Chiral low-energy constants $\mathbf{L_{10}}$ and $\mathbf{C_{87}}$ from hadronic τ decays

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Using recent precise hadronic τ-decay data on the $V-A$ spectral function and general properties of QCD such as analyticity, the operator product expansion (OPE) and chiral perturbation theory (χPT), we get accurate values for the QCD chiral order parameters L_{10}^r and C_{87}^r. At order p^4 we obtain $L_{10}^r(M_\rho) = -(5.22 \pm 0.06) \cdot 10^{-3}$, whereas at order p^6 we get $L_{10}^r(M_\rho) = -(4.06 \pm 0.39) \cdot 10^{-3}$ and $C_{87}^r(M_\rho) = (4.89 \pm 0.19) \cdot 10^{-3}$ GeV^{-2}.

1. Introduction

Hadronic τ-decay data are a very important source of information, both on perturbative and non-perturbative QCD. Of special interest in order to study non-perturbative QCD quantities is the difference of the vector and axial-vector spectral functions, because in the chiral limit the corresponding $V-A$ correlator is exactly zero in perturbation theory.

The τ data can be used to determine the parameters of χPT [1], the effective field theory of QCD at very low energies (a Taylor expansion in external momenta and quark masses). At lowest order, $\mathcal{O}(p^2)$, the SU(3) χPT Lagrangian has only two parameters, the pion decay constant f_π and the light quark condensate. At $\mathcal{O}(p^4)$ twelve more low-energy constants (LECs) appear ($L_{i=1,\cdots,10}$ and $H_{1,2}$), whereas at $\mathcal{O}(p^6)$ we have 90 (23) additional parameters C_i in the even (odd) intrinsic parity sector [2]. These LECs are related to order parameters of the spontaneous chiral symmetry breaking of QCD, and have to be determined phenomenologically or using non-perturbative techniques. For the L_i couplings this has been done to an acceptable accuracy, but the C_i LECs are less well known.

There has been a lot of recent activity to determine these chiral LECs from theory, using as much as possible QCD information [3–9]. This strong effort is motivated by the precision required in present phenomenological applications, which makes necessary to include corrections of $\mathcal{O}(p^6)$. The huge number of unknown couplings is the major source of theoretical uncertainty.

We present here an accurate determination of the χPT couplings L_{10} and C_{87} [10], using the most recent hadronic τ-decay data [11]. Estimates of L_{10} from τ-data have been done previously [12–14], altough our analysis is the first that includes the known two-loop χPT contributions and then also the first that provides C_{87}.

2. Theoretical Framework

The basic objects of the theoretical analysis are the two-point correlation functions of the non-strange vector and axial-vector quark currents

$$\Pi_{ij,\mathcal{J}}^{\mu\nu}(q) \equiv i \int \mathrm{d}^4x\, \mathrm{e}^{iqx} \langle 0|T\big(\mathcal{J}_{ij}^\mu(x)\mathcal{J}_{ij}^\nu(0)^\dagger\big)|0\rangle$$

$$= (-g^{\mu\nu}q^2 + q^\mu q^\nu)\Pi_{ij,\mathcal{J}}^{(1)}(q^2) + q^\mu q^\nu \Pi_{ij,\mathcal{J}}^{(0)}(q^2), \quad (1)$$

where $\mathcal{J}_{ij}^\mu$ denotes $V_{ud}^\mu = \overline{u}\gamma^\mu d$ and $A_{ud}^\mu = \overline{u}\gamma^\mu\gamma_5 d$. In particular we are interested in the difference $\Pi(s) \equiv \Pi_{ud,V}^{(0+1)} - \Pi_{ud,A}^{(0+1)}$, and we will work in the isospin limit ($m_u = m_d$) where $\Pi_{ud,V}^{(0)}(q^2) = 0$.

