Tau–Decay Determination of the Strange Quark Mass

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The recent ALEPH measurements of the inclusive Cabibbo–suppressed decay width of the τ and several moments of its invariant mass distribution are used to determine the value of the strange quark mass. We obtain, in the $\overline{\text{MS}}$ scheme, $m_s(M_\tau^2) = (119 \pm 24)$ MeV, which corresponds to $m_s(1\,\text{GeV}^2) = (164 \pm 33)$ MeV, $m_s(4\,\text{GeV}^2) = (114 \pm 23)$ MeV.

1. INTRODUCTION

The precise numerical value of the strange quark mass is a controversial issue, with important implications for low–energy phenomenology. The Particle Data Group [1] quotes a rather wide range of m_s values, reflecting the large uncertainties in the present determinations of this parameter from QCD Sum Rules and Lattice calculations.

The high precision data on tau decays [2] collected at LEP and CESR provide a very powerful tool to analyse strange quark mass effects in a cleaner environment. The QCD analysis of the inclusive tau decay width,

$$R_\tau \equiv \frac{\Gamma\left[\tau^- \to \nu_\tau + \text{hadrons}\,(\gamma)\right]}{\Gamma\left[\tau^- \to e^-\,\overline{\nu}_e\,\nu_\tau(\gamma)\right]}, \qquad (1)$$

has already made possible [3] an accurate measurement of the strong coupling constant at the τ mass scale, $\alpha_s(M_\tau^2)$, which complements and competes in accuracy with the high precision measurements of $\alpha_s(M_Z^2)$ performed at LEP. More recently, detailed experimental studies of the Cabibbo–suppressed width of the τ have started to become available [4,5], allowing to perform a systematic investigation of the corrections induced by the strange quark mass in the τ decay width [6–8].

What makes a m_s determination from τ data very interesting is that the hadronic input does not depend on any extra hypothesis; it is a purely experimental issue, which accuracy can be systematically improved. The major part of the uncertainty will eventually come from the theoretical side. However, owing to its inclusive character, the total Cabibbo–suppressed tau decay width can be rigorously analyzed within QCD, using the Operator Product Expansion (OPE). Therefore, the theoretical input is in principle under control and the associated uncertainties can be quantified.

2. THEORETICAL FRAMEWORK

The theoretical analysis of the inclusive hadronic tau decay width [9–12] involves the two–point correlation functions

$$\Pi_{ij,\mathcal{J}}^{\mu\nu}(q) \equiv i \int d^4x\, e^{iqx}\, \langle 0|T\left(\mathcal{J}_{ij}^\mu(x)\,\mathcal{J}_{ij}^\nu(0)^\dagger\right)|0\rangle$$

for the vector, $\mathcal{J}_{ij}^\mu = V_{ij}^\mu(x) \equiv \overline{q}_j \gamma^\mu q_i$, and axial–vector, $\mathcal{J}_{ij}^\mu = A_{ij}^\mu(x) \equiv \overline{q}_j \gamma^\mu \gamma_5 q_i$, colour–singlet quark currents $(i, j = u, d, s)$. These correlators have the Lorentz decompositions

$$\begin{aligned}
\Pi_{ij,V/A}^{\mu\nu}(q) &= \left(-g^{\mu\nu}\, q^2 + q^\mu q^\nu\right) \Pi_{ij,V/A}^T(q^2) \\
&\quad + q^\mu q^\nu\, \Pi_{ij,V/A}^L(q^2), \qquad (2)
\end{aligned}$$

where the superscript in the transverse and longitudinal components denotes the corresponding angular momentum $J = 1$ (T) and $J = 0$ (L) in the hadronic rest frame.

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(k,l)	$\mathcal{F}^{kl}_{L+T}(x)$	$\mathcal{F}^{kl}_L(x)$
(0,0)	$(1-x)^3\,(1+x)$	$(1-x)^3$
(1,0)	$\frac{1}{10}\,(1-x)^4\,(7+8x)$	$\frac{3}{4}\,(1-x)^4$
(2,0)	$\frac{2}{15}\,(1-x)^5\,(4+5x)$	$\frac{3}{5}\,(1-x)^5$
(1,1)	$\frac{1}{6}\,(1-x)^4\,(1+2x)^2$	$\frac{3}{20}\,(1-x)^4\,(1+4x)$
(1,2)	$\frac{1}{210}\,(1-x)^4\,(13+52x+130x^2+120x^3)$	$\frac{1}{20}\,(1-x)^4\,(1+4x+10x^2)$

Table 1
Explicit values of the relevant kinematical kernels.

