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M. J. SEVILLA, A. G. CAMACHO y P. ROMERO



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Relations between the Celestial Ephemeris Pole and several axes considered in the Earth rotation theories.

M.J. SEVILLA, P. ROMERO ,A.G.CAMACHO.

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Abstract: Relations between the nutational and polar motion for different axes involved in the Earth's rotation theories are pointed out.

Keywords: Earth's rotation, Nutation, Polar motion.

Introduction.

From the definition of the rotation, the angular momentum and the figure axes; and from a suitable definition of the Earth reference system; it is possible to obtain several geometrical relations between these axes. So if we obtain the expressions for the motion of an axis (rotation axis) , we can easily deduce the spatial motions (precession, nutation) and the terrestrial motions (polar motion) for the other axes and also for their mean (diurnal) positions . The position of a particular pole: "the Celestial Ephemeris Pole "has a special interest. Its corresponding axis has an important role in the theories of Earth's rotation, precession-nutation, and Earth reference systems.

1. Basic Assumptions.

Let us consider a general deformable Earth's model; and the following fundamental reference systems :a) A geocentric inertial frame $\{I\} = \{OXYZ\}$ (associated with a fixed ecliptic plane).b) An Earth fixed frame $\{T\} = \{Oxyz\}$ (rotating with angular velocity ω ; usually, it can be associated with the Tisserand axes, Oz being the mean figure axis).c) The Nutation reference system $\{N\} = \{Ox^*y^*z^*\}$, (with the Oz^* axis having a spatial constant direction close to the Oz axis direction, and ω_0 being the constant angular velocity of rotation with respect to the inertial frame $\{I\}$). So, we can write

$$\omega = \omega_0 + \delta\omega \quad (1-1)$$

where

$$\omega_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and

$$\delta\omega = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} \Omega, \quad (1-2)$$

in the $\{T\}$ system.

The Nutation system $\{N\}$, used by Jeffreys & Vicente (1957), Molodensky (1961), Shen & Manshina (1976), Smith (1977), Marh (1979) and Moritz(1981); enables us to obtain in an easy way the spatial positions (nutation) and the terrestrial positions (polar motion) for the different axes considered in the Earth rotation theories. Following Moritz (1980) ,we will obtain for a deformable model some geometrical relations between several important axes.

As the $\delta\omega$ vector is an infinitesimal vector, we can make the transformations between the (T) and (N) reference systems using an infinitesimal rotation matrix. So, the transformation equations

$$\underline{x} = R_1(\theta_3) R_2(\theta_2) R_3(\theta_1) \underline{x}^0,$$

can be written as

$$\underline{x} = [I + \Theta] \underline{x}^0, \quad (1-3)$$

where \underline{x} and \underline{x}^0 represent the position vectors in the (T) and (N) references respectively, I is the unit matrix and Θ is an infinitesimal rotation matrix given by

$$\Theta = \begin{bmatrix} 0 & \theta_3 & -\theta_2 \\ -\theta_3 & 0 & \theta_1 \\ \theta_2 & -\theta_1 & 0 \end{bmatrix}$$

If we define the vector

$$\underline{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix},$$

we can write the equations (1-3) in the form

$$\underline{x} = \underline{x}^0 - \underline{\theta} \wedge \underline{x}^0, \quad (1-4)$$

then, in a first order approximation, the inverse transformation equations are

$$\underline{x}^0 = \underline{x} + \underline{\theta} \wedge \underline{x}, \quad (1-5)$$

2. Polar motion equations.

As the Oz^0 axis of the (N) reference system is fixed with respect to the inertial frame (I), the nutational motion with respect to (N) of the rotation axis (R), the mean figure axis (z), the instantaneous figure axis (F), and the angular momentum axis (H), may be obtained (Moritz, 1980) as

$$\begin{aligned} \underline{n}_R &= \underline{e}_R^0 - \underline{e}_3^0, \\ \underline{n}_z &= \underline{e}_3^0 - \underline{e}_3^0, \\ \underline{n}_F &= \underline{e}_F^0 - \underline{e}_3^0, \\ \underline{n}_H &= \underline{e}_H^0 - \underline{e}_3^0 \end{aligned} \quad (2-1)$$

\underline{e}_3^0 , \underline{e}_R^0 , \underline{e}_z^0 , \underline{e}_F^0 , and \underline{e}_H^0 being the unit vectors along the $Oz_0 \equiv Ox_3^0$, R, z, F and H axes respectively.

