Vorticity constraints on a fluid/fluid interface

Cesar Dopazo

Fluid Mechanics Group, School of Engineering, University of Zaragoza, Maria de Luna, 3, 50015 Zaragoza, Spain and LITEC/CSIC, Maria de Luna, 3, 50015 Zaragoza, Spain

Antonio Lozano

LITEC/CSIC, Maria de Luna, 3, 50015 Zaragoza, Spain

Felix Barreras

Fluid Mechanics Group, School of Engineering, University of Zaragoza, Maria de Luna, 3, 50015 Zaragoza, Spain and LITEC/CSIC, Maria de Luna, 3, 50015 Zaragoza, Spain

(Received 28 October 1999; accepted 24 April 2000)

General relations among the components of the strain rate tensors and those of the tangential vorticities on the two sides of a liquid/gas interface are derived; kinematic constraints as well as the tangential-stress balance at the interface are used. For small gas to liquid dynamic viscosity ratios compared to unity simple expressions relating the liquid tangential vorticity components to the tangential velocity component perpendicular to them, the interface curvatures and the normal velocity surface-gradient components are obtained. Starting from the customary Eulerian vorticity equation, a transport equation for the vortex sheet strength is obtained. © 2000 American Institute of Physics. [S1070-6631(00)02208-X]

I. INTRODUCTION

Vorticity patterns near liquid/gas interfaces may be rather complicated. Wind-generated water waves and air-assisted atomization of liquids are but two examples in which the vortical flows on the two sides of a water/air interface interact and give rise to intricate patterns. As an example, in a liquid/gas mixing layer behind a thick trailing edge there is some evidence that the gas vortex shedding frequency is closely related to the interface oscillation frequency and that, under some conditions, the gas vorticity field induces the interfacial instability, leading to the subsequent breakup of filaments.

The vorticity dynamics on liquid/gas interfaces has been a subject of research for many years, with interest increasing recently.¹⁻⁴ The relation between the vorticity vectors on the two sides of a liquid/gas interface is of relevance, should the vorticity field be treated as a primary variable. The continuity of the component of vorticity normal to the interface is a consequence of the definition of the vorticity and the no-slip boundary condition.³ However, the relation among the tangential vorticity components has received little attention.

The notion of boundary-wall vorticity flux as defined by Lighthill⁵ has been generalized to fluid/fluid interfaces first by Lugt,⁶ and later used by Wu³ and Lundgren and Koumoutsakos.⁷ The two-dimensional (2-D) relations among liquid tangential vorticity, velocity, and curvature of a steady stress-free surface has been derived by Longuet-Higgins⁸ and cited, for example in Batchelor;⁹ the three-dimensional (3-D) version of this relation has been obtained by Lundgren¹⁰ for steady flow and extended by Wu³ to unsteady flow using the surface strain rate and general surface coordinates; recently, Longuet-Higgins¹¹ has derived the 3-D form in terms of the strain-rate tensor using orthogonal curvilinear coordinates.

Interfaces have been treated as vortex sheets in 2-D¹² and in 3-D¹³ and an inviscid vortex-sheet-strength transport equation has been derived. The extension of the vortex sheet notion to viscous flows has recently been presented.⁷ These formulations, however, have ignored the fact that the strain rate tensor presents a discontinuity across the fluid/fluid interface.

In this Brief Communication new exact relations among the tangential vorticity components involving the components of the strain rate tensors on the two sides of a liquid/gas interface are derived; new interfacial generalized expressions, analogous to those of Batchelor⁹ and Longuet-Higgins¹¹ for a single fluid, relating the liquid tangential vorticity components to the air tangential velocity components, normal interface curvatures, and normal air velocity gradient along the surface are obtained. Taking into account these results, a transport equation for the vortex sheet strength is derived starting from the customary vorticity equation and integrating over thin layers adjacent to the interface.