The semi-hadronic decay rate of the τ lepton, can be expressed as an integral of the spectral functions $\operatorname{Im}\Pi^T(s)$ and $\operatorname{Im}\Pi^L(s)$ over the invariant mass s of the final–state hadrons as follows:

$$R_\tau = 12\pi \int_0^{M_\tau^2} \frac{\mathrm{d}s}{M_\tau^2}\left(1 - \frac{s}{M_\tau^2}\right)^2$$
$$\times \left[\left(1 + 2\frac{s}{M_\tau^2}\right)\operatorname{Im}\Pi^T(s) + \operatorname{Im}\Pi^L(s)\right]. \quad (3)$$

Moreover, according to the quantum numbers content of the two–point function correlators

$$\Pi^J(s) \equiv |V_{ud}|^2\left[\Pi^J_{V,ud}(s) + \Pi^J_{A,ud}(s)\right]$$
$$+ |V_{us}|^2\left[\Pi^J_{V,us}(s) + \Pi^J_{A,us}(s)\right], \quad (4)$$

we can decompose R_τ into

$$R_\tau \equiv R_{\tau,V} + R_{\tau,A} + R_{\tau,S}, \quad (5)$$

where $R_{\tau,V}$ and $R_{\tau,A}$ correspond to the first two terms in Eq. (4), while $R_{\tau,S}$ contains the remaining Cabibbo–suppressed contributions.

The measurement of the invariant mass distribution of the final hadrons provides additional information on the QCD dynamics, through the moments [12]

$$R_\tau^{kl} \equiv \int_0^{M_\tau^2} ds\left(1 - \frac{s}{M_\tau^2}\right)^k\left(\frac{s}{M_\tau^2}\right)^l \frac{dR_\tau}{ds}, \quad (6)$$

which include $R_\tau \equiv R_\tau^{00}$ as a particular case.

Exploiting the analytic properties of $\Pi^J(s)$, we can express these moments as contour integrals in the complex s-plane running counter-clockwise around the circle $|s| = M_\tau^2$:

$$R_\tau^{kl} = -\pi i \oint_{|x|=1} \frac{\mathrm{d}x}{x}\left\{3\,\mathcal{F}^{kl}_{L+T}(x)\,D^{L+T}(M_\tau^2 x)\right.$$
$$\left. + 4\,\mathcal{F}^{kl}_L(x)\,D^L(M_\tau^2 x)\right\}. \quad (7)$$

We have used integration by parts to rewrite R_τ^{kl} in terms of the logarithmic derivatives

$$D^{L+T}(s) \equiv -s\frac{\mathrm{d}}{\mathrm{d}s}\left[\Pi^{L+T}(s)\right], \quad (8)$$

$$D^L(s) \equiv \frac{s}{M_\tau^2}\frac{\mathrm{d}}{\mathrm{d}s}\left[s\,\Pi^L(s)\right], \quad (9)$$

which satisfy homogeneous renormalization group equations. All kinematical factors have been absorbed into the kernels $\mathcal{F}^{kl}_{L+T}(x)$ and $\mathcal{F}^{kl}_L(x)$. Table 1 shows the explicit form of these kernels for the moments which we are going to analyze in the following sections.

For large enough $-s$, the contributions to $D^J(s)$ can be organized with the OPE in a series of local gauge–invariant scalar operators of increasing dimension $D = 2n$, times the appropriate inverse powers of $-s$. This expansion is expected to be well behaved along the complex contour $|s| = M_\tau^2$, except in the crossing point with the positive real axis [13]. As shown in Table 1, the region near the physical cut is strongly suppressed by a zero of order $3 + k$ at $s = M_\tau^2$. Therefore, the uncertainties associated with the use of the OPE near the time–like axis are very small. Inserting this series in (7) and evaluating the contour integral, one can rewrite R_τ^{kl} as an expansion in inverse powers of M_τ^2 [9],

$$R_\tau^{kl} \equiv 3\left[|V_{ud}|^2 + |V_{us}|^2\right] S_{\mathrm{EW}}\left\{1 + \delta'_{\mathrm{EW}} + \delta^{kl\,(0)}\right.$$
$$\left. + \sum_{D=2,4,\cdots}\left(\cos^2\theta_C\,\delta^{kl\,(D)}_{ud} + \sin^2\theta_C\,\delta^{kl\,(D)}_{us}\right)\right\},$$

where $\sin^2\theta_C \equiv |V_{us}|^2/[|V_{ud}|^2 + |V_{us}|^2]$ and we have pulled out the electroweak corrections $S_{\mathrm{EW}} = 1.0194$ [14] and $\delta'_{\mathrm{EW}} \simeq 0.0010$ [15].

