Eigenvalue spectra and stability of directed complex networks

— Supplemental Material —

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CONTENTS

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S1. OVERVIEW

This document contains additional information about the calculations performed to obtain the results detailed in the main paper.

To begin with, we perform the calculation discussed in Section III of the main text. Specifically, Section S2 of this document is dedicated to showing how one can compute the eigenvalue potential, defined in Eq. (7) of the main text, and to deriving the annealed approximation for the network given in Eq. (9). Given this annealed approximation, Section S3 then shows how the resolvent can be deduced by performing a saddle point approximation (valid for large N) of the eigenvalue potential $\Phi(\omega, \omega^\star)$.

We then move on to deriving the 'general results' in Section IV of the main text. Section S4 shows how each of the properties of the bulk of the eigenvalue spectrum (the boundary of the bulk spectrum, the leading eigenvalue of the bulk region and the density of eigenvalues within the bulk region) can be found. These results are valid for a general network degree distribution and are given in Section IV A of the main text. Similarly, Section S5 then shows how one can derive the density of eigenvalues in the case where the random matrix $\underline{\underline{a}}$ is fully symmetric (an undirected network) — i.e. the results in Section IV B of the main text. Then, Section S6 derives the general expression for the outlier eigenvalue discussed in Section IV C of the main text

In Section S7, we then perform series expansions of the more general results given in Section IV of the main text to obtain the small-s^2 (small-network-heterogeneity) modifications to the elliptic, circular and semi-circular laws, as well as the outlier eigenvalues, that were highlighted in Section V of the main text.

Finally, in Section S8, we discuss a couple of example degree distributions that were used to produce the figures in the main text, namely the dichotomous and uniform degree distributions. We discuss how the general results in Section IV of the main text can be used to find the properties of the eigenvalue spectrum in these special cases.

S2. EIGENVALUE POTENTIAL AND ANNEALED APPROXIMATION

In a similar fashion to Refs. [S1–S3], we evaluate the eigenvalue potential, defined in Eq. (7) of the main text, using the replica method [S4]. The replica method exploits the fact that $\ln x = \lim_{n\to 0}(x^n - 1)/n$ to evaluate the average of the logarithm in Eq. (7) by instead calculating the average of an n-fold replicated system.

It has been shown in other works [S2, S3] (see the Supplemental material in Ref. [S5] for a more detailed discussion) that in fact the replicas 'decouple', meaning that the logarithm and the ensemble average commute. That is

$$\Phi(\omega, \omega^\star) = -\frac{1}{N}\left\langle \ln\det\left[(\omega^\star \underline{\underline{\mathbb{1}}} - \underline{\underline{a}}^T)(\omega\underline{\underline{\mathbb{1}}} - \underline{\underline{a}})\right]\right\rangle = -\frac{1}{N}\ln\left\langle \det\left[(\omega^\star \underline{\underline{\mathbb{1}}} - \underline{\underline{a}}^T)(\omega\underline{\underline{\mathbb{1}}} - \underline{\underline{a}})\right]\right\rangle. \tag{S1}$$

The determinant above can be written as a Gaussian integral (after having performed a Hubbard-Stratonovich transformation [S6])

$$\det\left[(\mathbb{1}\omega^\star - \underline{\underline{a}}^T)(\mathbb{1}\omega - \underline{\underline{a}})\right]^{-1} = \int \prod_i \frac{d^2z_i d^2y_i}{2\pi^2} \exp\left[-\sum_i y_i^\star y_i\right]$$
$$\times \exp\left[i\sum_{ij} z_i^\star(a_{ji} - \omega^\star\delta_{ij})y_j\right] \exp\left[i\sum_{ij} y_i^\star(a_{ij} - \omega\delta_{ij})z_j\right]. \quad \text{(S2)}$$

Performing the ensemble average according to the distribution described in Eqs. (1) and (2) of the main text, we obtain

$$\begin{aligned}
&\left\langle \exp\left[ia_{ij}(z_j^\star y_i + z_j y_i^\star) + ia_{ji}(z_i^\star y_j + z_i y_j^\star)\right]\right\rangle \\
&= 1 + f_{ij}\left[\left\langle \exp\left[ia_{ij}(z_j^\star y_i + z_j y_i^\star) + ia_{ji}(z_i^\star y_j + z_i y_j^\star)\right]\right\rangle_\pi - 1\right] \\
&\approx \exp\left[\frac{k_i k_j}{pN}\left[\left\langle \exp\left[ia_{ij}(z_j^\star y_i + z_j y_i^\star) + ia_{ji}(z_i^\star y_j + z_i y_j^\star)\right]\right\rangle_\pi - 1\right]\right] \qquad \text{(S3)}
\end{aligned}$$

where we have used the approximation $f_{ij} \approx \frac{k_i k_j}{pN}$ [see Eq. (3) in the main text], which is valid for $\frac{k_i k_j}{pN} \ll 1$. Now, expanding the exponential and assuming that higher order moments of a_{ij} decay faster then $1/p$, we obtain

$$\begin{aligned}
&\left\langle \exp\left[ia_{ij}(z_j^\star y_i + z_j y_i^\star) + ia_{ji}(z_i^\star y_j + z_i y_j^\star)\right]\right\rangle_\pi \\
&\approx 1 + i\frac{\mu}{p}\left[(z_j^\star y_i + z_j y_i^\star) + (z_i^\star y_j + z_i y_j^\star)\right] \\
&- \frac{\sigma^2}{2p}\left[(z_i^\star y_j + z_i y_j^\star)^2 + (z_j^\star y_i + z_j y_i^\star)^2 + 2\Gamma(z_i^\star y_j + z_i y_j^\star)(z_j^\star y_i + z_j y_i^\star)\right] \\
&\approx \exp\Bigg[i\frac{\mu}{p}\left[(z_j^\star y_i + z_j y_i^\star) + (z_i^\star y_j + z_i y_j^\star)\right] \\
&- \frac{\sigma^2}{2p}\left[(z_i^\star y_j + z_i y_j^\star)^2 + (z_j^\star y_i + z_j y_i^\star)^2 + 2\Gamma(z_i^\star y_j + z_i y_j^\star)(z_j^\star y_i + z_j y_i^\star)\right]\Bigg]. \qquad \text{(S4)}
\end{aligned}$$

Finally, we obtain the following approximation for the eigenvalue potential

$$\begin{aligned}
\exp\left[-N\Phi(\omega)\right] = &\int \prod_i \left(\frac{d^2z_i d^2y_i}{2\pi^2}\right) \exp\left[-\sum_i y_i^\star y_i\right] \\
&\times \exp\left[-i\sum_i (z_i^\star y_i \omega^\star + z_i y_i^\star \omega) + i\frac{\mu}{p}\sum_{ij}\frac{k_i k_j}{pN}(z_i^\star y_j + z_i y_j^\star)\right] \\
&\times \exp\left[-\frac{\sigma^2}{2p}\sum_{ij}\frac{k_i k_j}{pN}\left[(z_i^\star y_j + z_i y_j^\star)^2 + \Gamma(z_i^\star y_j + z_i y_j^\star)(z_j^\star y_i + z_j y_i^\star)\right]\right]. \qquad \text{(S5)}
\end{aligned}$$

We note that this is exactly the same expression for $\Phi(\omega, \omega^\star)$ that we would have obtained if we had used the distribution for a_{ij} in Eq. (9) in the main text from the start. We therefore conclude that the annealed network approximation in Eq. (9) is valid when $p \gg 1$ and $\frac{k_i k_j}{pN} \ll 1$ for all i and j.

