\leftarrow g(q),\dots) &= \sum_\lambda \sum_{\mu,\nu} \sum_{a,b} \left(\varepsilon^{(\lambda)}_\mu(q)\right)^* \left[\mathcal{A}^{(\text{tree})}_{N+2}(g(q),g(-q),\dots)\right]^{\mu\nu}_{ab} \varepsilon^{(\lambda)}_\nu(q)\\ &= \sum_{\mu,\nu} d_{\mu\nu}(q) \sum_{a,b} \left[\mathcal{A}^{(\text{tree})}_{N+2}(g(q),g(-q),\dots)\right]^{\mu\nu}_{ab}\;,\end{aligned} \tag{106}$$

  where $\varepsilon^{(\lambda)}_\nu(q)$ is the gluon-polarization vector;

- $P=$ massive quark (s labels the spin; α,β are Dirac indices; i,j are color indices) yields

$$\begin{aligned}\mathcal{A}^{(\mathrm{tree})}_{N+2}(Q(q) \leftarrow Q(q), \dots) &= \sum_{s=1,2} \sum_{\alpha,\beta} \sum_{i,j} \overline{u}^{(s)}_{\alpha}(q) \left[\mathcal{A}^{(\mathrm{tree})}_{N+2}(Q(q), \overline{Q}(-q), \dots)\right]^{ij}_{\alpha\,\beta} u^{(s)}_{\beta}(q) \\ &= \mathrm{Tr}\left[(\not{q} + M) \sum_{i,j} \left[\mathcal{A}^{(\mathrm{tree})}_{N+2}(Q(q), \overline{Q}(-q), \dots)\right]^{ij}\right] , \qquad (107)\end{aligned}$$

where $u^{(s)}_{\beta}(q)$ is the customary Dirac spinor for spin 1/2 fermions;

- P= massive anti-quark (s labels the spin; α, β are Dirac indices; i, j are color indices) yields

$$\begin{aligned}\mathcal{A}^{(\mathrm{tree})}_{N+2}(\overline{Q}(q) \leftarrow \overline{Q}(q), \dots) &= -\sum_{s=1,2} \sum_{\alpha,\beta} \sum_{i,j} \overline{v}^{(s)}_{\alpha}(q) \left[\mathcal{A}^{(\mathrm{tree})}_{N+2}(Q(-q), \overline{Q}(q), \dots)\right]^{ij}_{\alpha\,\beta} v^{(s)}_{\beta}(q) \\ &= -\,\mathrm{Tr}\left[(\not{q} - M) \sum_{i,j} \left[\mathcal{A}^{(\mathrm{tree})}_{N+2}(Q(-q), \overline{Q}(q), \dots)\right]^{ij}\right] , \qquad (108)\end{aligned}$$

where $v^{(s)}_{\beta}(q)$ is the customary Dirac spinor for spin 1/2 anti-fermions.

Note that, as stated below Eq. (104), we sum over both particles and antiparticles. However, on the right-hand side of Eq. (104), $\sum_P$ can equivalently be defined to just refer to the sum over particles. According to this alternative definition, the antiparticle contribution $\widetilde{\mathcal{A}}^{(\mathrm{tree})}_{N+2}(\overline{P}(q) \leftarrow \overline{P}(q), \dots)$ is absent, and the corresponding particle contribution $\widetilde{\mathcal{A}}^{(\mathrm{tree})}_{N+2}(P(q) \leftarrow P(q), \dots)$ must be multiplied by a factor of 2.

10.2 Scattering amplitudes

To extend the discussion of Sect. 10.1 to scattering amplitudes, the only relevant point to be examined is the on-shell limit of the corresponding Green's functions (the introduction of the wave-function factors of the external lines is straightforward).