The dimension–zero contribution $\delta^{kl\,(0)}$ is the purely perturbative correction, neglecting quark masses, which, owing to chiral symmetry, is identical for the vector and axial–vector parts. The symbols $\delta_{ij}^{kl\,(D)} \equiv [\delta_{ij,V}^{kl\,(D)} + \delta_{ij,A}^{kl\,(D)}]/2$ stand for the average of the vector and axial–vector contributions from dimension $D \geq 2$ operators; they contain an implicit suppression factor $1/M_\tau^D$.

3. SU(3) BREAKING

The separate measurement of the Cabibbo–allowed and Cabibbo–suppressed decay widths of the τ [4] allows one to pin down the SU(3) breaking effect induced by the strange quark mass, through the differences

$$\delta R_\tau^{kl} \equiv \frac{R_{\tau,V+A}^{kl}}{|V_{ud}|^2} - \frac{R_{\tau,S}^{kl}}{|V_{us}|^2}$$
$$= 3\,S_{EW} \sum_{D \geq 2} \left[\delta_{ud}^{kl\,(D)} - \delta_{us}^{kl\,(D)} \right]. \qquad (10)$$

The leading contributions to δR_τ^{kl} are quark–mass corrections of dimension two [6,7]; they are the dominant SU(3) breaking effect, generating the wanted sensitivity to the strange quark mass. The corrections of $O(m^4)$ are very tiny [7]. The main $D = 4$ contribution comes from the SU(3)–breaking quark condensate

$$\delta O_4 \equiv \langle 0| m_s\,\bar{s}s - m_d\,\bar{d}d |0\rangle. \qquad (11)$$

Neglecting the small $O(m^4)$ terms and $D \geq 6$ contributions, δR_τ^{kl} can be written as [7]:

$$\delta R_\tau^{kl} \approx 24\,S_{EW} \left\{ \frac{m_s^2(M_\tau^2)}{M_\tau^2} \left(1 - \epsilon_d^2\right) \Delta_{kl}^{(2)}(a_\tau) \right.$$
$$\left. -2\pi^2 \frac{\delta O_4}{M_\tau^4} Q_{kl}(a_\tau) \right\}, \qquad (12)$$

where $\epsilon_d \equiv m_d/m_s = 0.053 \pm 0.002$ [16] and $a_\tau \equiv \alpha_s(M_\tau^2)/\pi$.

The perturbative QCD expansions $\Delta_{kl}^{(2)}(a_\tau)$ and $Q_{kl}(a_\tau)$ are known to $O(a_\tau^2)$. Moreover, the $O(a_\tau^3)$ contributions to $\Delta_{kl}^{(2)}(a_\tau)$ coming from the longitudinal correlator $D^L(s)$ have been also computed. Using the value of the ($\overline{\text{MS}}$) strong coupling determined by the total hadronic τ decay width [3], $\alpha_s(M_\tau^2) = 0.35 \pm 0.02$, one gets numerical results shown in Table 2 [7].

(k,l)	$\Delta_{kl}^{(2)}(a_\tau)$	$Q_{kl}(a_\tau)$
(0,0)	2.0 ± 0.5	1.08 ± 0.03
(1,0)	2.4 ± 0.7	1.52 ± 0.03
(2,0)	2.7 ± 1.0	1.93 ± 0.02
(1,1)	-0.39 ± 0.26	-0.41 ± 0.02
(1,2)	0.07 ± 0.06	-0.02 ± 0.01

Table 2
Numerical values [7] of the relevant perturbative expansions for $\alpha_s(M_\tau^2) = 0.35 \pm 0.02$.