II. VORTICITY AND STRAIN RATE TENSOR RELATIONSHIPS ON AN INTERFACE

Let $\mathbf{r}$ be an infinitesimal fluid element on an interface, $S(\mathbf{x}, t) = 0$, separating fluid 1 from fluid 2. The time rate of change of $\mathbf{r}$ is expressed as

$$\frac{d\mathbf{r}}{dt} = \vec{s} \cdot \mathbf{r} + \frac{1}{2} \boldsymbol{\omega} \times \mathbf{r}, \quad (1)$$

where $\vec{s}$ is the strain rate tensor and $\boldsymbol{\omega}$ is the vorticity vector. $\mathbf{r}$ can indistinctly be considered as a part of either fluid 1 or 2. Equating (1) for the two fluids leads to

FIG. 1. Sketch of the interface.

$$(\vec{s}^{(2)} - \vec{s}^{(1)}) \cdot \mathbf{r} + \frac{1}{2}(\boldsymbol{\omega}^{(2)} - \boldsymbol{\omega}^{(1)}) \times \mathbf{r} = 0, \quad (2)$$

where $\vec{s}^{(\alpha)}$ and $\boldsymbol{\omega}^{(\alpha)}$ are the strain rate tensor and the vorticity vector, respectively, for fluid α ($\alpha=1,2$), at the interface.

For incompressible fluids 1 and 2,

$$s_{jj}^{(1)} = 0 = s_{jj}^{(2)}. \quad (3)$$

For Newtonian fluids, $\tau'_{ij} = 2\mu s_{ij}$, with constant surface tension, T , the balance of tangential stresses at the interface yields

$$\begin{aligned} 2\mu^{(2)}[\vec{s}^{(2)} \cdot \mathbf{n} - (\mathbf{n} \cdot \vec{s}^{(2)} \cdot \mathbf{n})\mathbf{n}] \\ = 2\mu^{(1)}[\vec{s}^{(1)} \cdot \mathbf{n} - (\mathbf{n} \cdot \vec{s}^{(1)} \cdot \mathbf{n})\mathbf{n}], \end{aligned} \quad (4)$$

where $\mu^{(\alpha)}$ is the dynamic viscosity coefficient of fluid α and $\mathbf{n}$ is the unit vector normal to the interface, pointing from fluid 1 to fluid 2.

A system of orthogonal curvilinear coordinates, (ξ_1, ξ_2, ξ_3) , as defined by Batchelor,⁹ is shown in Fig. 1. $\mathbf{a}$ and $\mathbf{b}$ are unit vectors along ξ_1 and ξ_2 , respectively, on a plane tangent to $S(\mathbf{x}, t) = 0$.

The definition of the vorticity vector, $\boldsymbol{\omega} = \nabla \times \mathbf{u}$, together with the no slip boundary condition at the interface, $\mathbf{u}^{(1)} = \mathbf{u}^{(2)}$, directly implies that

$$\omega_3^{(1)} = \omega_3^{(2)}, \quad (5)$$

where $\omega_3^{(\alpha)} = \boldsymbol{\omega}^{(\alpha)} \cdot \mathbf{n}$ is the vorticity component normal to the interface, namely,

$$\omega_3^{(\alpha)} = \frac{1}{h_1 h_2} \left(\frac{\partial h_2 u_2^{(\alpha)}}{\partial \xi_1} - \frac{\partial h_1 u_1^{(\alpha)}}{\partial \xi_2} \right), \quad (6)$$

where h_1 and h_2 are the positive scale factors along ξ_1 and ξ_2 , respectively.

Restricting (2) for $\mathbf{r} = \mathbf{a}$ and $\mathbf{r} = \mathbf{b}$, respectively, and using (3) and (4) leads to the new relations

FIG. 2. Vortex sheet elementary volume.

$$\vec{s}^{(1)} = \begin{pmatrix} s_{11}^{(2)} & s_{12}^{(2)} & \frac{\mu^{(2)}}{\mu^{(1)}} s_{13}^{(2)} \\ s_{12}^{(2)} & s_{22}^{(2)} & \frac{\mu^{(2)}}{\mu^{(1)}} s_{23}^{(2)} \\ \frac{\mu^{(2)}}{\mu^{(1)}} s_{13}^{(2)} & \frac{\mu^{(2)}}{\mu^{(1)}} s_{23}^{(2)} & -(s_{11}^{(2)} + s_{22}^{(2)}) \end{pmatrix}, \quad (7)$$

$$\omega_1^{(1)} = \omega_1^{(2)} + 2 \left(1 - \frac{\mu^{(2)}}{\mu^{(1)}} \right) s_{23}^{(2)}, \quad (8)$$

$$\omega_2^{(1)} = \omega_2^{(2)} + 2 \left(\frac{\mu^{(2)}}{\mu^{(1)}} - 1 \right) s_{13}^{(2)}. \quad (9)$$

Equation (7) indicates that the strain rate tensor for fluid 1 is fully determined by the five independent components of the strain rate tensor for fluid 2 and by the viscosity ratio, $\mu^{(2)}/\mu^{(1)}$. On the other hand, the vorticity vector of fluid 1 is completely known, should the vorticity vector of fluid 2, the strain rate tensor components $s_{13}^{(2)}$ and $s_{23}^{(2)}$, as well as the viscosity ratio, be prescribed. If $\mu^{(2)} = \mu^{(1)}$, then $\boldsymbol{\omega}^{(1)} = \boldsymbol{\omega}^{(2)}$.