Considering the off-shell Green's function $\mathcal{A}^{(1-\mathrm{loop})}_{N}$, we introduce the following decomposition:

$$\mathcal{A}^{(1-\mathrm{loop})}_{N} = \mathcal{A}^{(1-\mathrm{loop;\,ex.})}_{N} + \mathcal{A}^{(1-\mathrm{loop;\,in.})}_{N} \ , \qquad (109)$$

where $\mathcal{A}^{(1-\mathrm{loop;\,ex.})}_{N}$ is the contribution from one-loop insertions on the N external lines, while $\mathcal{A}^{(1-\mathrm{loop;\,in.})}_{N}$ is the remaining contribution (i.e. one-loop insertions on internal lines). In explicit form, we have

$$\mathcal{A}^{(1-\mathrm{loop;\,ex.})}_{N}(p_1, \dots, p_N) = \sum_{j=1}^{N} \mathcal{A}^{(1-\mathrm{loop})}_{2}(p_j, -p_j) \frac{i\, D_j(p_j)}{p_j^2 - M_j^2 + i0} \mathcal{A}^{(\mathrm{tree})}_{N}(p_1, \dots, p_N) \qquad (110)$$

where $D_j(p_j)$ is the spin-polarization factor of the particle in the internal line with momentum p_j.

As is well known, $\mathcal{A}^{(1-\mathrm{loop;\,ex.})}_{N}$ cannot directly be evaluated on-shell owing to the kinematical singularity arising from its external-line propagators (the propagators with momentum p_j in Eq. (110)). Thus, to calculate the one-loop scattering amplitude, $\mathcal{A}^{(1-\mathrm{loop;\,ex.})}_{N}$

has to be first evaluated off-shell, then it has to be renormalized (mass and wave-function renormalization), before considering its on-shell limit.

In contrast, the one-loop contribution $\mathcal{A}_N^{(1-\text{loop; in.})}$ can directly be computed in the on-shell limit. In particular, we can write a duality relation in the form of Eq. (101):

$$\mathcal{A}_N^{(1-\text{loop; in.})} = - \,\widetilde{\mathcal{A}}_N^{(1-\text{loop; in.})} \quad . \tag{111}$$

Here, the integrand of the phase-space integral on the right-hand side contains a sum of *on-shell* tree-level Feynman diagrams (the N external lines are on-shell, and the two additional lines from cutting the loop are also on-shell). The algebraic computation of the integrand is thus completely analogous to the computation of the (on-shell) tree-level scattering amplitude with $N+2$ external legs. Having performed the tree-level computation of the integrand, the result can be integrated over the single-particle phase-space to obtain the full one-loop term $\mathcal{A}_N^{(1-\text{loop; in.})}$.

We point out that the integrand of the phase-space integral on the right-hand side of Eq. (111) is *not* equal (modulo the replacement of Feynman with dual propagators) to the tree-level scattering amplitude with $N+2$ external legs. This is because a subset of the diagrams that enter the complete tree-level scattering amplitude is not included. This subset has been removed by considering only $\mathcal{A}_N^{(1-\text{loop; in.})}$, i.e. by removing $\mathcal{A}_N^{(1-\text{loop; ex.})}$ from the complete one-loop expression $\mathcal{A}_N^{(1-\text{loop})}$.

This 'missing' subset of tree-level diagrams can be reinserted in the duality relation. However, as discussed below, this makes more delicate the on-shell limit.

We consider the internal-line contribution $\mathcal{A}_N^{(1-\text{loop; in.})}$ before setting the external lines on-shell. We can write the following duality relation:

$$\begin{aligned} \mathcal{A}_N^{(1-\text{loop; in.})}(p_1, \dots, p_N) &= + \frac{1}{2} \int \frac{d^d q}{(2\pi)^{d-1}} \sum_P \delta_+(q^2 - M_P^2)\; \sigma(P) \\ \times \Big\{ & \widetilde{\mathcal{A}}_{N+2}^{(\text{tree})}(P(q) \leftarrow P(q), p_1, \dots, p_N) \\ & - \sum_{j=1}^{N} \widetilde{\mathcal{A}}_4^{(\text{tree})}(P(q) \leftarrow P(q), p_j, -p_j) \frac{i\, D_j(p_j)}{p_j^2 - M_j^2 + i0} \mathcal{A}_N^{(\text{tree})}(p_1, \dots, p_N) \Big\} \; . \end{aligned} \tag{112}$$

The derivation of this equation is simple. We first use Eq. (109) to express $\mathcal{A}_N^{(1-\text{loop; in.})}$ as difference of $\mathcal{A}_N^{(1-\text{loop})}$ and $\mathcal{A}_N^{(1-\text{loop; ex.})}$. Then we use Eq. (110) to rewrite $\mathcal{A}_N^{(1-\text{loop; ex.})}$ in terms of $\mathcal{A}_2^{(1-\text{loop})}$. Finally, we express the full one-loop Green's functions $\mathcal{A}_N^{(1-\text{loop})}$ and $\mathcal{A}_2^{(1-\text{loop})}$ in terms of the duality relation (104).