Similarly to obtain the polar motion with respect to (T) we have

$$\begin{aligned}\underline{\omega}_R &= \underline{\omega}_R^0 - \underline{\omega}_2^0, \\ \underline{\omega}_z &= \underline{\omega}_z^0 - \underline{\omega}_2^0, \\ \underline{\omega}_F &= \underline{\omega}_F^0 - \underline{\omega}_2^0, \\ \underline{\omega}_H &= \underline{\omega}_H^0 - \underline{\omega}_2^0.\end{aligned}\tag{2-2}$$

And we get for any axis

$$\underline{\omega} = \underline{n} - \underline{n}_2^0.\tag{2-3}$$

3. Unit vectors along the different axes.

3.1 Instantaneous rotation axis.

Let us consider the instantaneous rotation vector of the Earth with respect to the inertial space. We write it as

$\underline{\omega}$ in the (T) reference system

$\underline{\omega}^0$ in the (N) reference system.

So, using the transformation equation (1-5), we have

$$\underline{\omega} = \underline{\omega}^0 - \underline{\theta} \wedge \underline{\omega}^0.\tag{3-1}$$

On the other hand, we can decompose $\underline{\omega}^0$ into two rotations : R_1 (rotation with angular velocity $\dot{\theta}$ of (T) with respect to (N)) and R_2 (rotation with angular velocity ω_0 of (N) with respect to (I)). Then we have

$$\underline{\omega}^0 = \dot{\theta} + \underline{\omega}_0.\tag{3-2}$$

To study the polar motion and the nutational motion, without considering the variation of the rotation speed, we can take $\theta_0 = 0$ then $\underline{\theta}$ is reduced to $\underline{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ 0 \end{bmatrix}$.

If we now define the complex number $w = \theta_1 + i \theta_2$, then we can write $\underline{\theta} = w \underline{\omega}_1^0$; and also, if we set $w = a \cdot \exp(i(\varphi t + \psi))$, we have

$$\dot{\theta} = \dot{w} \underline{\omega}_1^0 = i \varphi w \underline{\omega}_1^0 = i \varphi \underline{\theta}.$$

So the expression (3-2) becomes

$$\underline{\omega}^0 = i \varphi \underline{\theta} + \underline{\omega}_0.\tag{3-3}$$

and the unit vector along the rotation axis results

$$\underline{\omega}_R^0 = \frac{\underline{\omega}^0}{|w\omega_1^0|} = \frac{\underline{\omega}^0}{a} = \underline{\omega}_3^0 + i \frac{r}{a} \underline{\theta}.\tag{3-4}$$

3.2 Mean figure axis

The mean figure axis is the Oz axis of the (T) reference. Then, using (1-5), in the (N) reference it can be expressed as

$$\begin{aligned}\text{with } \underline{\omega}_3^0 &= \underline{\omega}_3 + \underline{\theta} \wedge \underline{\omega}_3 \\ \underline{\omega}_3 &= \underline{\omega}_3^0 = [0 \ 0 \ 1]^T, \\ \text{and } \underline{\theta} \wedge \underline{\omega}_3 &= -i \underline{\theta}. \\ \text{Then } \underline{\omega}_2^0 &= \underline{\omega}_3^0 - i \underline{\theta}.\end{aligned}\tag{3-5}$$

3.3 Instantaneous figure axis.

The instantaneous figure axis will be the one corresponding to the eigenvector of the Earth inertia tensor. This is, the one which direction cosines (f_1, f_2, f_3) relatives to the Earth fixed frame (T) oblige

$$(f_1, f_2, f_3) \cdot C \begin{vmatrix} f_1 \\ f_2 \\ f_3 \end{vmatrix} = \text{maximum}.$$