The rather large theoretical uncertainties of

$$\Delta_{kl}^{(2)}(a_\tau) \equiv \frac{1}{4} \left\{ 3\,\Delta_{kl}^{L+T}(a_\tau) + \Delta_{kl}^{L}(a_\tau) \right\}, \qquad (13)$$

have their origin in the bad perturbative behaviour of the longitudinal contribution. The most important higher–order corrections can be resummed [6], using the renormalization group, but the resulting "improved" series is still rather badly behaved. For instance,

$$\Delta_{00}^{L}(0.1) = 1.5891 + 1.1733 + 1.1214 + 1.2489 + \cdots$$

which has $O(a^2)$ and $O(a^3)$ contributions of the same size. On the contrary, the $J = L + T$ series converges very well:

$$\Delta_{00}^{L+T}(0.1) = 0.7824 + 0.2239 + 0.0831 + \cdots$$

Fortunately, the longitudinal contribution to $\Delta_{kl}^{(2)}(a_\tau)$ is parametrically suppressed by a factor $1/3$. Thus, the combined final expansion looks still acceptable for the first few terms:

$$\Delta_{00}^{(2)}(0.1) = 0.9840 + 0.4613 + 0.3427$$
$$+ \left(0.3122 - 0.000045\,c_3^{L+T}\right) + \cdots \qquad (14)$$

Nevertheless, after the third term the series appears to be dominated by the longitudinal contribution, and the bad perturbative behaviour becomes again manifest. Taking the unknown $O(a^3)$ coefficient of the $D^{L+T}(s)$ perturbative series as $c_3^{L+T} \sim c_2^{L+T}\left(c_2^{L+T}/c_1^{L+T}\right) \approx 323$, the fourth term becomes 0.298; i.e. a 5% reduction only.

Since the longitudinal series seems to reach an asymptotic behaviour at $O(a^3)$, the central values of $\Delta_{kl}^{(2)}(a_\tau)$ have been evaluated adding to

the fully known $O(a^2)$ result one half of the longitudinal $O(a^3)$ contribution. To estimate the associated theoretical uncertainties, we have taken one half of the size of the last known perturbative contribution plus the variation induced by a change of the renormalization scale in the range $\xi \in [0.75, 2]$ (added in quadrature).

The SU(3)–breaking condensate δO_4 could be extracted from the τ decay data, together with m_s, through a combined fit of different δR_τ^{kl} moments. However, this is not possible with the actual experimental accuracy. We can estimate the value of δO_4 using the constraints provided by chiral symmetry. To lowest order in Chiral Perturbation Theory, δO_4 is fully predicted in terms of the pion decay constant and the pion and kaon masses: $\delta O_4 \simeq -f_\pi^2 \left(m_K^2 - m_\pi^2\right) \simeq -1.9 \times 10^{-3}\,\mathrm{GeV}^4$. Taking into account the leading $O(p^4)$ corrections through the ratio of quark vacuum condensates [17,18]

$$v_s \equiv \frac{\langle 0|\bar{s}s|0\rangle}{\langle 0|\bar{d}d|0\rangle} = 0.8 \pm 0.2\,, \tag{15}$$

one gets the improved estimate,

$$\begin{aligned}\delta O_4 &\simeq -\frac{m_s}{2\hat{m}}\left(v_s - \epsilon_d\right)f_\pi^2\, m_\pi^2 \\ &\simeq -(1.5 \pm 0.4) \times 10^{-3}\,\mathrm{GeV}^4\,,\end{aligned} \tag{16}$$

where we have used the known quark mass ratio [16] $m_s/\hat{m} = 24.4 \pm 1.5$.

Strictly speaking, δO_4 and v_s are scale dependent. This dependence cancels with the $O(m^4)$ contributions [7] and is then of $O(p^8)$ in the chiral expansion. The numerical effect is smaller than the accuracy of (16) and has been neglected together with the tiny $O(m^4)$ corrections.

4. NUMERICAL ANALYSIS

The ALEPH collaboration has measured [4] the weighted differences δR_τ^{kl} for five different values of (k,l). The experimental results are shown in Table 3, together with the corresponding $m_s(M_\tau^2)$ values. Since the QCD counterparts to the moments $(k,l) = (1,1)$ and $(1,2)$ have theoretical uncertainties larger than 100%, we only use the moments $(k,l) = (0,0)$, $(1,0)$, and $(2,0)$.