The duality relation (112) involves the phase-space integration of complete tree-level Green's functions, namely $\mathcal{A}_N^{(\text{tree})}(p_1, \dots, p_N)$, and (the duality-propagator version of) $\mathcal{A}_{N+2}^{(\text{tree})}(P(q) \leftarrow P(q), p_1, \dots, p_N)$ and $\mathcal{A}_4^{(\text{tree})}(P(q) \leftarrow P(q), p_j, -p_j)$. The integrand factor in the curly bracket on the right-hand side is well defined in the on-shell limit. However, the two terms in the curly bracket are separately singular in the on-shell limit. The singularity is purely kinematical; it simply arises from the propagators of the lines with momenta equal to the momenta p_j of the external lines. Various procedures can be devised to introduce an intermediate regularization of the separate singularities, so as to directly evaluate the two terms close to on-shell kinematical configurations.

11 Final remarks

Applying directly the Cauchy residue theorem in the complex plane of any of the space-time coordinates of the loop momentum we have derived a duality relation between one-loop integrals and single-cut phase-space integrals. The calculation of the residues is elementary, but introduces several subtleties. The location in the complex plane of the pole of the cut propagator modifies the original $+i0$ Feynman prescription of the uncut propagators. One-loop integrals are then written as a linear combination of N single-cut phase-space integrals, with propagators regularized by a new complex Lorentz-covariant prescription, named dual prescription. It is defined through a future-like auxiliary vector η. This simple modification compensates for the absence of multiple-cut contributions that appear in the FTT. The dependence on η cancels, as expected, in the sum of all the single-cut contributions, leading to η-independent results.

We have generalized the duality relation for internal massive propagators and unstable particles. Real masses just modify the position of the poles in the complex plane by a translation parallel to the real axis, and thus do not affect the dual prescription. Unstable particles introduce a finite imaginary contribution in their propagators. The poles of the complex-mass propagators are located at a finite imaginary distance from the real axis, and the $+i0$ prescription of the usual Feynman propagators can be removed when propagators of unstable particles are cut.

Particular care has to be taken with gauge propagators in both the FTT and the duality relation owing to the presence of unphysical extra gauge poles. We have discussed this issue, and have identified the different gauge choices where the duality relation can be applied in its original form.

Finally, we have extended the duality relation from Feynman integrals to Green's functions and scattering amplitudes. One-loop scattering amplitudes can be obtained starting from tree-level scattering amplitudes (or, more precisely, from Feynman diagrams that enter the computation of tree-level scattering amplitudes), where (some of) the internal propagators are replaced by dual propagators. This tree-level counterpart is then integrated over a single-particle phase space to get the one-loop scattering amplitude. Since efficient methods for the numerical calculation of tree-level amplitudes exist and have been automated, the duality relation is also suitable for the automation of the numerical evaluation of one-loop amplitudes. The numerical evaluation can also be extended to the level of physical cross-sections at next-to-leading order [3, 5], since the single-particle phase space integration of the duality relation can directly be combined with the integration over the multi-particle phase-space of the physical process. The duality relation can also be used to obtain one-loop results in analytic form, starting from corresponding tree-level results.

The extension of the duality relation from one-loop to two-loop Feynman diagrams is under investigation [5].

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A Appendix: Derivation of the duality relation

In Sect. 4 we have illustrated the derivation of the duality relation in Eqs. (32) and (33) by using the residue theorem. The derivation is simple. However, it involves some subtle points. These points are discussed in detail in this Appendix.

Applying the residue theorem in the complex plane of the variable q_0, the computation of the one-loop integral $L^{(N)}$ reduces to the evaluation of the residues at N poles, according to Eqs. (27) and (28).