From the analytic structure of the correlator $\Pi(s)$ in the complex s-plane and its OPE one can get the following two sum rules (see ref. [10] for a

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careful derivation)

$$-8\,L_{10}^{\text{eff}} \equiv \int_{s_{\text{th}}}^{s_0} \frac{ds}{s}\frac{1}{\pi}\,\text{Im}\,\Pi(s) = \frac{2f_\pi^2}{m_\pi^2} + \Pi(0) \quad (2)$$

$$16\,C_{87}^{\text{eff}} \equiv \int_{s_{\text{th}}}^{s_0} \frac{ds}{s^2}\frac{1}{\pi}\,\text{Im}\,\Pi(s) = \frac{2f_\pi^2}{m_\pi^4} + \frac{d\Pi}{ds}(0)\ ,(3)$$

that represent the starting point of our work. The interest of these two relations stems from the fact that the effective parameters L_{10}^{eff} and C_{87}^{eff} can be extracted from the data and the r.h.s can be rigorously calculated within χPT in terms of the LECs that we want to determine. From the results of ref. [15] we get

$$2f_\pi^2/m_\pi^2 + \Pi(0) = -8L_{10}^r(\mu) + G_{1L}^4(\mu)$$
$$+ G_{0L}^6(\mu) + G_{1L}^6(\mu) + G_{2L}^6(\mu) + \mathcal{O}(p^8)$$
$$2f_\pi^2/m_\pi^4 + \Pi'(0) = H_{1L}^4$$
$$+ 16\,C_{87}^r(\mu) + H_{1L}^6(\mu) + H_{2L}^6(\mu) + \mathcal{O}(p^8)\,,(4)$$

where the functions $G_{nL}^m(\mu), H_{nL}^m(\mu)$ are corrections of order p^m generated at the n-loops level. We omit their explicit analytic form [10] for simplicity, but it is important to say that $G_{0L,1L}^6(\mu)$ contain some LECs that will represent the main source of uncertainty for L_{10}^r.

3. Determination of Effective Couplings

We will use the recent ALEPH data on hadronic τ decays [11], that provide the most precise measurement of the $V-A$ spectral function.

The relations (2) and (3) are exactly satisfied only at $s_0 \to \infty$, but we are forced to take finite values of s_0 neglecting in this way the rest of the integral[2]. From the s_0-sensitivity of the effective parameters one can assess the size of this theoretical error (quark-hadron duality violation -DV-).

In Fig. 1, we plot the value of L_{10}^{eff} obtained for different values of s_0, with the one-sigma experimental error band, and we can see a quite stable result at $s_0 \gtrsim 2$ GeV^2 (solid lines). The weight function $1/s$ decreases the impact of the high-energy region, minimising the DV; the resulting integral appears then to be much better behaved than the sum rules with s^n ($n \geq 0$) weights.

[2]Equivalently, we are assuming that the OPE is a good approximation for $\Pi(s)$ at any $|s|=s_0$, what is not expected to happen near the real axis and that produces the DV.
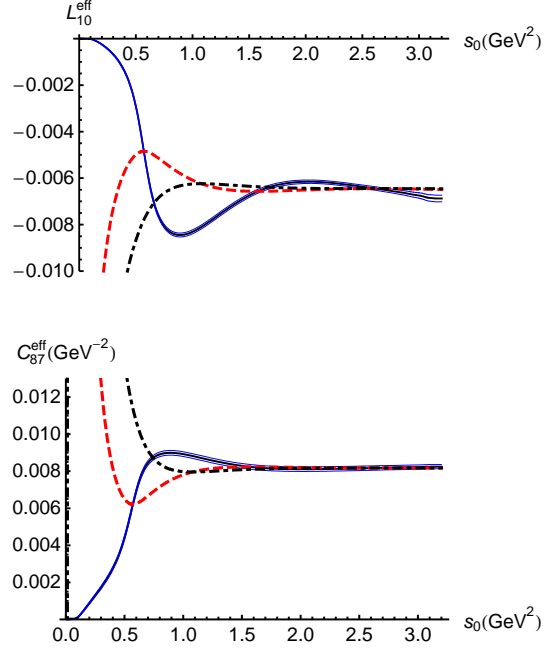

Figure 1. $L_{10}^{\text{eff}}(s_0)$ and $C_{87}^{\text{eff}}(s_0)$ from different sum rules. For clarity, we do not include the error bands associated with the modified weights.