S3. FINDING THE RESOLVENT

We note from previous works [S5, S7–S9] that low-rank perturbations to a random matrix produce outlier eigenvalues, but they do not affect the bulk of the eigenvalue distribution. Noting that the introduction of a non-zero value of μ is equivalent to a rank-1 perturbation in the annealed approximation, we can set $\mu = 0$ in Eq. (S5) and continue the calculation to find the bulk eigenvalue density. We return later to the outlier eigenvalue that emerges as a result of setting a non-zero value of μ in Section S6.

A. Introduction of order parameters

We now imagine that we group each node with all other nodes that share the same degree k_α in a group labelled by the index α. In the annealed network approximation the problem therefore reduces to that of finding the eigenvalue spectrum of a random matrix with block-structured statistics (as discussed in the main text). This enables us to follow along the lines of previous works that have studied the eigenvalue spectra of block-structured random matrices (see Ref. [S1] in particular).

We begin by rewriting Eq. (S5) as

$$\begin{aligned}\exp\left[-N\Phi(\omega)\right] = \int \prod_{\alpha i}\left(\frac{d^2 z_i^\alpha d^2 y_i^\alpha}{2\pi^2}\right)\exp\left[-\sum_{\alpha i} y_i^{\alpha\star} y_i^\alpha\right] \\ \times \exp\left[-i\sum_{\alpha i}\left(z_i^{\alpha\star} y_i^\alpha \omega^\star + z_i^\alpha y_i^{\alpha\star}\omega\right) + i\frac{\mu}{p}\sum_{\alpha\beta ij}\frac{k_\alpha k_\beta}{pN}\left(z_i^{\alpha\star} y_j^\beta + z_i^\alpha y_j^{\beta\star}\right)\right] \\ \times \exp\left[-\frac{\sigma^2}{2p}\sum_{\alpha\beta ij}\frac{k_\alpha k_\beta}{pN}\left[\left(z_i^{\alpha\star} y_j^\beta + z_i^\alpha y_j^{\beta\star}\right)^2 + \Gamma\left(z_i^{\alpha\star} y_j^\beta + z_i^\alpha y_j^{\beta\star}\right)\left(z_j^{\beta\star} y_i^\alpha + z_j^\beta y_i^{\alpha\star}\right)\right]\right],\end{aligned} \tag{S6}$$

where the indices i and j now only run over the nodes in the groups α and β respectively. We introduce the following 'order parameters', defining $x_\alpha = k_\alpha/p$

$$\begin{aligned} u = \frac{1}{N}\sum_{\alpha i} x_\alpha z_i^{\alpha\star} z_i^\alpha, \quad v_\alpha = \frac{1}{N_\alpha}\sum_i y_i^{\alpha\star} y_i^\alpha, \\ w_\alpha = \frac{1}{N_\alpha}\sum_i z_i^{\alpha\star} y_i^\alpha, \quad w_\alpha^\star = \frac{1}{N_\alpha}\sum_i y_i^{\alpha\star} z_i^\alpha,\end{aligned} \tag{S7}$$

where N_α is the number of nodes with degree k_α. We impose these definitions in the integral Eq. (S6) using Dirac delta functions in their complex exponential representation. We can thus rewrite Eq. (S6) as

$$\exp\left[-N\Phi(\omega)\right] = \int \mathcal{D}\left[\cdots\right]\exp\left[N(\Psi + \Theta + \Omega)\right], \tag{S8}$$

where $\mathcal{D}[\cdots]$ denotes integration over all of the order parameters and their conjugate ('hatted') variables, and where

$$\Psi = i\hat{u}u + i\sum_\alpha \gamma_\alpha\left(\hat{v}_\alpha v_\alpha + \hat{w}_\alpha w_\alpha^\star + \hat{w}_\alpha^\star w_\alpha\right),$$

$$
\Theta = -\sum_{\alpha} v_\alpha - \sigma^2 u \sum_{\alpha} \gamma_\alpha x_\alpha v_\alpha - i \sum_{\alpha} \gamma_\alpha (w_\alpha \omega^\star + w_\alpha^\star \omega) - \frac{\Gamma\sigma^2}{2} \sum_{\alpha\beta} \gamma_\alpha \gamma_\beta x_\alpha x_\beta (w_\alpha w_\beta + w_\alpha^\star w_\beta^\star),
$$

$$
\Omega = \sum_{\alpha} \gamma_\alpha \ln \left[\int \left(\frac{d^2 z^\alpha d^2 y^\alpha}{2\pi^2} \right) \exp \Big\{ - i [\hat{u} x_\alpha z^{\alpha\star} z^\alpha + \hat{v}_\alpha y^{\alpha\star} y^\alpha + \hat{w}_\alpha y^{\alpha\star} z^\alpha + \hat{w}_\alpha^\star z^{\alpha\star} y^\alpha] \Big\} \right]. \tag{S9}
$$

Here, we have defined $\gamma_\alpha = N_\alpha/N$. In the limit $N \to \infty$, this quantity will be given by the degree distribution of the network, i.e. $\gamma_\alpha \to \gamma(k_\alpha)$.

We note that the integrals over y_i and z_i in Eq. (S9) are uncoupled for different values of i as a result of introducing the order parameters in Eq. (S7). We note also that we have neglected terms involving $N^{-1}\sum_i (y_i^\alpha)^2$, $N^{-1}\sum_i (z_i^\alpha)^2$, $N^{-1}\sum_i z_i^\alpha y_i^\alpha$ and similar terms involving the complex conjugates of z_i^α and y_i^α, which do not contribute in the thermodynamic limit (see Refs. [S2, S3, S5, S10] for further discussion).