C , being the inertia tensor for the deformed Earth. Then, the following expressions must be satisfied

$$\begin{aligned} (A + c_{11} - \lambda) f_1 + c_{12} f_2 + c_{13} f_3 &= 0, \\ c_{12} f_1 + (A + c_{22} - \lambda) f_2 + c_{23} f_3 &= 0, \\ c_{13} f_1 + c_{23} f_2 + (A + c_{33} - \lambda) f_3 &= 0. \end{aligned}$$

In a first order approximation we can neglect products as $c_{ij} f_i$, and also take $f_3 = 1$, in this way we obtain

$$f_1 = \frac{c_{13}}{C-A}, \quad f_2 = \frac{c_{23}}{C-A}, \quad f_3 = 1.$$

Using the complex numbers

$$\begin{aligned} f &= f_1 + i f_2, \\ C &= c_{13} + i c_{23}, \end{aligned}$$

we can write

$$f = \frac{C}{C-A},$$

In order to express the \underline{f} vector relative to the (N) reference, we use the relation (1-3) getting

$$\underline{\underline{e}}_F^0 = \underline{f} + \underline{\theta} \wedge \underline{f}$$

where

$$\underline{\theta} \wedge \underline{f} = -i \underline{\theta},$$

as $\underline{\underline{e}}_I^0 = \underline{\underline{e}}_I^0$, the unit vector for the instantaneous figure axis results

$$\underline{\underline{e}}_F^0 = f_1 \underline{\underline{e}}_1^0 + f_2 \underline{\underline{e}}_2^0 + \underline{\underline{e}}_3^0 - i \underline{\theta}. \quad (3-6)$$

3.4 Angular-momentum axis.

For a deformable Earth model and in an Earth fixed frame, the angular momentum vector \underline{H} is given by

$$\underline{H} = \underline{C} \underline{\omega} + \underline{h},$$

where \underline{C} is the inertia tensor, and \underline{h} is the relative angular momentum vector.

Using expressions (3-1) and (3-3) the angular velocity $\underline{\omega}$ can be written as

$$\underline{\omega} = \underline{\omega}^0 - \underline{\theta} \wedge \underline{\omega} = \underline{\omega}_0 + \dot{\underline{\theta}} - \underline{\theta} \wedge (\underline{\omega}_0 + \dot{\underline{\theta}}) = \underline{\omega}_0 + \dot{\underline{\theta}} - \underline{\theta} \wedge \underline{\omega},$$

then

$$\underline{H} = C \underline{\omega}_0 + i (r + \Omega) \underline{C} \underline{\theta} + \underline{h} = C \Omega \underline{e}_3 + i A (r + \Omega) \underline{\theta} + \underline{h} + \Omega \underline{c},$$

and, in the (N) reference system, it will be

$$\underline{H}^0 = \underline{H} + \underline{\theta} \wedge \underline{H} = C \Omega \underline{e}_3 + i (r - \epsilon_0 - C \Omega/A) \underline{\theta} + \underline{h} + \Omega \underline{c}.$$

If we write $\epsilon = (C-A)\Omega/A$, and we put $\epsilon_1 = \epsilon_0^0$, H^0 finally becomes

$$\underline{H}^0 = C \Omega \underline{e}_3^0 + i A (\epsilon - \epsilon_1) \underline{\theta} + \underline{h} + \Omega \underline{c} \quad (3-7)$$

and, the unit vector \underline{e}_n^0 along the \underline{H} axis

$$\underline{e}_n^0 = \frac{\underline{H}^0}{|\underline{H}^0|} = \underline{e}_3^0 + i \frac{A(\epsilon - \epsilon_1)}{C\Omega} \underline{\theta} + \frac{\underline{h}}{C\Omega} + \frac{\underline{c}}{C}. \quad (3-8)$$

4. Final formulae to obtain the polar motion and the spatial nutations.

If we define the complex numbers

$$\begin{aligned} n &= n_1 + i n_2, \\ p &= p_1 + i p_2, \end{aligned}$$

giving the nutations and polar motion respectively for any axis, and using the expressions (3-4), (3-5), (3-6), and (3-8), then relations (2-1) and (2-2) enables us to write

$$\begin{aligned} n_R &= i \frac{r}{\Omega} n, \\ n_3 &= -i \frac{r}{\Omega} n, \\ n_F &= \frac{r}{C-A} - i \frac{r}{\Omega} n, \\ n_H &= i \frac{A(\epsilon - \epsilon_1)}{C\Omega} n + \frac{\underline{h}}{C\Omega} + \frac{\underline{c}}{C}, \end{aligned} \quad (4-1)$$

$$\begin{aligned} p_R &= i \frac{r+\Omega}{\Omega} n, \\ p_3 &= 0, \\ p_F &= \frac{r}{C-A}, \\ p_H &= i \frac{A(r+\Omega)}{C\Omega} n + \frac{\underline{h}}{C\Omega} + \frac{\underline{c}}{C}. \end{aligned}$$