The evaluation of the residues in Eq. (28) is a key point in the derivation of the duality relation. To make this point as clear as possible, we first introduce the notation $q_{i0}^{(+)}$ to explicitly denote the location of the i-th pole, i.e. the location of the pole with negative imaginary part (see Eq. (12)) that is produced by the propagator $G(q_i)$. We further simplify our notation with respect to the explicit dependence on the subscripts that label the momenta. We write $G(q_j) = G(q_i + (q_j - q_i))$, where q_i depends on the loop momentum while $(q_j - q_i) = k_{ji}$ is a linear combination of the external momenta (see Eq. (2)). Therefore, to carry out the explicit computation of the i-th residue in Eq. (28), we re-label the momenta by $q_i \to q$ and $q_j \to q + k_j$, and we simply evaluate the term

$$\left[\mathrm{Res}_{\{q_0 = q_0^{(+)}\}} \, G(q) \right] \, \left[\prod_j \, G(q + k_j) \right]_{q_0 = q_0^{(+)}} \quad , \tag{113}$$

where (see Eq. (12))

$$q_0^{(+)} = \sqrt{\mathbf{q}^2 - i0} \ \ . \tag{114}$$

In the next paragraphs, we follow the steps of Sect. 4 (see Eqs. (29) and (30)) and we separately compute the residue of $G(q)$ and its prefactor – the associated factor arising from the propagators $G(q + k_j)$.

The computation of the residue of $G(q)$ gives

$$\begin{aligned} \mathrm{Res}_{\{q_0=q_0^{(+)}\}}\, G(q) &= \lim_{q_0\,\to\, q_0^{(+)}} \left\{ (q_0 - q_0^{(+)})\, \frac{1}{q_0^2 - \mathbf{q}^2 + i0} \right\} = \frac{1}{2\, q_0^{(+)}} \\ &= \frac{1}{2\sqrt{\mathbf{q}^2}} = \int dq_0\; \delta_+(q^2) \;\; , \end{aligned} \tag{115}$$

thus leading to the result in Eq. (29). Note that the first equality in the second line of Eq. (115) is obtained by removing the $i0$ prescription from the previous expression. This is fully justified. The term $(q_0^{(+)})^{-1} = (\sqrt{\mathbf{q}^2 - i0})^{-1}$ becomes singular when $\mathbf{q}^2 \to 0$, and this corresponds to an end-point singularity in the integration over $\mathbf{q}$: therefore the $i0$ prescription has no regularization effect on such end-point singularity. The second equality in the second line of Eq. (115) simply follows from the definition of the on-shell delta function $\delta_+(q^2)$.

We now consider the evaluation of the residue prefactor (the second square-bracket factor in Eq. (113)). We first recall that the $i0$ prescription of the Feynman propagators has played an important role in the application (see Eqs. (27) and (113)) of the residue theorem to the computation of the loop integral: having selected the pole with negative imaginary part, $q_0 = q_0^{(+)}$, the prescription eventually singled out the on-shell mode with positive definite energy, $q_0 = |\mathbf{q}|$ (see Eq. (115)). However, we observe that the result in Eq. (115) can be obtained by removing (neglecting) the $i0$ prescription either in $q_0^{(+)}$ ($q_0^{(+)} \to |\mathbf{q}|$) or in $G(q)$ ($G(q) \to 1/q^2$):

$$\mathrm{Res}_{\{q_0=q_0^{(+)}\}}\, G(q) = \mathrm{Res}_{\{q_0=|\mathbf{q}|\}}\, \frac{1}{q^2} = \int dq_0\; \delta_+(q^2) \;\; . \tag{116}$$

Hence, the $i0$ prescription has no effect on the actual calculation of the residue of the propagator $G(q)$ in Eq. (113). On the basis of this observation, we might assume that the $i0$ prescription also has no effect on the calculation of the residue prefactor in Eq. (113), since the propagators $G(q+k_j)$ are not singular when evaluated at the poles of $G(q)$. We might thus compute the residue prefactor by removing the $i0$ prescription; under this assumption we obtain

$$\left[\prod_j G(q+k_j) \right]_{q_0=q_0^{(+)}} \to \left[\prod_j \frac{1}{(q+k_j)^2} \right]_{q_0=|\mathbf{q}|} \;\; . \tag{117}$$