Carrying out the integrals over the variables y_i and z_i in the expression for Ω in Eq. (S9) one obtains

$$
\Omega = -\sum_{\alpha} \gamma_\alpha \ln \left[\hat{w}_\alpha \hat{w}_\alpha^\star - x_\alpha \hat{u} \hat{v}_\alpha \right]. \tag{S10}
$$

B. Saddle point integration and evaluation of the order parameters

We now suppose that $N \gg 1$, and carry out the integral in Eq. (S8) in the saddle-point approximation. To do this, we extremise the expression $\Psi + \Theta + \Omega$. Extremising with respect to the conjugate variables $\hat{u}, \hat{v}, \hat{w}_\alpha$ and $\hat{w}_\alpha^\star$, we find

$$
\begin{aligned}
& iu = -\sum_{\alpha} \gamma_\alpha x_\alpha \frac{\hat{v}_\alpha}{f_\alpha}, \quad iv_\alpha = -x_\alpha \frac{\hat{u}}{f_\alpha}, \quad iw_\alpha = \frac{\hat{w}_\alpha}{f_\alpha}, \quad iw_\alpha^\star = \frac{\hat{w}_\alpha^\star}{f_\alpha}, \\
& f_\alpha = \hat{w}_\alpha \hat{w}_\alpha^\star - x_\alpha \hat{u} \hat{v}_\alpha.
\end{aligned} \tag{S11}
$$

We now instead extremise $\Psi + \Theta + \Omega$ in Eq. (S8) with respect to u, v, w_α and $w_\alpha^\star$. We find

$$
\begin{aligned}
& i\hat{u} = \sigma^2 v, \quad i\gamma_\alpha \hat{v}_\alpha = 1 + \sigma^2 \gamma_\alpha x_\alpha u, \\
& i\hat{w}_\alpha = i\omega + \Gamma\sigma^2 \sum_{\beta} \gamma_\beta x_\alpha x_\beta w_\beta^\star, \quad i\hat{w}_\alpha^\star = i\omega^\star + \Gamma\sigma^2 \sum_{\beta} \gamma_\beta x_\alpha x_\beta w_\beta,
\end{aligned} \tag{S12}
$$

where we introduce $v = \sum_\alpha \gamma_\alpha x_\alpha v_\alpha$.

C. Expression for the resolvent

In a similar fashion to Ref. [S1], we observe that the quantities w_α and $w_\alpha^\star$, when evaluated at the saddle point, are related to the resolvent [see Eq. (8) of the main text]:

$$
G(\omega) = \frac{\partial \Phi(\omega, \omega^\star)}{\partial \omega} = i \sum_{\alpha} \gamma_\alpha w_\alpha^\star,
$$

$$G^\star(\omega) = \frac{\partial \Phi(\omega,\omega^\star)}{\partial \omega^\star} = i\sum_\alpha \gamma_\alpha w_\alpha. \tag{S13}$$

So, if we can solve Eqs. (S11) and (S12) for $i\sum_\alpha \gamma_\alpha w_\alpha^\star$ as a function of only ω and $\omega^\star$, we can obtain the eigenvalue density of the bulk region using Eq. (6) of the main text.
First, eliminating $\hat{u}$, one sees that

$$v = \sigma^2 v \sum_\alpha \gamma_\alpha \frac{x_\alpha^2}{f_\alpha}. \tag{S14}$$

This equation has two solutions.

First solution:
One solution is $v = 0$, implying $\hat{u} = 0$ [see Eq. (S12)], and hence $f_\alpha = \hat{w}_\alpha \hat{w}_\alpha^\star = -1/(w_\alpha w_\alpha^\star)$. We therefore obtain from Eqs. (S11) and (S12)

$$\begin{aligned} 1 &= i\omega w_\alpha^\star + \Gamma\sigma^2 \sum_\beta \gamma_\beta x_\alpha x_\beta w_\alpha^\star w_\beta^\star, \\ 1 &= i\omega^\star w_\alpha + \Gamma\sigma^2 \sum_\beta \gamma_\beta x_\alpha x_\beta w_\alpha w_\beta. \end{aligned} \tag{S15}$$

From this, one can solve for the resolvent $G(\omega) = \frac{1}{N}\sum_\alpha \gamma_\alpha w_\alpha^\star$, which we see is an analytic function of ω. This means that the eigenvalue density vanishes in regions of the complex plane for which $v = 0$ is the only valid solution [as a result of Eq. (6) of the main text].

Second solution:
The other solution to the first of Eq. (S14) is

$$\sum_\alpha \gamma_\alpha \frac{x_\alpha^2}{f_\alpha} = \frac{1}{\sigma^2}. \tag{S16}$$

Now, we see from the expression for iu and $i\hat{v}_\alpha$ in Eqs. (S11) and (S12) that when the second Eq. (S16) is satisfied, $u \to \infty$. This means that $i\hat{v}_\alpha \to \sigma^2 x_\alpha u$ and hence

$$\hat{v}_\alpha \hat{u} \to x_\alpha^2 g(\omega,\omega^\star), \tag{S17}$$

where $g(\omega,\omega^\star)$ is an arbitrary function to be found that is independent of α. Hence, we obtain the following simultaneous equations, which enable us to find the resolvent $G(\omega,\omega^\star) = \sum_\alpha \gamma_\alpha (i w_\alpha^\star)$

$$\begin{aligned} \frac{1}{\sigma^2} &= \sum_\alpha \gamma_\alpha \frac{x_\alpha^2}{f_\alpha}, \\ -w_\alpha w_\alpha^\star &= 1/f_\alpha + x_\alpha^2 \frac{g(\omega,\omega^\star)}{f_\alpha^2}, \\ i w_\alpha &= \frac{\omega}{f_\alpha} - \Gamma\sigma^2 \frac{x_\alpha}{f_\alpha} \sum_\beta i w_\beta^\star \gamma_\beta x_\beta, \\ i w_\alpha^\star &= \frac{\omega^\star}{f_\alpha} - \Gamma\sigma^2 \frac{x_\alpha}{f_\alpha} \sum_\beta i w_\beta \gamma_\beta x_\beta. \end{aligned} \tag{S18}$$

In principle, one can solve these along with Eq. (S16) to find $g(\omega, \omega^\star)$, iw_α and $iw_\alpha^\star$ as functions of ω and $\omega^\star$. In this case, the resolvent is no longer necessarily an analytic function of ω. Therefore, in the region of the complex plane where Eq. (S16) is satisfied, the eigenvalue density is non-zero.