If we put $u = \omega_1 + i\omega_2$, from (4-1) we obtain

$$\begin{aligned}
 p_R &= \frac{u}{\Omega}, \\
 p_z &= 0, \\
 p_F &= \frac{c}{c-A}, \\
 p_H &= \frac{A}{c} p_R + \frac{h}{c\Omega} + \frac{c}{c}, \\
 n_R &= \frac{\Omega}{\Omega + \Omega} p_R, \\
 n_z &= -\frac{\Omega}{\Omega + \Omega} p_R, \\
 n_F &= \frac{c}{c-A} \frac{\Omega}{\Omega + \Omega} p_R, \\
 n_H &= \left[\frac{A}{c} \frac{\Omega - \Omega_E}{\Omega + \Omega_E} \right] p_R + \frac{h}{c\Omega} + \frac{c}{c}.
 \end{aligned} \tag{4-2}$$

From a dynamical theory of the Earth's rotation formulated in terms of the components of the polar motion for the rotation axis (as the Liouville equations) we can obtain p_R . Calculating the variations of the Earth inertia tensor components c and the angular momentum h for the deformable earth model considered, and using expressions (4-2) we can easily obtain the nutation and polar motion for any axis.

Lunisolar Torque

We are looking for a general relation between the complex number $w = \theta_1 + i\theta_2$ and the lunisolar torque $L = L_1 + iL_2$ for a deformable model. Let us consider the fundamental equation of motion

$$\underline{H}^0 + \underline{\omega}_0 \wedge \underline{H}^0 = \underline{L}^0. \tag{5-1}$$

For a deformable body, \underline{H}^0 is given by (3-7). Then

$$\begin{aligned}
 \underline{H}^0 &= -A \Omega (\Omega - \Omega_E) \underline{\theta} + i \Omega \underline{h} + i \Omega \underline{\epsilon}, \\
 \text{and } \underline{\omega}_0 \wedge \underline{H}^0 &= \Omega \underline{\epsilon}_3 \wedge \underline{H}^0 = -A \Omega (\Omega - \Omega_E) \underline{\theta} + i \Omega \underline{h} + i \Omega \underline{\epsilon}.
 \end{aligned}$$

Substituting these expressions into (5-1) we get

$$-A(\Omega + \Omega_E)(\Omega - \Omega_E) \underline{\theta} + i(\Omega + \Omega_E) \underline{h} + i\Omega(\Omega + \Omega_E) \underline{\epsilon} = \underline{L}^0,$$

and from this, using the same complex notation as above, we obtain

$$w = \frac{-L}{A(\Omega + \Omega_E)(\Omega - \Omega_E)} + \frac{i(h + \Omega c)}{A(\Omega - \Omega_E)}.$$

The substitution of this relation into (4-1) gives for the forced motion ($L=0$)

$$\begin{aligned}
 n_x^{\text{for.}} &= \frac{iL}{A\Omega(r+\Omega)(r-r_E)} - \frac{r(h+2c)}{A\Omega(r-r_E)}, \\
 n_z^{\text{for.}} &= \frac{iL}{A(r+\Omega)(r-r_E)} + \frac{h+2c}{A(r-r_E)}, \\
 n_r^{\text{for.}} &= \frac{c}{C-A} + \frac{iL}{A(r+\Omega)(r-r_E)} + \frac{h+2c}{A(r-r_E)}, \\
 n_h^{\text{for.}} &= \frac{-iL}{C\Omega(r+\Omega)}, \\
 p_x^{\text{for.}} &= \frac{-iL}{A\Omega(r-r_E)} - \frac{(r+\Omega)(h+2c)}{A\Omega(r-r_E)}, \\
 p_z^{\text{for.}} &= 0, \\
 p_r^{\text{for.}} &= \frac{c}{C-A}, \\
 p_h^{\text{for.}} &= \frac{-iL}{C\Omega(r-r_E)} - \frac{h+2c}{A(r-r_E)}. \tag{5-2}
 \end{aligned}$$

By means of these relations we can see the influence of the deformations in the forced polar motion. Obviously, it results that the forced nutation for the angular-momentum vector does not depend on the internal constitution of the earth model, and that the forced nutation for this axis is zero.