The experimental errors quoted in Table 3 do not include the present uncertainty in $|V_{us}|$. To

(k,l)	δR_τ^{kl}	$m_s(M_\tau^2)$ (MeV)
$(0,0)$	0.394 ± 0.137	$143 \pm 31 \pm 18$
$(1,0)$	0.383 ± 0.078	$121 \pm 17 \pm 18$
$(2,0)$	0.373 ± 0.054	$106 \pm 12 \pm 21$
$(1,1)$	0.010 ± 0.029	–
$(1,2)$	0.006 ± 0.015	–

Table 3

Measured [4] moments δR_τ^{kl} and corresponding $m_s(M_\tau^2)$ values [7]. The first error is experimental and the second theoretical.

estimate the corresponding error in m_s, we take the following numbers from ALEPH: $R_{\tau,V+A}^{00} = 3.486 \pm 0.015$, $R_{\tau,S}^{00} = 0.1610 \pm 0.0066$, $|V_{ud}| = 0.9751 \pm 0.0004$ and $|V_{us}| = 0.2218 \pm 0.0016$. This gives $\delta R_\tau^{00} = 0.394 \pm 0.135 \pm 0.047$, where the second error comes from the uncertainty in $|V_{us}|$ and translates into an additional uncertainty of 10 MeV in the strange quark mass. We will put the same $|V_{us}|$ uncertainty to the other two moments, for which the ALEPH collaboration does not quote the separate values of $R_{\tau,V+A}^{kl}$ and $R_{\tau,S}^{kl}$.

Taking the information from the three moments into account, we get our final result [7]:

$$\begin{aligned}m_s(M_\tau^2) &= (119 \pm 12 \pm 18 \pm 10)\,\mathrm{MeV} \\ &= (119 \pm 24)\,\mathrm{MeV}\,.\end{aligned} \tag{17}$$

The first error is experimental, the second reflects the QCD uncertainty and the third one is from the present uncertainty in $|V_{us}|$. Since the three moments are highly correlated, we have taken the smaller individual errors as errors of the final average. Our determination (17) corresponds to

$$m_s(1\,\mathrm{GeV}^2) = (164 \pm 33)\,\mathrm{MeV} \tag{18}$$

and

$$m_s(4\,\mathrm{GeV}^2) = (114 \pm 23)\,\mathrm{MeV}\,. \tag{19}$$

5. COMPARISON WITH ALEPH

The ALEPH collaboration has performed a phenomenological analysis of the δR_τ^{kl} moments in Table 3, which results in larger m_s values [4]:

$$m_s(M_\tau^2) = \begin{cases} 149^{+24_{\mathrm{exp}}+21_{\mathrm{th}}}_{-30_{\mathrm{exp}}-25_{\mathrm{th}}} \pm 6_{\mathrm{fit}}\,\mathrm{MeV}, \\ 176^{+37_{\mathrm{exp}}+24_{\mathrm{th}}}_{-48_{\mathrm{exp}}-28_{\mathrm{th}}} \pm 8_{\mathrm{fit}} \pm 11_{J=0}\,\mathrm{MeV}. \end{cases}$$

To derive these numbers, ALEPH has used our published results in refs. [6], [9] and [12]. Since we have analyzed the same data with improved theoretical input [7], it is worthwhile to understand the origin of the numerical difference.

ALEPH makes a global fit to the five measured moments, including the last two which are unreliable (100% theoretical errors). In view of the asymptotic behaviour of $\Delta_{kl}^{L}(a_\tau)$, they truncate this perturbative series at $O(a_\tau)$, neglecting the known and positive $O(a_\tau^2)$ and $O(a_\tau^3)$ contributions. Thus, they use a smaller value of $\Delta_{kl}^{(2)}(a_\tau)$ and, therefore, get a larger result for m_s (the first value above) because the sensitivity to this parameter is through the product $m_s^2(M_\tau^2)\Delta_{kl}^{(2)}(a_\tau)$. Since they put rather conservative errors, their result is nevertheless consistent with ours.

ALEPH has made a second analysis subtracting the $J = L$ contribution. Unfortunately, only the pion and kaon contributions are known. Using the positivity of the longitudinal spectral functions, this pole contributions provide lower bounds on $\mathrm{Im}\Pi_{ud}^{L}(s)$ and $\mathrm{Im}\Pi_{us}^{L}(s)$, which translate into lower limits on the corresponding $J = L$ contribution to δR_τ^{kl}. Subtracting this contribution, one gets upper bounds on $\delta R_{\tau,L+T}^{kl}$ [7] which imply $m_s(M_\tau^2) < 202$ MeV [7].

However, besides subtracting the pion and kaon poles, ALEPH makes a tiny ad-hoc correction to account for the remaining unknown $J = L$ contribution, and quotes the resulting number as a $m_s(M_\tau^2)$ determination [the second value above]. Since they add a generous uncertainty, their number does not disagree with ours. However, it is actually an upper bound on $m_s(M_\tau^2)$ and not a determination of this parameter.

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