Noting Eq. (S13) and Eq. (6) from the main text, one then obtains the eigenvalue density via

$$\rho(\omega) = \frac{1}{\pi}\mathrm{Re}\left[\sum_\alpha \gamma_\alpha \frac{\partial i w_\alpha^\star}{\partial \omega^\star}\right]. \tag{S19}$$

S4. GENERAL RESULTS FOR THE BULK REGION

We have discussed two solutions to Eq. (S14): one that corresponds to the region of the complex plane in which the bulk of the eigenvalue spectrum of $\underline{\underline{a}}$ resides and one that corresponds to the region of the complex plane where there are no eigenvalues. Noting these two solutions [given in Eqs. (S15) and (S18)] along with Eq. (S19), we are now in a position to derive the results given in Section IV A of the main text.

A. Replacing the index α with k

So as to make clear the effective block structure of the random matrix that we were considering, we introduced the index α, which indexed nodes with the same degree k_α. We now use the fact that $\gamma_\alpha = \gamma(k_\alpha)$ in the large N limit and drop the index α to obtain the expressions in Section IV A of the main text. For example, the expressions in Eqs. (S18) become

$$\begin{aligned}
\frac{1}{\sigma^2} &= \sum_k \gamma(k)\frac{(k/p)^2}{f_k}, \\
-w_k w_k^\star &= 1/f_k + (k/p)^2 \frac{g(\omega,\omega^\star)}{f_k^2}, \\
iw_k &= \frac{\omega}{f_k} - \Gamma\sigma^2 \frac{(k/p)}{f_k}\sum_{k'}\gamma(k')(k'/p)(iw_{k'}^\star), \\
iw_k^\star &= \frac{\omega^\star}{f_k} - \Gamma\sigma^2 \frac{(k/p)}{f_k}\sum_{k'}\gamma(k')(k'/p)(iw_{k'}).
\end{aligned} \tag{S20}$$

B. Boundary of the bulk region

We first derive the results in Section IV A 1 of the main text for the boundary of the bulk region. Because the two solutions to Eq. (S14) correspond to the region inside the bulk of the eigenvalue spectrum and the outside, the boundary of the bulk region is given by the set of points $\omega = \omega_x + i\omega_y$ that simultaneously satisfy both Eqs. (S15) and (S18). The simultaneous solution of these equations yields

$$\frac{1}{\sigma^2} = \sum_k \gamma(k)(k/p)^2 |iw_k^\star|^2,$$

$$iw_k^\star = \frac{1}{i\omega + \Gamma\sigma^2(k/p)\sum_{k'}\gamma(k')(k'/p)[iw_{k'}^\star]}. \tag{S21}$$

Let $\sum_k \gamma(k)(k/p)(iw_k^\star) = A_x + iA_y$, where both A_x and A_y are real. Then we obtain

$$\begin{aligned}
\frac{1}{\sigma^2} &= \sum_k \gamma(k)\frac{(k/p)^2}{[\omega_x - \Gamma\sigma^2(k/p)A_x]^2 + [\omega_y - \Gamma\sigma^2(k/p)A_y]^2}, \\
A_x &= \sum_k \gamma(k)\frac{(k/p)\omega_x - \Gamma\sigma^2(k/p)^2 A_x}{[\omega_x - \Gamma\sigma^2(k/p)A_x]^2 + [\omega_y - \Gamma\sigma^2(k/p)A_y]^2}, \\
A_y &= \sum_k \gamma(k)\frac{-(k/p)\omega_y + \Gamma\sigma^2(k/p)^2 A_y}{[\omega_x - \Gamma\sigma^2(k/p)A_x]^2 + [\omega_y - \Gamma\sigma^2(k/p)A_y]^2}.
\end{aligned} \tag{S22}$$

Now defining

$$h = \sum_k \gamma(k)\frac{k/p}{[\omega_x - \Gamma\sigma^2(k/p)A_x]^2 + [\omega_y - \Gamma\sigma^2(k/p)A_y]^2}, \tag{S23}$$

and noting that $A_x = \omega_x h/(1+\Gamma)$ and $A_y = -\omega_y h/(1-\Gamma)$, one arrives at the expressions in Eq. (10) of the main text.

C. Leading eigenvalue of the bulk region

The leading eigenvalue of the bulk region can be obtained by finding the point on the boundary of the bulk region with $\omega_y = 0$. Making this substitution in Eqs. (S22), one readily obtains Eqs. (13) in the main text.

D. Eigenvalue density in the bulk region

We now derive the results in Section IV A 3 of the main text for the eigenvalue value density inside the bulk region. Defining $m = \sum_k \gamma(k)\frac{(k/p)}{f_k}$, one finds after some rearrangement of the last two of Eqs. (S18) that

$$iw_k^\star = \frac{\omega^\star}{f_k} - \Gamma\sigma^2\frac{(k/p)}{f_k}\frac{(\omega - \Gamma\omega^\star)}{1-\Gamma^2}m. \tag{S24}$$

Thus, substituting this into the second of Eqs. (S18), one obtains

$$f_k = \left|\omega^\star - \Gamma\sigma^2(k/p)\frac{(\omega - \Gamma\omega^\star)}{1-\Gamma^2}m\right|^2 - (k/p)^2 g(\omega, \omega^\star). \tag{S25}$$

Using Eq. (S24) in combination with Eq. (S25) and Eq. (S19), one obtains Eq. (14) in the main text. Also substituting Eqs. (S24) and (S25) the first of Eqs. (S18) and the definition of m above, one obtains Eqs. (15) in the main text.

S5. GENERAL RESULTS FOR SYMMETRIC MATRICES

In the case of a symmetric matrix, all the eigenvalues lie on the real axis. For this reason, it is no longer useful to consider the eigenvalue density, as defined in Eq. (4) of the main text. This definition has the normalisation $\int d^2\omega \rho(\omega) = 1$, where the integral is taken over the whole complex plane. Instead, we now define the real eigenvalue density to be normalised such that $\int d\omega_x \rho_x(\omega_x) = 1$, where we still have

$$\rho_x(\omega_x) = \left\langle \frac{1}{N} \sum_i \delta(\omega - \lambda_i) \right\rangle, \tag{S26}$$

but now the delta functions are taken to have only a real argument. In this case, one instead obtains the eigenvalue density from the trace of the resolvent matrix via [S10, S11]

$$\rho_x(\omega_x) = -\frac{1}{\pi} \lim_{\epsilon \to 0} \mathrm{Im}\left[G(\omega_x + i\epsilon)\right]. \tag{S27}$$

The definition of G remains the same as in Eq. (5) of the main text. However, because we only consider real values of $\omega = \omega_x$, the resolvent must be the analytic [S12]. When this is the case, $iw_k^\star$ satisfies Eqs. (S15). Solving Eqs. (S15) for $iw_k^\star$ and substituting this into Eq. (S13), one thus obtains Eqs. (16) and (17) in the main text.