For an air/water interface at ambient temperature $\mu^{(2)}/\mu^{(1)} = 0.016$; then $\omega_1^{(1)} \approx \omega_1^{(2)} + 2s_{23}^{(2)}$ and $\omega_2^{(1)} \approx \omega_2^{(2)} - 2s_{13}^{(2)}$. Depending on the signs of $\omega_1^{(2)}$, $\omega_2^{(2)}$, $s_{13}^{(2)}$, $s_{23}^{(2)}$, the water tangential vorticity at an interface may be either larger or smaller than that on the air side. Writing the explicit expressions for $\boldsymbol{\omega}$ and $\vec{s}$, it is easy to show that

$$\omega_1^{(1)} \approx \frac{2}{h_2} \frac{\partial u_3^{(2)}}{\partial \xi_2} - \frac{2u_2^{(2)}}{h_2 h_3} \frac{\partial h_2}{\partial \xi_3}, \quad (10)$$

$$\omega_2^{(1)} \approx -\frac{2}{h_1} \frac{\partial u_3^{(2)}}{\partial \xi_1} + \frac{2u_1^{(2)}}{h_1 h_3} \frac{\partial h_1}{\partial \xi_3}, \quad (11)$$

where the second terms on the right-hand sides of (10) and (11) are related to the normal curvatures of the ξ_1 and ξ_2 axis, $K_1 = -(1/h_1 h_3)(\partial h_1 / \partial \xi_3)$ and $K_2 = -(1/h_2 h_3) \times (\partial h_2 / \partial \xi_3)$, respectively.¹¹

III. INTERFACIAL VORTEX SHEET DYNAMICS

Taking the curl of the momentum equation,

$$\frac{D\mathbf{u}}{Dt} = \mathbf{A} = -\frac{1}{\rho} \nabla p + \mathbf{g} - \nu \nabla \times \boldsymbol{\omega}, \quad (12)$$

the vorticity transport equation is readily written as

$$\frac{D}{Dt} \boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla) \mathbf{u} + \nabla \times \mathbf{A}. \quad (13)$$

Integration of (13) over a fluid volume, V , containing the interface (Fig. 2) and use of the Reynolds transport theorem yields

$$\frac{D}{Dt} \int_V \boldsymbol{\omega} dV = \int_V (\boldsymbol{\omega} \cdot \nabla) \mathbf{u} dV + \int_V (\nabla \times \mathbf{A}) dV. \quad (14)$$

If the vorticity on the two sides of an interface were concentrated within thin vortex sheets of infinitesimal thicknesses, δ_1 and δ_2 , V would consist of two volumes, $V_1 = \delta_1 dS$ and $V_2 = \delta_2 dS$, as shown in Fig. 2. dV can be expressed as $dS dn$, dn being the length differential along the normal to the interface. Therefore

$$\begin{aligned} \int_V \boldsymbol{\omega} dV &= \int_{V_1} \boldsymbol{\omega}^{(1)} dV + \int_{V_2} \boldsymbol{\omega}^{(2)} dV \\ &= dS \left(\int_{-\delta_1}^0 \boldsymbol{\omega}^{(1)} dn + \int_0^{\delta_2} \boldsymbol{\omega}^{(2)} dn \right) \\ &= dS (\boldsymbol{\gamma}^{(1)} + \boldsymbol{\gamma}^{(2)}) = \boldsymbol{\gamma} dS, \end{aligned} \quad (15)$$

where $\boldsymbol{\gamma}^{(1)} = \int_{-\delta_1}^0 \boldsymbol{\omega}^{(1)} dn$ and $\boldsymbol{\gamma}^{(2)} = \int_0^{\delta_2} \boldsymbol{\omega}^{(2)} dn$ are the vortex sheet strengths on fluids 1 and 2, respectively. The total vortex sheet strength vector is $\boldsymbol{\gamma} = \boldsymbol{\gamma}^{(1)} + \boldsymbol{\gamma}^{(2)}$.