The expression on the right-hand side of Eq. (117) is well-defined, but, when inserted (through Eqs. (113) and (28)) in Eq. (27), it leads to an ill-defined result: the integration over $\mathbf{q}$ is singular at any phase-space points where the denominator factors $(q+k_j)^2$ vanish. To recover a well-defined result, we have to reintroduce the $i0$ prescription in the residue prefactor. We might thus maintain the $i0$ prescription in the Feynman propagators $G(q+k_j)$ and still keeping q_0 at its on-shell value $q_0 = |\mathbf{q}|$; then we obtain

$$\left[\prod_j G(q+k_j) \right]_{q_0=q_0^{(+)}} \to \left[\prod_j \frac{1}{(q+k_j)^2 + i0} \right]_{q_0=|\mathbf{q}|} \;\; . \tag{118}$$

Inserting (through Eqs. (113) and (28)) Eq. (115) and the right-hand side of Eq. (118) into Eq. (27), we arrive at a well-defined result for the one-loop integral, since the singularities from the propagators $1/(q+k_j)^2$ are now regularized by the Feynman $i0$ prescription. However, this result for the one-loop integral is exactly equal (see Eqs. (20) and (22)) to the sole 1-cut contribution, $L_{1-\mathrm{cut}}$, of the FTT. The ensuing contradiction with the FTT can be resolved only if the total contribution from multiple cuts, $L_{2-\mathrm{cut}} + L_{3-\mathrm{cut}} + \ \ldots$, to the FTT vanishes; this is obviously unlikely, and it is actually not true as shown by the explicit one-loop calculations performed in Sect. 5.

The discussion of the previous paragraph illustrates that the evaluation of the one-loop integrals by the direct application of the residue theorem (as in Eq. (27)) involves some subtleties. The subtleties mainly concern the correct treatment of the Feynman $i0$ prescription in the calculation of the residue prefactors. A consistent treatment requires the *strict* computation of the residue prefactor in Eq. (113): the $i0$ prescription in both $G(q+k_j)$ and $q_0^{(+)}$ has to be dealt with by considering the imaginary part $i0$ as a *finite* (thus, for instance, $2i0 \neq i0$), though possibly small, quantity; the limit of infinitesimal values of $i0$ has to be taken only at the *very* end of the computation, thus leading to the interpretation of the ensuing $i0$ prescription as mathematical distribution. Applying this strict procedure, we obtain

$$\left[\prod_j G(q+k_j)\right]_{q_0=q_0^{(+)}} = \left[\prod_j \frac{1}{(q+k_j)^2+i0}\right]_{q_0=q_0^{(+)}} = \prod_j \frac{1}{2q_0^{(+)}k_{j0} - 2\mathbf{q}\cdot\mathbf{k}_j + k_j^2}$$
$$= \prod_j \frac{1}{2|\mathbf{q}|k_{j0} - 2\mathbf{q}\cdot\mathbf{k}_j + k_j^2 - i0k_{j0}/|\mathbf{q}|} = \left[\prod_j \frac{1}{2qk_j + k_j^2 - i0k_{j0}/q_0}\right]_{q_0=|\mathbf{q}|} . \tag{119}$$

The last equality on the first line of Eq. (119) simply follows from setting $q_0 = q_0^{(+)}$ in the expression on the square-bracket (note, in particular, that $q^2 = -i0$). The first equality on the second line follows from $2q_0^{(+)} \simeq 2|\mathbf{q}| - i0/|\mathbf{q}|$ (i.e. from expanding $q_0^{(+)}$ at small values of $i0$).

The result in Eq. (119) for the residue prefactor is well-defined and leads to a well-defined (i.e. non singular) expression once it is inserted in Eq. (27). The possible singularities from each of the propagators $1/(q+k_j)^2$ are regularized by the displacement produced by the associated imaginary amount $i0k_{j0}/q_0$. Performing the limit of infinitesimal values of $i0$, only the sign of the $i0$ prescription (and not its actual magnitude) is relevant. Therefore, since q_0 is positive, in Eq. (119) we can perform the replacement $i0k_{j0}/q_0 \to i0\,\eta k_j$, where η^μ is the vector $\eta^\mu = (\eta_0, \mathbf{0})$ with $\eta_0 > 0$; we finally obtain

$$\left[\prod_j G(q+k_j)\right]_{q_0=q_0^{(+)}} = \left[\prod_j \frac{1}{(q+k_j)^2 - i0\,\eta k_j}\right]_{q_0=|\mathbf{q}|} , \tag{120}$$

which is the result in Eq. (30) (To be precise, Eq. (30) is recovered by reintroducing the original labels of the momenta of the loop integral according to the replacements $q \to q_i$, $k_j \to q_j - q_i$, see the discussion above Eq. (113)).