S6. GENERAL RESULTS FOR THE OUTLIER EIGENVALUES

So far, we have discussed how one can deduce the properties of the bulk region of the eigenvalue spectrum of $\underline{\underline{a}}$, to which most of the eigenvalues of confined. We did this by setting $\mu = 0$, which only has the effect of removing the outlier eigenvalue [S1, S5, S7] without affecting the bulk region. We now reintroduce a non-zero value of μ and deduce the location of the outlier eigenvalue in the complex plane.

We begin with Eq. (19) of the main text, which is simply the definition of an eigenvalue of the matrix $\underline{\underline{a}}$,

$$\det\left[\lambda_{\mathrm{outlier}} \underline{\underline{\mathbb{1}}} - \underline{\underline{z}} - p^{-1} \underline{\underline{\mu}}\right] = 0, \tag{S28}$$

where we define $\underline{\underline{z}} = \underline{\underline{a}} - p^{-1}\underline{\underline{\mu}}$, with $\underline{\underline{\mu}} = p\left\langle \underline{\underline{a}} \right\rangle$.

The elements of the matrix $\underline{\underline{\mu}}$, as was shown in Section S2 [see also Eq. (9) of the main text], can be replaced by $\mu \frac{k_i k_j}{N}$. This is a rank-1, block-structured matrix. Let us group nodes with the same degree and introduce a block index (superscript) so that $\mu_{ij}^{kl} = \frac{kl}{N}$ is the element in ith row, the jth column of the block in the kth row of blocks and the lth row of blocks.

As was mentioned in the main text, we see that we can rewrite Eq. (S28) as

$$\det\left[\underline{\underline{\mathbb{1}}} - p^{-1} \underline{\underline{G_0 \mu}}\right] = 0, \tag{S29}$$

where we write $\underline{\underline{G_0}} = \left[\lambda_{\mathrm{outlier}} \underline{\underline{\mathbb{1}}} - \underline{\underline{z}}\right]^{-1}$.

We follow the reasoning in Ref. [S1], which also deals with block structured matrices (see Section S3 of the Supplemental Material of this reference in particular). There, it is demonstrated for general block-structured matrices that the resolvent matrix $\underline{\underline{G_0}}$ is diagonal and has elements $G_{ij}^{kl} = \delta_{ij}\delta_{kl}G_k$, where $G_k = iw_k^\star$ is the contribution to the resolvent corresponding to the kth block. The resolvent is evaluated outside the bulk region, since we are dealing with an outlier eigenvalue, so $iw_k^\star$ is given by the expression in Eq. (S15).

Now, we use Sylvester's determinant identity

$$\det\left[\mathbb{1}_m + \underline{\underline{AB}}\right] = \det\left[\mathbb{1}_k + \underline{\underline{BA}}\right], \tag{S30}$$

which is valid for combinations of $m \times k$ matrices $\underline{\underline{A}}$ and $k \times m$ matrices $\underline{\underline{B}}$. Noting that $\underline{\underline{\mu}}$ can be written as a product of two vectors of dimension $N \times 1$ and $1 \times N$, i.e. $\underline{\underline{\mu}} = \mu(pN)^{-1}\underline{v}\underline{v}^T$ with $v_i^k = k$, one finds from Eq. (S29)

$$\begin{aligned}\det\left[\underline{\underline{\mathbb{1}}}_N - p^{-1}\underline{\underline{G_0}}\underline{\underline{\mu}}\right] = \det\left[\underline{\underline{\mathbb{1}}}_1 - \mu(p^2N)^{-1}\underline{v}^T\underline{\underline{G_0}}\underline{v}\right] = 0,\\ \Rightarrow 1 - \mu\sum_k \gamma(k)(k/p)^2(iw_k^\star) = 0,\end{aligned} \tag{S31}$$

where we have used $N_k/N \to \gamma(k)$ when $N \to \infty$, where N_k is the number of nodes with degree k. Using Eqs. (S15), one thus obtains Eqs. (21) of the main text.

S7. CORRECTIONS TO KNOWN RESULTS FOR NON-ZERO NETWORK HETEROGENEITY

In Sections S4, S5 and S6, we derived general expressions for the bulk region and the outlier eigenvalue, which are valid for an arbitrary network degree distribution $\gamma(k)$. These expressions, while useful, are not always easy to evaluate and they do not offer us an intuitive understanding of the effects of a non-trivial complex network structure on the eigenvalue spectrum.

In this section, we derive the approximations to the eigenvalue spectrum discussed in Section V of the main text. These approximations are valid for small values of the network heterogeneity s^2, which is defined as

$$s^2 = \sum_k \gamma(k)(k-p)^2/p^2. \tag{S32}$$

A. Bulk region

1. $\Gamma = 0$*: Universal circular law and bulk density*

In this section, we derive Eqs. (26)–(29) in the main text, which gives the eigenvalue density in the case $\Gamma = 0$. Inside the bulk region of the eigenvalue spectrum, the boundary of which is given by Eq. (24) of the main text, the resolvent is given by Eqs. (14) and (15) in the main text. In the special case $\Gamma = 0$, one obtains

$$G(\omega, \omega^\star) = \sum_k \gamma(k)\frac{\omega^\star}{|\omega|^2 - (k/p)^2 g(|\omega|)}, \tag{S33}$$

where the function $g(|\omega|)$ is obtained by solving

$$\frac{1}{\sigma^2} = \sum_k \gamma(k) \frac{(k/p)^2}{|\omega|^2 - (k/p)^2 g(|\omega|)}. \tag{S34}$$

Noting that $\rho(|\omega|) = \frac{1}{\pi}\mathrm{Re}\left\{\frac{\partial}{\partial\omega^\star}G(\omega,\omega^\star)\right\}$ [see Eq. (6) of the main text], we obtain for the eigenvalue density

$$\rho(|\omega|) = \frac{1}{\pi}\mathrm{Re}\left\{\sum_k \gamma(k)\frac{1}{|\omega|^2 - (k/p)^2 g(|\omega|)} - \sum_k \gamma(k)\left[\omega - (k/p)^2\frac{\partial g}{\partial \omega^\star}\right]\frac{\omega^\star}{[|\omega|^2 - (k/p)^2 g(|\omega|)]^2}\right\}. \tag{S35}$$

Differentiating Eq. (S34), we also obtain

$$\frac{\partial g}{\partial\omega^\star}\sum_k \gamma(k)\frac{(k/p)^4}{[|\omega|^2 - (k/p)^2 g]^2} = \omega\sum_k \gamma(k)\frac{(k/p)^2}{[|\omega|^2 - (k/p)^2 g]^2}. \tag{S36}$$

Eliminating $\partial g/\partial\omega^\star$ from Eq. (S35) using Eq. (S36), we then arrive Eq. (26) in the main text.