The left-hand side of (14) can then be expanded as

$$\frac{D}{Dt} (\boldsymbol{\gamma} dS) = dS \frac{D \boldsymbol{\gamma}}{Dt} + \boldsymbol{\gamma} \frac{D(dS)}{Dt}. \quad (16)$$

From purely kinematic considerations

$$\frac{D(dS)}{Dt} = -dS \{ [(\mathbf{n} \cdot \nabla) \mathbf{u}] \cdot \mathbf{n} - \nabla \cdot \mathbf{u} \}. \quad (17)$$

In the coordinate system of Fig. 1, the right-hand side of (17) can be expressed as

$$dS (S_{11} + S_{22}) = \frac{dS}{h_1 h_2} \left(\frac{\partial h_2 u_1}{\partial \xi_1} + \frac{\partial h_1 u_2}{\partial \xi_2} \right) - dS u_3 K, \quad (18)$$

where $K = -\nabla \cdot \mathbf{n} = K_1 + K_2$ is the mean curvature.

Superscripts are not required in (18), since the functions multiplying dS are continuous at the interface.

The first term on the right-hand side of (14) can be sequentially rewritten as

$$\begin{aligned} \int_V (\boldsymbol{\omega} \cdot \nabla) \mathbf{u} dV &= \int_V \vec{s} \cdot \boldsymbol{\omega} dV \\ &= dS \left(\int_{-\delta_1}^0 \vec{s}^{(1)} \cdot \boldsymbol{\omega}^{(1)} dn \right. \\ &\quad \left. + \int_0^{\delta_2} \vec{s}^{(2)} \cdot \boldsymbol{\omega}^{(2)} dn \right). \end{aligned} \quad (19)$$

Although $\vec{s}$ is discontinuous at the interface, as shown in Sec. II, and, therefore, the vorticity fields on the two sides of the interface can, in general, undergo different stretching rates, $\boldsymbol{\omega}^{(1)}$ and $\boldsymbol{\omega}^{(2)}$ are, in the vortex sheet approximation,

tangent to the interface. Equation (19) thus only involves the components of $\vec{s}^{(1)}$ and $\vec{s}^{(2)}$ which are continuous across the interface and can be finally expressed in its customary form $dS(\vec{s} \cdot \boldsymbol{\gamma}) = dS(\boldsymbol{\gamma} \cdot \nabla) \mathbf{u}$.

The last term on the right hand side of (14) can be rephrased as

$$\int_V \nabla \times \mathbf{A} dV = \int_S \mathbf{n} \times \mathbf{A} dS = -\mathbf{n} \times [[\mathbf{A}]] dS, \quad (20)$$

where $[[\mathbf{A}]] = \mathbf{A}^{(1)} - \mathbf{A}^{(2)}$ is the fluid acceleration jump across the vortex sheet. From (12), one can readily obtain

$$\begin{aligned} -\mathbf{n} \times [[\mathbf{A}]] &= \mathbf{n} \times \left[\left[\frac{1}{\rho} \nabla p \right] \right] + [[\nu \nabla \boldsymbol{\omega} \cdot \mathbf{n}]] - (\boldsymbol{\sigma}^{(1)} \\ &\quad + \boldsymbol{\sigma}^{(2)}). \end{aligned} \quad (21)$$

The last term is the sum of interfacial vorticity fluxes, given by

$$\begin{aligned} (-1)^{\alpha-1} \boldsymbol{\sigma}^{(\alpha)} &= \nu^{(\alpha)} \mathbf{n} \cdot \nabla \boldsymbol{\omega}^{(\alpha)} \\ &= \nu^{(\alpha)} (\nabla \boldsymbol{\omega}^{(\alpha)}) \cdot \mathbf{n} + \mathbf{n} \\ &\quad \times \left(\mathbf{A}^{(\alpha)} + \frac{1}{\rho^{(\alpha)}} \nabla p^{(\alpha)} - \mathbf{g} \right), \end{aligned} \quad (22)$$

particularized at $S(\mathbf{x}, t) = 0$, and corresponds to the net vorticity creation rate on the interface.³