There are some possible strategies to estimate the value of L_{10}^{eff} and his error. One is to give the predictions fixing s_0 at the so-called "duality points", two points where the first and second Weinberg sum rules (WSR) [16] happen to be satisfied. In this way we get $L_{10}^{\text{eff}} = -(6.50 \pm 0.13) \cdot 10^{-3}$, where the uncertainty covers the values obtained at the two "duality points".

If we assume that the integral (2) oscillates around his asymptotic value with decreasing oscillations and we perform an average between the maxima and minima of the oscillations we get $L_{10}^{\text{eff}} = -(6.5 \pm 0.2) \cdot 10^{-3}$.

Another way of estimating the DV uses appropriate oscillating functions defined in [17] which mimic the real quark-hadron oscillations above the data. These functions are defined such that they match the data at ~ 3 GeV^2, go to zero with decreasing oscillations and satisfy the two WSRs.

We find in this way $L_{10}^{\text{eff}} = -(6.50 \pm 0.12) \cdot 10^{-3}$, where the error spans the range generated by the different functions used.

Finally we can take advantage of the WSRs to construct modified sum rules with weight factors $w(s)$ proportional to $(1 - s/s_0)$, in order to suppress numerically the role of the suspect region around $s \sim s_0$ [18]. Fig. 1 shows the results obtained with $w_1(s) \equiv (1-s/s_0)/s$ (dashed line) and $w_2(s) \equiv (1-s/s_0)^2/s$ (dot-dashed line). These weights give rise to very stable results over a quite wide range of s_0 values. One gets $L_{10}^{\text{eff}} = -(6.51 \pm 0.06) \cdot 10^{-3}$ using $w_1(s)$ and $L_{10}^{\text{eff}} = -(6.45 \pm 0.06) \cdot 10^{-3}$ using $w_2(s)$.

Taking into account all the previous discussion, we quote as our final conservative result:

$$L_{10}^{\text{eff}} = -(6.48 \pm 0.06) \cdot 10^{-3}. \tag{5}$$

We have made a completely analogous analysis to determine C_{87}^{eff}. The results are shown in Fig. 1. The solid lines, obtained from Eq. (3), are much more stable than the corresponding results for L_{10}^{eff}, due to the $1/s^2$ factor in the integrand. The dashed and dot-dashed lines have been obtained with the modified weights $w_3(s) \equiv \frac{1}{s^2}\left(1 - \frac{s^2}{s_0^2}\right)$ and $w_4(s) \equiv \frac{1}{s^2}\left(1 - \frac{s}{s_0}\right)^2\left(1 + 2\frac{s}{s_0}\right)$. The agreement among the different estimates is quite remarkable, and our final conservative result is

$$C_{87}^{\text{eff}} = (8.18 \pm 0.14) \cdot 10^{-3} \, \text{GeV}^{-2}. \tag{6}$$

4. Determination of L_{10}^r and C_{87}^r

The χPT coupling $L_{10}^r(\mu)$ can be obtained from L_{10}^{eff}, using the relation (4). At $\mathcal{O}(p^4)$ the determination is straightforward and one gets

$$L_{10}^r(\mu = M_\rho) = -(5.22 \pm 0.06) \cdot 10^{-3}. \tag{7}$$

At order p^6, the numerical relation is more involved because it gets small corrections from other LECs. It is useful to classify the $\mathcal{O}(p^6)$ contributions through their ordering within the $1/N_C$ expansion. The tree-level term $G_{0L}^6(\mu)$ contains the only $\mathcal{O}(p^6)$ correction in the large–N_C limit, $4m_\pi^2(C_{61}^r - C_{12}^r - C_{80}^r)$, that is numerically small because of the m_π^2 suppression and can be estimated with a moderate accuracy [5, 8, 15, 19, 20].