Nutation series for the Celestial Ephemeris Pole.

The nutation series must be calculated for the Celestial Ephemeris Pole, which is the pole having no quasi-diurnal motions with respect to the inertial frame (I) neither with respect to the Earth fixed frame (T). This corresponds to the position resulting of the forced nutation of the mean figure axis

$$n_c = n_z^{\text{for.}},$$

(as its corresponding forced polar motion is zero, $p_z^{\text{for.}}=0$)

The conversion from the nutation system (N) to the Ecliptic system, to whom the classical Nutation theories refer, may be calculated by means of the following transformation

$$\Delta\theta + i\Delta\gamma = -i n_e^{\text{int}}.$$

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Sevilla, M.J., Romero, P., Canacho,A.B.
Instituto de Astronomia y Geodesia
Facultad de Ciencias Matematicas
Ciudad Universitaria
Madrid 28040 , Spain.

Amplitudes of the polar motion and spatial nutations from the Sasao, Okubo, and Saito's equations.

P.ROMERO, M.J. SEVILLA, A.G. CAMACHO

Romero,P.; Sevilla,M.J.; Camacho,A.G., 1985 : Amplitudes of the polar motion and spatial nutations from the Sasao, Okubo and Saito's equations. Proceedings of the Tenth International Symposium on Earth Tides, pp. 337-346.

Abstract : One of the most suitable formulation of the deformable earth's rotational motion is the Sasao, Okubo and Saito theory developped for the Molodensky Earth's model. Their resulting equations, expressed in terms of the components of polar motion for the rotation axis, have been solved in order to obtain the amplitudes of polar motion, and from them the corresponding modifications to the amplitudes for the less realistic rigid Earth's model. Also, the amplitudes of polar motion and spatial nutations for several different axes, including the Celestial Ephemeris Pole, have been obtained.

Keywords : Earth's rotation, polar motion, nutation.

1. The Sasao, Okubo, and Saito's equations.

The Sasao, Okubo, and Saito equations (1980) describe the rotational motion for the Molodensky's Earth model (elastic mantle and liquid inner core). The most suitable form of these equations is given by Moritz (1982) as follows

$$\begin{aligned} A \dot{u} - i(C-A) \Omega u + A_c(v + i \Omega v) + \Omega(c + i \Omega c) &= L, \\ A_c \dot{u} + A_c \dot{v} + i C_c \Omega v + \Omega \dot{c}^e &= 0, \\ c = 1/3 \theta^{-1} a^2 \Omega k_0(u-f) - 1/3 \theta^{-1} a^2 \Omega k_1 \beta v, \\ c^e = \theta^{-1} A_c [\tau(u-f) + \beta v], \end{aligned} \tag{1-1}$$

where, $u = \omega_i + i \omega_3$ corresponds to the rotation of the whole Earth with respect to the inertial space and $v = \chi_i + i \chi_3$ to the rotation of the core with respect to the mantle ; A , C , A_c , C_c are the moments of inertia of the mantle and core respectively ; $c = c_{13} + i c_{33}$ and $c^e = c_{13}^e + i c_{33}^e$ are the variations of the inertia tensor of the whole Earth and the core respectively ; $L = L_1 + i L_2$ is the lunisolar torque related with the complex number $f = f_1 + i f_2$ by the relation

$$iL = (C - A) \Omega f, \tag{1-2}$$

τ , β are parameters given by Sasao & alii (1980), with numerical values are depending on the geophysical Earth model adopted; and k_0 and k_1 are the static and dynamic Love's number related with the Love's number k by