In the following we explain in more detail the origin of the η dependence in the $i0$ prescription of the dual propagators. The explicit calculation performed in this Appendix leads to the introduction of the future-like vector $\eta^\mu = (\eta_0, \mathbf{0})$ (see Eqs. (119) and (120)). As discussed in Sect. 4, different future-like vectors can be introduced by applying the residue theorem in different systems of coordinates. To clarify this point, we explicitly show the application of the residue theorem in light-cone coordinates (see Eq. (6)) rather than in space-time coordinates (as in Eq. (27)). The one-loop integral can then be evaluated as follows:

$$\begin{aligned} L^{(N)}(p_1, p_2, \dots, p_N) &= \int_{(q_-, \mathbf{q}_\perp)} \int dq_+ \prod_{i=1}^{N} G(q_i) \\ &= -2\pi i \int_{(q_-, \mathbf{q}_\perp)} \sum \mathrm{Res}_{\{\mathrm{Im}\; q_+ < 0\}} \left[\prod_{i=1}^{N} G(q_i) \right] \;, \end{aligned} \tag{121}$$

where we have applied the residue theorem by closing the integration contour at ∞ in the lower half-plane of the complex variable q_+ (see Figs. 2 and 3). We can now compute the residues in Eq. (121) by closely following the analogous computation in Eqs. (113), (115) and (119).

The analogue of the term in Eq. (113) is

$$\left[\mathrm{Res}_{\{q_+ = q_+^{(+)}\}} \, G(q) \right] \left[\prod_j G(q + k_j) \right]_{q_+ = q_+^{(+)}} \;, \tag{122}$$

where $q_+^{(+)}$ denotes the location (in the q_+ plane) of the pole with negative imaginary part that is produced by the propagator $G(q)$. Thus (see Eq. (12)), we have

$$q_+^{(+)} = \frac{\mathbf{q}_\perp{}^2 - i0}{2q_-} \;, \quad \text{with} \quad q_- > 0 \;, \tag{123}$$

where the requirement of negative imaginary part leads to the constraint $q_- > 0$.

The computation of the residue of $G(q)$ gives

$$\begin{aligned} \mathrm{Res}_{\{q_+ = q_+^{(+)}\}} \, G(q) &= \theta(q_-) \lim_{q_+ \to q_+^{(+)}} \left\{ (q_+ - q_+^{(+)}) \frac{1}{2q_+ q_- - \mathbf{q}_\perp{}^2 + i0} \right\} \\ &= \theta(q_-) \, \frac{1}{2q_-} = \int dq_+ \; \delta_+(q^2) \;. \end{aligned} \tag{124}$$

We see that the residue produces the same factor as in Eq. (115).

The residue prefactor is evaluated by using the same procedure as in Eqs. (119) and (120). We obtain

$$\begin{aligned} \left[\prod_j G(q + k_j) \right]_{q_+ = q_+^{(+)}} &= \prod_j \frac{1}{2q_+^{(+)} k_{j-} + 2q_- k_{j+} - 2\mathbf{q}_\perp \cdot \mathbf{k}_{\perp j} + k_j^2} \\ = \left[\prod_j \frac{1}{2qk_j + k_j^2 - i0 k_{j-}/q_-} \right]_{q_+ = \mathbf{q}_\perp{}^2/q_-} &= \left[\prod_j \frac{1}{(q + k_j)^2 - i0 \, \eta k_j} \right]_{q_+ = \mathbf{q}_\perp{}^2/q_-} \;. \end{aligned} \tag{125}$$

The last equality in this equation has been found by performing the limit of infinitesimal values of $i0$, analogously to Eq. (120). Since q_- is positive, we have thus implemented the replacement $i0k_{j-}/q_- \to i0\,\eta k_j$ where, in the present case, we have introduced the future-like vector $\eta^\mu = (\eta_+, \mathbf{0}_\perp, \eta_- = 0)$ with $\eta_+ = \eta_0\sqrt{2} > 0$.