Now we turn our attention to the small-s^2 expansion in Eq. (29) of the main text. Noting that $g(|\omega|)$ fully determines the eigenvalue density, we merely need to find an approximation for $g(|\omega|)$ up to first order in s^2 and insert this into Eq. (S33) to obtain the eigenvalue density.

We suppose that we can approximate $g \approx g_0 + s^2 g_1$. We also make the substitution $k = p(1+\Delta_k)$, so that $\sum_k \gamma(k)(k-p)^2/p^2 = \sum_k \gamma(k)\Delta_k^2 = s^2$. Expanding the summand of Eq. (S34) as a series in Δ_k, carrying out the sums over k and equating terms of the same order in s^2 on either side, we then have

$$\begin{aligned} \frac{1}{|\omega|^2 - g_0} &= \frac{1}{\sigma^2}, \\ |\omega|^4 + 3g_0|\omega|^2 + g_1|\omega|^2 - g_0 g_1 &= 0. \end{aligned} \tag{S37}$$

One can solve these equations simultaneously to obtain

$$\begin{aligned} g_0 &= |\omega|^2 - \sigma^2, \\ g_1 &= \frac{|\omega|^2}{\sigma^2}(3\sigma^2 - 4|\omega|^2). \end{aligned} \tag{S38}$$

Now, expanding the right-hand side of Eq. (S33) in a similar way, we obtain

$$G(\omega,\omega^\star) \approx \frac{\omega^\star}{|\omega|^2 - g_0} + s^2\omega^\star\frac{g_0|\omega|^2 + 3g_0^2 + g_1|\omega|^2 - g_0 g_1}{(|\omega|^2 - g_0)^3}. \tag{S39}$$

Substituting the expressions for g_0 and g_1 in Eq. (S38) into Eq. (S39), we finally obtain

$$G(\omega,\omega^\star) = \frac{\omega^\star}{\sigma^2}\left[1 + s^2\left(3 - 4\frac{|\omega|^2}{\sigma^2}\right)\right], \tag{S40}$$

from which one recovers Eq. (29) of the main text using Eq. (6), after dividing through by an appropriate normalising factor [which doesn't affect the approximation to order $O(s^2)$].

2. $\Gamma \neq 0$ *and* $\Gamma \neq 1$*: Modified elliptic law for small* s^2

Now, we demonstrate how Eq. (30) of the main text [the small-s^2 approximation for the boundary of the bulk of the eigenvalue spectrum for arbitrary Γ] can be derived from Eqs. (10). Our aim is to find ω_y as a function of ω_x to first order in s^2 in such a way that the leading eigenvalue of the bulk region is also correctly predicted to first order in s^2 [this is given in Eq. (31) of the main text]. To this end, we perform a similar expansion as in the previous subsection. We do this by expanding the quantities $h(\omega_x)$ and $\omega_y(\omega_x)$ to first order in s^2.

Expanding Eqs. (10) of the main text, we have up to first order in s^2

$$\begin{aligned}\frac{1}{\sigma^2} &\approx \frac{1}{\omega_x^2\left[1-\frac{\Gamma\sigma^2}{(1+\Gamma)}(h_0+s^2h_1)\right]^2+\left[\omega_y^{(0)}+\frac{s^2}{2}\omega_y^{(1)}\right]^2\left[1+\frac{\Gamma\sigma^2}{(1-\Gamma)}(h_0+s^2h_1)\right]^2}+s^2\frac{f_\sigma}{\sigma^2},\\ h &\approx \frac{1}{\omega_x^2\left[1-\frac{\Gamma\sigma^2}{(1+\Gamma)}(h_0+s^2h_1)\right]^2+\left[\omega_y^{(0)}+\frac{s^2}{2}\omega_y^{(1)}\right]^2\left[1+\frac{\Gamma\sigma^2}{(1-\Gamma)}(h_0+s^2h_1)\right]^2}+s^2\frac{f_h}{\sigma^2}.\end{aligned} \tag{S41}$$

We note that we have included contributions of the order s^2 to the denominator of the leading order terms so as to correctly preserve the critical value of ω_y^2 at which $\omega_y^2=0$, in a similar way to the procedure in Refs. [S13, S14]. Eliminating h in Eqs. (S41), we then find (writing simply $\left[\omega_y^{(0)}+\frac{s^2}{2}\omega_y^{(1)}\right]^2\approx\omega_y^2$, understanding that ω_y^2 is approximate to first order in s^2)

$$\omega_x^2\left[\frac{1}{(1+\Gamma)}-s^2\frac{\Gamma}{(1+\Gamma)}(f_h-f_\sigma)\right]^2+\omega_y^2\left[\frac{1}{(1-\Gamma)}+s^2\frac{\Gamma}{(1-\Gamma)}(f_h-f_\sigma)\right]^2\approx\sigma^2(1+s^2f_\sigma). \tag{S42}$$

This hints at the form of the solution. Clearly, we will end up with some sort of modified ellipse. Now, by performing the expansion of Eqs. (10) in the main text, as in the previous section, and comparing coefficients of s^2 with Eqs. (S41), one obtains

$$\begin{aligned} f_h &= f_\sigma+\frac{2\Gamma}{\sigma^2}\left[\frac{\omega_y^2}{(1-\Gamma)^2}-\frac{\omega_x^2}{(1+\Gamma)^2}\right]-1,\\ f_\sigma &= 1-\frac{4\Gamma}{\sigma^2}\left[\frac{\omega_y^2}{(1-\Gamma)^2}-\frac{\omega_x^2}{(1+\Gamma)^2}\right]-\frac{\Gamma^2}{\sigma^2}\left[\frac{\omega_y^2}{(1-\Gamma)^2}+\frac{\omega_x^2}{(1+\Gamma)^2}\right]\\ &\quad+\frac{4\Gamma^2}{\sigma^4}\left[\frac{\omega_y^2}{(1-\Gamma)^2}-\frac{\omega_x^2}{(1+\Gamma)^2}\right]^2,\end{aligned} \tag{S43}$$

where we have used

$$\frac{1}{\sigma^2}\approx h_0\approx\frac{1}{\omega_x^2\left[1-\frac{\Gamma\sigma^2}{(1+\Gamma)}h\right]^2+\omega_y^2\left[1+\frac{\Gamma\sigma^2}{(1-\Gamma)}h\right]^2} \tag{S44}$$

in the coefficient of s^2 (this does affect our approximation to first-order in s^2).