$$k = k_0 + k_1 \frac{v}{f-v}, \tag{1-3}$$

If we write

$$k_s = \frac{3G(C-A)}{a^5 \Omega^2},$$

$$s = \frac{k_0}{k_s} \frac{C-A}{A}, \quad (1-4)$$

$$\epsilon_c = \frac{1 - k_0/k_s}{1+s} \frac{C-A}{A}$$

the two last equations of (1-1) become

$$c = \frac{k_0}{k_s} \frac{C-A}{\Omega} (u - f) - \frac{k_1}{k_s} \frac{C-A}{\Omega} v, \quad (1-5)$$

$$\epsilon_c = \frac{A_c \tau}{\Omega} (u - f) + \frac{A_c \beta}{\Omega} v,$$

and the substitution of (1-5) and (1-2) into (1-1) gives the final equations

$$A(1+s)u - i\epsilon_c(1+s)Au + i[A_c - (C-A)k_1/k_s]\Omega v + [A_c - (C-A)k_1/k_s]v = (1 - k_0/k_s)L + ik_0/k_s L/\Omega$$

$$A(1+\tau)u + A_c(1+\beta)v + iC_c\Omega v = iA_c\tau L/(C-A)\Omega. \quad (1-6)$$

2. Free motion. Proper frequencies.

Let us suppose the solutions for the free motion in the form

$$u = a e^{\lambda st}, \quad (2-1)$$

$$v = b e^{\lambda st},$$

then, with $L = 0$, we can express equations (1-6) as

$$A(1+s)(s-\epsilon_c)u + [A_c - (C-A)k_1/k_s](s+\Omega)v = 0,$$

$$A_c(1+\tau)su + [A_c(1+\beta)s + C_c\Omega]v = 0,$$

therefore, the proper frequencies λ must verify

$$A(1+s)(s-\epsilon_c)[A_c(1+\beta)s + C_c\Omega] - A_c(1+\tau)[A_c - (C-A)k_1/k_s]s(s+\Omega) = 0. \quad (2-2)$$

In a first order approximation, we take $\beta = 0$ and $\epsilon = (C_c - A_c)/C_c = 0$, then (2-2) is reduced to

$$[A(s-\epsilon_c) - \frac{1+\tau}{1+s}[A_c - (C-A)\frac{k_1}{k_s}]]s(s+\Omega) = 0,$$

with the solutions r_1^0 and r_2^0 being

$$r_1^0 = \frac{r_c A}{1 - \frac{1+\gamma}{1+s} [A_c - (C-A) \frac{k_1}{k_s}]},$$

$$r_2^0 = -\Omega.$$

In a second order approximation, we take

$$r_1 = r_1^0 + \zeta_1 \epsilon + \tau_1 \beta,$$

$$r_2 = r_2^0 + \zeta_2 \epsilon + \tau_2 \beta,$$

and it results

$$r_1 = \frac{A}{A - \frac{1+\gamma}{1+s} [A_c - (C-A) \frac{k_1}{k_s}]} r_c,$$

$$r_2 = -[1 + \frac{A(\Omega+r_c)(\epsilon-\beta)}{A(\Omega+r_c) - \frac{1+\gamma}{1+s} \Omega [A_c - (C-A) \frac{k_1}{k_s}]}] \Omega,$$

or

$$r_1 = \frac{A}{A_m} \frac{k_s}{k_s} \frac{C-A}{A} \Omega = \frac{A}{A_m} \frac{k_s}{k_s} r_2,$$

$$r_2 = -[1 + \frac{A}{A_m} (\epsilon - \beta)] \Omega,$$
(2-3)

corresponding to the Chandler Wobble (CW) and to the Nearly Diurnal Free Wobble (NDFW) respectively.

With (2-3) the solutions for the free motion are

$$u^{\text{free}} = a_1 e^{\exp(i\omega_1 t)} + a_3 e^{\exp(i\omega_3 t)},$$

$$v^{\text{free}} = b_1 e^{\exp(i\omega_1 t)} + b_3 e^{\exp(i\omega_3 t)}$$
(2-4)

where the amplitudes a_1 and a_3 must be determined by observation.