It is important to note that, owing to the on-shell condition $\delta_+(q^2)$, Eqs. (120) and (125) have the same form, although the corresponding auxiliary vectors η^μ are different: though $\eta_0 > 0$ in both equations, η is time-like ($\eta^2 > 0$) in Eq. (120), whereas it is light-like ($\eta^2 = 0$) in Eq. (125).

We also note that the use of the residue theorem in the complex plane q_0 at fixed values of q_- and $\mathbf{q}_\perp$ leads to a residue prefactor with exactly the same light-like vector η^μ as in Eq. (125).

The main features of the calculation presented in this Appendix are very general: they are valid in any system of coordinates that can be used to apply the residue theorem. The residue of $G(q)$ always replaces the Feynman propagator with the corresponding on-shell propagator $\delta_+(q^2)$ (see Eqs. (29), (115) and (124)); the residue prefactor generates dual propagators with an auxiliary vector η that depends on the specific system of coordinates that has been actually employed (see Eqs. (30), (120) and (125)).

We conclude this Appendix by briefly describing the derivation (by means of the residue theorem) of the generalized duality relation stated in Eq. (85). The generalized one-loop integral on the left-hand side contains both Feynman and advanced propagators. Before applying the residue theorem, we can specify how the infinitesimal limit '$i0 \to 0$' is performed in the two different types of propagators. We rewrite the advanced propagator as $G_A(q) = [\,q^2 - i\rho\,\mathrm{sign}(q_0)\,]^{-1}$ and, evaluating the one-loop integral, we perform first the limit $i0 \to 0$ (at fixed ρ) in the Feynman propagators and then the limit $i\rho \to 0$ in the advanced propagators. We apply the residue theorem by closing the integration contour at ∞ in the lower half-plane of the complex variable q_0, such that the poles of the advanced propagators do not contribute. Performing the limit $i0 \to 0$, the Feynman propagators behave exactly as in the case of the duality relation in Eqs. (32) and (33), while the advanced propagators remain unchanged (since ρ is kept finite). Finally, we perform the infinitesimal limit $i\rho \to 0$. We thus obtain Eq. (85), whereas the advanced propagators have not been altered by going from the one-loop integral on the left-hand side to the phase-space integral on the right-hand side.

B Appendix: An algebraic relation

Here, we provide a proof of the relation (71). More generally, we consider a set of n real variables λ_i, with $i = 1, 2, \dots, n$, that fulfill the constraint

$$\sum_{i=1}^{n} \lambda_i = 0 \quad . \tag{126}$$

We shall prove the following relation:

$$\theta(\lambda_1)\,\theta(\lambda_1+\lambda_2)\,\dots\,\theta(\lambda_1+\lambda_2+\cdots+\lambda_{n-1}) + \text{cyclic perms.} = 1 \quad . \tag{127}$$

Equation (71) simply follows from setting $\lambda_i = \eta p_i$ and is just a consequence of momentum conservation, namely Eq. (126). Note that the future-like nature of the vector η plays no role in Eq. (71).

To simplify the notation we define $\theta(i,j) \equiv \theta(\lambda_i+\lambda_{i+1}+\ldots+\lambda_j)$; with λ_i taken modulo n: $\lambda_{n+i} = \lambda_i$. The relation (127) is equivalent to

$$F_n(\lambda_1, \cdots, \lambda_n) = \sum_{i=1}^{n} \theta(i)\theta(i,i+1)\cdots\theta(i,i+n-2) = 1 \ . \tag{128}$$

Then, we proceed by induction. Assuming that Eq. (128) is valid for $n-1$ real variables we shall prove that it is valid for n.