Expanding Eq. (S42) to first order in s^2 using Eqs. (S43), quartic terms in ω_x and ω_y cancel and one obtains

$$\frac{\omega_x^2}{(1+\Gamma)^2}\left[1-s^2(1+2\Gamma-\Gamma^2)\right]+\frac{\omega_y^2}{(1-\Gamma)^2}\left[1-s^2(1-2\Gamma-\Gamma^2)\right]=\sigma^2. \tag{S45}$$

Finally, noting that $1/(1+x)^2 \approx 1+2x$ for small x, we arrive at Eq. (30) in the main text.

We note that if we set $\omega_y = 0$ in Eq. (S45), we obtain $\omega_x = \sqrt{(1+\Gamma) + s^2(1+3\Gamma+\Gamma^2-\Gamma^3)} \approx (1+\Gamma) + s^2(1+3\Gamma+\Gamma^2-\Gamma^3)/2$. We can compare this result with the expansion of Eq. (13) of the main text. Letting $\lambda_{\text{edge}} \approx \lambda_0 + s^2\lambda_1$ and $A = A_0 + s^2 A_1$, we obtain by comparing coefficients of s^2 in this expansion

$$\begin{aligned}
\frac{1}{\sigma^2} &= \frac{1}{(\lambda_0 - A_0\Gamma\sigma^2)^2}, \\
0 &= \lambda_0^2 + A_0\Gamma\sigma^2\lambda_0 - 2\lambda_0\lambda_1 + 2A_1\Gamma\sigma^2\lambda_0 + 2A_0\Gamma\sigma^2\lambda_1 - 2A_0A_1\Gamma^2\sigma^4, \\
A_0 &= \frac{1}{\lambda_0 - A_0\Gamma\sigma^2}, \\
A_1 &= \frac{A_0\Gamma\sigma^2\lambda_0 - \lambda_0\lambda_1 + A_1\Gamma\sigma^2\lambda_0 + A_0\Gamma\sigma^2\lambda_1 - A_0A_1\Gamma^2\sigma^4}{(\lambda_0 - A_0\Gamma\sigma^2)^3}.
\end{aligned} \tag{S46}$$

Solving these simultaneously, we obtain

$$\begin{aligned}
A_0 &= \frac{1}{\sigma}, \\
\lambda_0 &= (1+\Gamma)\sigma, \\
A_1 &= -\frac{(1+\Gamma)^2}{2\sigma}, \\
\lambda_1 &= \frac{\sigma}{2}(1+3\Gamma+\Gamma^2-\Gamma^3),
\end{aligned} \tag{S47}$$

which agrees with the expression in Eq. (31) of the main text. This means that the expansion we performed to obtain the modified elliptic law in Eq. (30) of the main text correctly preserved the point at which $\omega_y \to 0$ to first order in s^2, as desired.

3. $\Gamma = 1$: *Modified semi-circular law*

We now derive the modified semi-circular law in Eq. (33) of the main text. This is a first-order (in s^2) approximation to the eigenvalue density along the real axis in the case where $\Gamma = 1$, where we also preserve the point $\omega_x = \omega_c$ at which the eigenvalue density goes to zero to first order in s^2.

We begin with Eqs. (17) in the main text and expand in a similar way to the previous subsection to obtain

$$\begin{aligned}
A &\approx \frac{1}{\omega_x - \sigma^2 A} + s^2\frac{A\sigma^2\omega_x}{(\omega_x - \sigma^2 A)^3}, \\
G &\approx \frac{1}{\omega_x - \sigma^2 A} + s^2\frac{A^2\sigma^4}{(\omega_x - \sigma^2 A)^3}.
\end{aligned} \tag{S48}$$

Now the aim is to obtain an expression for A that is accurate to first order in s^2 and that preserves the square root singularity of the eigenvalue density at the edge of the bulk of the eigenvalue spectrum. The procedure we use is similar to Ref. [S13].

We begin by noting that the zeroth-order approximation for A satisfies

$$\sigma^2 A_0^2 - \omega_x A_0 + 1 = 0. \tag{S49}$$

Using this, we can rewrite the coefficient of s^2 in the first of Eqs. (S48) to obtain

$$\sigma^2 A^2 - \omega_x A + 1 - s^2\omega_x(1-\omega_x A)A \approx 0. \tag{S50}$$

We can thus solve this quadratic expression for A and find

$$A = \frac{1}{2(\sigma^2 + s^2\omega_x^2)}\left[\omega_x(1+s^2) - \sqrt{\omega_x^2(1-2s^2+s^4) - 4\sigma^2}\right]. \tag{S51}$$

Similarly, we find for the trace of the resolvent

$$G \approx A - s^2\sigma^2 A^3. \tag{S52}$$

Noting Eq. (16) in the main text , we see that one only obtains a non-zero eigenvalue density when the argument of the radical in Eq. (S51) is negative. We thus see that the critical value of ω^2 at which the eigenvalue density switches from non-zero to zero is (to first order in s^2)

$$\omega_c^2 \approx 4\sigma^2(1+2s^2). \tag{S53}$$

Since we only wish to preserve this critical value to leading order in s^2, we can ignore the term proportional to s^4 in Eq. (S51). We can thus rewrite A to leading order in s^2 as

$$A = \frac{2}{\omega_c^2}\left[1 - 2s^2\left(\frac{2\omega^2}{\omega_c^2} - 1\right)\right]\left[\omega(1+s^2) - (1-s^2)\sqrt{\omega^2-\omega_c^2}\right]. \tag{S54}$$

We use this expression to find an approximation for A^3 that is valid to first order in s^2 and that preserves the singularity at ω_c^2. We find

$$s^2\sigma^2 A^3 \approx s^2\frac{2}{\omega_c^4}\left[\omega^3 + 3\omega(\omega^2-\omega_c^2) - (4\omega^2-\omega_c^2)\sqrt{\omega^2-\omega_c^2}\right]. \tag{S55}$$

Finally, substituting this expression into Eq. (S52) and using Eq. (16) in the main text, we arrive at the modified semi-circular law in Eq. (33).