3 Amplitudes for the forced motion.

Let us take the solutions of equations (1-6) in the form (1-2); then, using (2-1) we obtain

$$A(1+s)(\sigma-\sigma_c) u + [A_c - (C-A) \frac{k_1}{k_s}] (\sigma+\Omega) v = -[\frac{1}{\Omega} - \frac{\sigma+\Omega}{k_s}] (C-A) \Omega f,$$

$$A_c(1+\tau) \sigma u + [A_c (1+\beta) \sigma + C_c \Omega] v = A_c \tau \sigma f,$$
(3-1)

the determinant Δ of this system being

$$\Delta = A(1+s)(s-s_c) [A_c(1+\beta)s + C_c\Omega] - A_c(1+\tau)[A_c - (C-A) \frac{k_1}{k_s}] s(s+\Omega),$$

and with the same procedure used by Moritz (1980) in the integration of equations for the Poincaré's model, we write

$$\Delta = K (s-s_1) (s-s_2),$$

with

$$K = A(1+s) A_c (1+\beta) - A_c(1+\tau) [A_c - (C-A) \frac{k_1}{k_s}],$$

hence

$$\Delta = A A_c [1 - \frac{A_c}{A}] (s-s_1) (s-s_2) = A_c A_m (s-s_1) (s-s_2), \quad (3-2)$$

From (3-1) we obtain the solutions

$$u = \Delta^{-1} [-[A_c(1+\beta)s + C_c\Omega][1 - \frac{s+\Omega}{\Omega} \frac{k_0}{k_s}] - [A_c - (C-A) \frac{k_1}{k_s}] \frac{s+\Omega}{\Omega} \frac{A_c}{C-A} \tau s] (C-A)\Omega f,$$

$$v = \Delta^{-1} [A_c (1+\tau) (1 - \frac{k_0}{\Omega} \frac{s+\Omega}{k_s}) + A A_c (1+s) (s - s_c) \tau f], \quad (3-3)$$

with Δ given by (3-2).

We now try to obtain q relating the amplitudes for the forced action of the Molodensky's model with the amplitudes corresponding to a rigid Earth. This is

$$q = u / u_o, \quad (3-4)$$

u_o being obtained from (3-3) with $k_0 = k_1 = 0$, $A_c = 0$, and $s_o = s_c$. To avoid the indetermination appearing with this procedure let us write

$$u = - \frac{[A_c - (C-A) \frac{k_1}{k_s}] (s+\Omega)}{A(1+s)(s-s_c)} v = - \frac{[1 - \frac{\Omega+\Omega}{\Omega} \frac{k_0}{k_s}] (C-A)\Omega}{A(1+s)(s-s_c)} f,$$

and from this expression we obtain

$$u_o = - (C-A)\Omega f / A(s-s_c). \quad (3-5)$$

Finally, using (3-2), (3-3) and (3-5) we obtain q as

$$q = \frac{A (s-s_c)}{A_c A_m (s-s_1) (s-s_2)} ([A_c(1+\beta)s + C_c\Omega][1 - \frac{s+\Omega}{\Omega} \frac{k_0}{k_s}] + [A_c - (C-A) \frac{k_1}{k_s}] \frac{s+\Omega}{\Omega} \frac{A_c}{C-A} \tau s). \quad (3-6)$$

4 Precession, nutation and polar motion.

Let us consider the following relations between the different axes for a deformable Earth's model (Sevilla, Romero, Camacho, 1985):

$$\begin{aligned}
 p_R &= u/\Omega, & n_R &= p_R (\varepsilon/\varepsilon+\Omega), \\
 p_Z &= 0, & n_Z &= -p_R (\Omega/\varepsilon+\Omega), \\
 p_F &= c/C-A, & n_F &= c/C-A - p_R (\Omega/\varepsilon+\Omega) = p_H + n_Z, \\
 p_H &= p_R A/C + h/C\Omega + c/C, & n_H &= p_R (A/C - \Omega/\varepsilon+\Omega) + h/C\Omega + c/C = p_H + n_Z,
 \end{aligned} \tag{4-1}$$