We select the last two variables λ_{n-1} and λ_n, and consider separately the two cases: $\theta(n-1,n) = 1$ and $\theta(n-1,n) = 0$. If $\theta(n-1,n) = 1$, the first term in the sum of Eq. (128) vanishes because $\theta(1,n-2) = 0$ owing to momentum conservation. For the second and subsequent terms and as far as $i \leq n-2$, we will have $\theta(i,n-2)\theta(i,n) = \theta(i,n-2)$. The $n-1$–th term is proportional to $\theta(n-1,n) = 1$ and hence is the product of $n-2$ theta functions as well. Then we apply Eq. (128) with the set of $n-1$ elements $\{\lambda_1, \cdots, \lambda_{n-2}, \lambda_{n-1}+\lambda_n\}$, where most of the terms cancel each other by applying Eq. (128) with $n-1$ elements, and we find

$$F_n(\lambda_1, \cdots, \lambda_n) = 1 + \theta(n,1)\theta(n,2)\cdots\theta(n,n-2)(\theta(n)-1) \ . \tag{129}$$

If λ_n is positive then $F_n = 1$ as we wanted to demonstrate. If λ_n is negative then $\theta(n,n-2) = 0$ because $\lambda_{1,n-2}$ is also negative, leading to $F_n = 1$ again.

In the other case, namely for $\theta(n-1,n) = 0$, the $n-1$–th term of the sum in Eq. (128) obviously vanishes, and the first term of that sum is proportional to $\theta(1,n-2) = 1$. Then we apply Eq. (128) to the first term with the set of $n-1$ elements $\{\lambda_1, \cdots, \lambda_{n-2}+\lambda_{n-1}, \lambda_n\}$ and find

$$\begin{aligned} F_n(\lambda_1, \cdots, \lambda_n) &= 1 + \sum_{i=2}^{n-3} \theta(i)\cdots\theta(i,n-3)\theta(i,n-1)\cdots\theta(i,i+n-2)(\theta(i,n-2)-1) \\ &+ \theta(n-2,n-1)\theta(n-2,n)\cdots\theta(n-2,n-4)(\theta(n-2)-1) \\ &+ \theta(n)\theta(n,1)\cdots\theta(n,n-3)(\theta(n,n-3)-1) \ , \end{aligned} \tag{130}$$

but owing to momentum conservation

$$\begin{aligned} &\theta(i,n)(\theta(i,n-2)-1) = -\theta(n-1,i-1)\theta(i,n) = 0 \ , \\ &\theta(n-2,n)(\theta(n-2)-1) = -\theta(n-1,n-3)\theta(n-2,n) = 0 \ , \\ &\theta(n,n-3)(\theta(n,n-3)-1) = -\theta(n,n-3)\theta(n-2,n-1) = 0 \ , \end{aligned} \tag{131}$$

and thus $F_n = 1$. This completes the proof of Eq. (128).

References

[1] R. P. Feynman, Acta Phys. Polon. **24** (1963) 697.

[2] R. P. Feynman, *Closed Loop And Tree Diagrams,* in *Magic Without Magic*, ed. J. R. Klauder, (Freeman, San Francisco, 1972), p. 355, in *Selected papers of Richard Feynman*, ed. L. M. Brown (World Scientific, Singapore, 2000) p. 867.

[3] S. Catani, presented at the Workshop *HP^2: High Precision for Hard Processes at the LHC*, Sept. 2006, Zurich, Switzerland (http : //www–theorie.physik.unizh.ch/research_groups/particle/hp2/); G. Rodrigo, presented at the *International Linear Collider (ILC) Workshop (ILC-ECFA and GDE Joint Meeting)*, Nov. 2006, Valencia, Spain (http : //ific.uv.es/ ilc/ECFA–GDE2006/); T. Gleisberg, presented at the Conference *Heidelberg 07 (DPG Spring Meeting)*, March 2007, Heidelberg, Germany (http : //www.dpg–tagungen.de/program/heidelberg/).

[4] T. Gleisberg, PhD Thesis, University of Dresden.

[5] S. Catani, T. Gleisberg, F. Krauss, G. Rodrigo and J. Winter, in preparation.

[6] A. Brandhuber, B. Spence and G. Travaglini, JHEP **0601** (2006) 142.

[7] E. Witten, Commun. Math. Phys. **252** (2004) 189.

[8] F. Cachazo, P. Svrcek and E. Witten, JHEP **0410** (2004) 074.

[9] See Z. Bern, L. J. Dixon and D. A. Kosower, Annals Phys. **322** (2007) 1587 and references therein.

[10] A. Denner, S. Dittmaier, M. Roth and D. Wackeroth, Nucl. Phys. B **560** (1999) 33; A. Denner, S. Dittmaier, M. Roth and L. H. Wieders, Nucl. Phys. B **724** (2005) 247.