B. Outlier eigenvalues: approximate expression for small s^2 and general Γ

In this subsection, we begin with Eqs. (21) in the main text and derive the approximate expression for the outlier eigenvalue in Eq. (32) of the main text, which is accurate to first order in s^2. In a similar spirit to the previous section, we imagine that we can write $A \approx A_0 + s^2 A_1$ and $\lambda_{\text{outlier}} \approx \lambda_0 + s^2\lambda_1$. We then expand Eqs. (21) and equate terms with the same power of s^2 to obtain

$$\begin{aligned}
A_0 &= \frac{1}{\lambda_0 - A_0\Gamma\sigma^2},\\
A_1 &= \frac{A_0\Gamma\lambda_0\sigma^2 - \lambda_0\lambda_1 + A_1\Gamma\lambda_0\sigma^2 + A_0\Gamma\lambda_1\sigma^2 - A_0A_1\Gamma^2\sigma^4}{[\lambda_0 - A_0\Gamma\sigma^2]^3},\\
\frac{1}{\mu} &= \frac{1}{\lambda_0 - A_0\Gamma\sigma^2},\\
0 &= \frac{\lambda_0^2 - \lambda_0\lambda_1 + A_1\Gamma\lambda_0\sigma^2 + A_0\Gamma\lambda_1\sigma^2 - A_0A_1\Gamma^2\sigma^4}{[\lambda_0 - A_0\Gamma\sigma^2]^3}.
\end{aligned} \tag{S56}$$

From the first and third of these equations, one thus finds $A_0 = 1/\mu$ and $\lambda_0 = \mu + \Gamma\sigma^2/\mu$. Substituting these expressions into the second and fourth of Eqs. (S56), one finds

$$A_1 = \frac{1}{\mu} + s^2 \frac{\Gamma\sigma^2}{\mu^3},$$
$$\lambda_1 = \mu + \frac{\Gamma\sigma^2}{\mu}. \tag{S57}$$

Combining the expressions for λ_0 and λ_1 above, we arrive at Eq. (32) of the main text.

S8. SOME EXAMPLES THAT ARE VALID FOR ANY VALUE OF s^2

A. Dichotomous degree distribution

To produce the solid lines in Figs. 1 and 2a, a dichotomous degree distribution was used with

$$\gamma(k) = \frac{1}{2}\left(\delta_{k,k_1} + \delta_{k,k_2}\right). \tag{S58}$$

In the case of Fig. 2a, all that one is required to calculate from this distribution is the mean degree and heterogeneity, which are given by respectively

$$p = \frac{1}{2}(k_1 + k_2),$$
$$s^2 = \frac{1}{2p^2}\left[(k_1 - p)^2 + (k_2 - p)^2\right]. \tag{S59}$$

In the case of Fig. 1, one must solve Eqs. (10) of the main text. That is, for each value of ω_x, one must first solve the following simultaneous equations (this is best done numerically) for h and ω_y

$$h = \frac{1}{2p}\left[\frac{k_1}{\omega_x^2[1 - \frac{\Gamma\sigma^2 k_1}{(1+\Gamma)p}h]^2 + \omega_y^2[1 + \frac{\Gamma\sigma^2 k_1}{(1-\Gamma)p}h]^2} + \frac{k_2}{\omega_x^2[1 - \frac{\Gamma\sigma^2 k_2}{(1+\Gamma)p}h]^2 + \omega_y^2[1 + \frac{\Gamma\sigma^2 k_2}{(1-\Gamma)p}h]^2}\right],$$
$$\frac{1}{\sigma^2} = \frac{1}{2p^2}\left[\frac{k_1^2}{\omega_x^2[1 - \frac{\Gamma\sigma^2 k_1}{(1+\Gamma)p}h]^2 + \omega_y^2[1 + \frac{\Gamma\sigma^2 k_1}{(1-\Gamma)p}h]^2} + \frac{k_2^2}{\omega_x^2[1 - \frac{\Gamma\sigma^2 k_2}{(1+\Gamma)p}h]^2 + \omega_y^2[1 + \frac{\Gamma\sigma^2 k_2}{(1-\Gamma)p}h]^2}\right]. \tag{S60}$$

B. Uniform degree distribution

The dichotomous distribution discussed above is fairly straightforward to implement. The uniform distribution used in Figs. 2b, 3, 4, 5 and 6 requires some additional manipulation.

The degree distribution is given by

$$\gamma(k) = \frac{1}{2[\sqrt{3}sp]} \sum_{l=p-[\sqrt{3}sp]}^{p+[\sqrt{3}sp]} \delta_{k,l}. \tag{S61}$$

We take for example the calculation of the edge of the bulk of the eigenvalue spectrum for Fig. 4. The solid lines in any of the aforementioned figures can be calculated in a similar way.

We begin with Eqs. (13) of the main text, from which we obtain

$$\frac{1}{\sigma^2} = \frac{1}{2[\sqrt{3}sp]} \sum_{k=p-[\sqrt{3}sp]}^{p+[\sqrt{3}sp]} \frac{(k/p)^2}{(\lambda_{\text{edge}} - \Gamma\sigma^2 kA/p)^2},$$
$$A = \frac{1}{2[\sqrt{3}sp]} \sum_{k=p-[\sqrt{3}sp]}^{p+[\sqrt{3}sp]} \frac{k/p}{(\lambda_{\text{edge}} - \Gamma\sigma^2 kA/p)}. \tag{S62}$$

For large p, these sums can be approximated by integrals such that we can write (using the substitution $x = k/p$)

$$\frac{1}{\sigma^2} = \frac{1}{2\sqrt{3}s} \int_{1-\sqrt{3}s}^{1+\sqrt{3}s} dx \frac{x^2}{(\lambda_{\text{edge}} - \Gamma\sigma^2 xA)^2},$$
$$A = \frac{1}{2\sqrt{3}s} \int_{1-\sqrt{3}s}^{1+\sqrt{3}s} dx \frac{x}{(\lambda_{\text{edge}} - \Gamma\sigma^2 xA)}. \tag{S63}$$

These are standard integrals that can be evaluated. One thus has to solve the following simultaneous equations numerically to find A and λ_{edge} (letting $x_1 = 1 - \sqrt{3}s$ and $x_2 = 1 + \sqrt{3}s$ for shorthand)

$$\frac{1}{\sigma^2} = \frac{1}{(x_2 - x_1)(\Gamma\sigma^2 A)^3} \left\{ \frac{\lambda_{\text{edge}}^2}{\lambda_{\text{edge}} - \Gamma\sigma^2 A x_2} - \frac{\lambda_{\text{edge}}^2}{\lambda_{\text{edge}} - \Gamma\sigma^2 A x_1} + \Gamma\sigma^2 A (x_2 - x_1) + 2\lambda_{\text{edge}} \ln\left[\frac{\lambda_{\text{edge}} - \Gamma\sigma^2 A x_2}{\lambda_{\text{edge}} - \Gamma\sigma^2 A x_1}\right] \right\},$$
$$A = -\frac{1}{(x_2 - x_1)(\Gamma\sigma^2 A)^2} \left\{ \lambda_{\text{edge}} \ln\left[\frac{\lambda_{\text{edge}} - \Gamma\sigma^2 A x_2}{\lambda_{\text{edge}} - \Gamma\sigma^2 A x_1}\right] + A\Gamma\sigma^2 (x_2 - x_1) \right\}. \tag{S64}$$

These simultaneous equations can be solved numerically to yield λ_{edge} as a function of (for example) Γ, as is plotted in Figs. 4 of the main text.

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