At NLO $G_{0L}^6(\mu)$ contributes with a term of the form $m_K^2(C_{62}^r - C_{13}^r - C_{81}^r)$. In the absence of information about these LECs we will adopt the conservative range $|C_{62}^r - C_{13}^r - C_{81}^r| \leq |C_{61}^r - C_{12}^r - C_{80}^r|/3$, which generates the uncertainty that will dominate our final error on L_{10}^r. Also at this order in $1/N_C$ there is the one-loop correction $G_{1L}^6(\mu)$ that is proportional to L_9^r, which is better known [21]. Calculating the $1/N_C^2$ suppressed two-loop function $G_{2L}^6(\mu)$ and taking all these contributions into account we finally get the wanted $\mathcal{O}(p^6)$ result:

$$\begin{aligned} L_{10}^r(M_\rho) &= -(4.06 \pm 0.04_{L_{10}^{\text{eff}}} \pm 0.39_{\text{LECs}}) \cdot 10^{-3} \\ &= -(4.06 \pm 0.39) \cdot 10^{-3}. \end{aligned} \tag{8}$$

where the error has been split into its two main components. Repeating the same process with C_{87}^r (where the only LEC involved is L_9^r) we get

$$C_{87}^r(M_\rho) = (4.89 \pm 0.19) \cdot 10^{-3} \, \text{GeV}^{-2}. \tag{9}$$

5. Summary

Using general properties of QCD and the measured $V - A$ spectral function [11] we have determined the chiral LECs $L_{10}^r(M_\rho)$ and $C_{87}^r(M_\rho)$ rather accurately, with a careful analysis of the theoretical uncertainties.

There are other determinations of L_{10} from τ data in the literature. Our result for L_{10}^{eff} agrees with [12, 13], but our estimation includes a more careful assessment of the theoretical errors. The $3.2\,\sigma$ discrepancy between the estimation of ref. [14] and ours is caused by an underestimation of the systematic error associates with the duality-point approach used in that reference. In [13] also C_{87}^{eff} is determined with a good agreement with our result again. The extraction of $L_{10}^r(\mu)$ from L_{10}^{eff} has only been done previously in ref. [12], at $\mathcal{O}(p^4)$.

Our determinations of $L_{10}^r(M_\rho)$ and $C_{87}^r(M_\rho)$ agree within errors with the large–N_C estimates based on lowest-meson dominance [4, 6, 15, 22] $L_{10} \approx -3f_\pi^2/(8M_V^2) \approx -5.4 \cdot 10^{-3}$ and $C_{87} \approx 7f_\pi^2/(32M_V^4) \approx 5.3 \cdot 10^{-3} \, \text{GeV}^{-2}$ and with the result of ref. [7] for C_{87}, based on Padé Approximants. These predictions, however, are unable to fix the scale dependence which is of higher-order

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in $1/N_C$. More recently, the resonance chiral theory Lagrangian [6, 23] has been used to analyse the correlator $\Pi(s)$ at NLO order in the $1/N_C$ expansion. Matching the effective field theory description with the short-distance QCD behaviour, the two LECs are determined, keeping full control of their μ dependence. The theoretically predicted values $L_{10}^r(M_\rho) = -(4.4 \pm 0.9) \cdot 10^{-3}$ and $C_{87}^r(M_\rho) = (3.6 \pm 1.3) \cdot 10^{-3}$ GeV^{-2} [9] are in perfect agreement with our determinations, although less precise. A recent lattice estimate [24] finds $L_{10}^r(M_\rho) = -(5.2 \pm 0.5) \cdot 10^{-3}$ at order p^4, in good agreement with our result (7).

Using the results of ref. [25], the SU(2) χPT LEC $\bar{l}_5$ can be extracted from $L_{10}^r(\mu)$. We find $\bar{l}_5 = 13.30 \pm 0.11$ at $\mathcal{O}(p^4)$ and $\bar{l}_5 = 12.24 \pm 0.21$ at $\mathcal{O}(p^6)$.

Recent analyses of the decay $\pi^+ \to l^+ \nu \gamma$ at $\mathcal{O}(p^6)$ have provided accurate values for the combinations $L_9 + L_{10}$ [20] and $\bar{l}_5 - \bar{l}_6$ [26], that can be combined with our results to get $L_9^r(M_\rho) = (5.5 \pm 0.4) \cdot 10^{-3}$ and $\bar{l}_6 = 15.22 \pm 0.39$ to order p^6, that are in perfect agreement with refs. [21, 27].

Acknowledgements

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