Commonly, the use of the transport equation for $\boldsymbol{\gamma}$ is restricted to the case of inviscid fluids and only the first term on the right-hand side of (21) is retained.^{7,12,13} It can easily be seen³ that

$$\mathbf{n} \times \left[\left[\frac{1}{\rho} \nabla p \right] \right] = \mathbf{n} \times \nabla \left[\left[\frac{p}{\rho} \right] \right] \quad (23)$$

and that

$$\left[\left[\frac{p}{\rho} \right] \right] = \frac{[[p]]}{\bar{\rho}(1+A)} - \frac{2A}{\bar{\rho}(1-A^2)} p^{(2)}, \quad (24)$$

where $\bar{\rho} = (\rho_1 + \rho_2)/2$, $A = (\rho_1 - \rho_2)/2\bar{\rho}$ is the Atwood ratio and $[[p]] = p^{(1)} - p^{(2)}$ is given by the balance of normal stresses at the interface, namely,

$$[[p]] = -2(\mu^{(1)} - \mu^{(2)})(S_{11} + S_{22}) - TK. \quad (25)$$

Assuming that the surface tension coefficient, T , is constant on the interface, one obtains for inviscid fluids

$$\nabla \left[\left[\frac{p}{\rho} \right] \right] = \frac{T}{\bar{\rho}(1+A)} \nabla K - \frac{2A}{\bar{\rho}(1-A^2)} \nabla p^{(2)}. \quad (26)$$

Putting (16) and (26) together leads to

$$\begin{aligned} \frac{D \boldsymbol{\gamma}}{Dt} &= -(S_{11} + S_{22}) \boldsymbol{\gamma} + (\boldsymbol{\gamma} \cdot \nabla) \mathbf{u} + \mathbf{n} \\ &\quad \times \left[-\frac{T}{\bar{\rho}(1+A)} \nabla K + \frac{2A \rho^{(2)}}{\bar{\rho}(1-A^2)} (\mathbf{A}^{(2)} - \mathbf{g}) \right], \end{aligned} \quad (27)$$

which is analogous to (64) in Wu,³ where the nomenclature is slightly changed.

IV. CONCLUSIONS

Simple and exact relations among the components of the strain rate tensors and those of the tangential vorticities on the two sides of a liquid/gas interface have been derived. The gas to liquid dynamic viscosity ratio enters the expressions. These relations can be used as boundary conditions for vorticity-based formulations or as consistency checks in numerical simulations.

A vortex sheet-strength transport equation has been derived in a simple manner for the case where the vorticities on the two sides of an interface can be assumed to be concentrated within thin layers adjacent to it.

ACKNOWLEDGMENTS

The authors are indebted to the referee for pointing out some relevant references and the correct sequence of historical developments presented in Sec. I; his remarks were also essential to obtaining the final form of (27).

¹S. I. Green, "Introduction to vorticity," in *Fluid Vortices*, edited by S. I. Green (Kluwer, Dordrecht, 1995), pp. 1–34.

²E. P. Rood, "Vorticity interactions with a free surface," in Ref. 1, pp. 687–730.

³J. Z. Wu, "A theory of three-dimensional interfacial vorticity dynamics," *Phys. Fluids* **7**, 2375 (1995).

⁴T. Sarpkaya, "Vorticity, free surface and surfactants," *Annu. Rev. Fluid Mech.* **28**, 83 (1996).

⁵M. J. Lighthill, "Introduction: Boundary layer theory," in *Laminar Boundary Layer*, edited by L. Rosenhead (Oxford University Press, Oxford, 1963), pp. 46–113.

⁶H. J. Lugt, "Local flow properties at a viscous free surface," *Phys. Fluids* **30**, 3647 (1987).

⁷T. Lundgren and P. Koumoutsakos, "On the generation of vorticity at a free surface," *J. Fluid Mech.* **382**, 351 (1999).

⁸M. S. Longuet-Higgins, "Mass transport in water waves," *Philos. Trans. R. Soc. London, Ser. A* **245**, 535 (1953).

⁹G. K. Batchelor, *An Introduction to Fluid Dynamics* (Cambridge University Press, Cambridge, 1967).

¹⁰T. S. Lundgren, "A free surface vortex method with weak viscous effects," in *Mathematical Aspects of Fluid Dynamics*, edited by R. E. Caflish (SIAM, Philadelphia, 1989), pp. 68–79.

¹¹M. S. Longuet-Higgins, "Vorticity and curvature at a free surface," *J. Fluid Mech.* **356**, 149 (1998).

¹²G. Tryggvason, "Numerical simulation of the Rayleigh–Taylor instability," *J. Comput. Phys.* **75**, 253 (1988).

¹³A. Lozano, A. García-Olivares, and C. Dopazo, "The instability growth leading to a liquid sheet breakup," *Phys. Fluids* **10**, 2188 (1998).