Before applying these formulae to deduce the nutation and polar motion we must obtain h and c for the Molodensky's model. As we have taken Tisserand axes for the mantle, h is reduced to the relative angular-momentum produced by the core. Thus

$$h = C^c \chi = (A_c \chi_1, A_c \chi_2, C_c \chi_3),$$

in a first order $\chi_3 = 0$, then the complex number h is

$$h = A_c v. \tag{4-2}$$

To obtain c we use (1-5), this is

$$c = \frac{k_o}{k_s} \frac{C-A}{\Omega} u - \frac{k_1}{k_s} \frac{C-A}{\Omega} v - \frac{k_o}{k_s} \frac{C-A}{\Omega} f. \tag{4-3}$$

5 Polar motion.

For obtaining the free motion, $L = f = 0$, we use the solution (2-4) to obtain from (4-2) and (4-3)

$$\begin{aligned}
 h &= \sum_{j=1}^3 a_j b_j e^{\exp(i\varphi_j t)}, \\
 c &= \sum_{j=1}^3 \left(\frac{k_o}{k_s} \frac{C-A}{\Omega} a_j - \frac{k_1}{k_s} \frac{C-A}{\Omega} b_j \right) e^{\exp(i\varphi_j t)}. \tag{5-1}
 \end{aligned}$$

Then, substituting (2-4) and (5-1) into (4-1) it results

$$\begin{aligned}
 p_R^{\text{free}} &= \sum_{j=1}^3 \frac{a_j}{\Omega} e^{\exp(i\varphi_j t)}, \\
 p_Z^{\text{free}} &= 0, \\
 p_H^{\text{free}} &= \sum_{j=1}^3 \left(\frac{k_o}{k_s} \frac{a_j}{\Omega} - \frac{k_1}{k_s} \frac{b_j}{\Omega} \right) e^{\exp(i\varphi_j t)}, \\
 p_H^{\text{free}} &= \sum_{j=1}^3 \left[\left(\frac{A}{C} + \frac{k_o}{k_s} \frac{C-A}{C} \right) a_j + \left(\frac{A_c}{C} - \frac{k_1}{k_s} \frac{C-A}{C} \right) b_j \right] \frac{1}{\Omega} e^{\exp(i\varphi_j t)}.
 \end{aligned} \tag{5-2}$$

For the forced motion, $L \neq 0$, from (3-4) and (3-1) we have the solutions

$$u = q u_o, \quad (5-3)$$

$$v = q' u + r f, \quad (5-4)$$

with q given by (3-6), and q' and r being

$$q' = -\frac{A_c(1+\tau)\epsilon}{A_c(1+\beta)\epsilon + C_c\theta}, \quad (5-5)$$

$$r = \frac{A\tau\epsilon}{A_c(1+\beta)\epsilon + C_c\theta},$$

as q , q' , r are depending on the frequency ω_j , we write q_j , q'_j , and r_j . Now, let us write the lunisolar torque L (Moritz, 1980) as follows

$$L = (C-A) \Omega^2 \sum_j B_j \epsilon \exp.(-i(\omega_j t + \beta_j)), \quad (5-6)$$

let us write also $\epsilon = -\dot{\omega}_j$, then using (5-6) and (1-2), the expression (3-5) giving u_o becomes

$$u_o = i \Omega \sum_j \frac{r_E}{\omega_j + \epsilon_E} B_j \epsilon \exp.(-i(\omega_j t + \beta_j)), \quad (5-7)$$

and from (5-3) and (5-7) we have

$$u = i \Omega \sum_j \frac{r_E}{\omega_j + \epsilon_E} q_j B_j \epsilon \exp.(-i(\omega_j t + \beta_j)). \quad (5-8)$$

In the same way, we obtain from (1-2), (5-6), (5-8) and (5-4)

$$v = i \Omega \sum_j \left(\frac{r_E}{\omega_j + \epsilon_E} q_j q'_j + r_j \right) B_j \epsilon \exp.(-i(\omega_j t + \beta_j)). \quad (5-9)$$

We use expressions (5-8) and (5-9) to obtain with (4-2) and (4-3)

$$h = i \Omega A_c \sum_j \left(\frac{r_E}{\omega_j + \epsilon_E} q_j q'_j + r_j \right) B_j \epsilon \exp.(-i(\omega_j t + \beta_j)), \quad (5-10)$$

$$c = i(C-A) \sum_j \left[\left(\frac{k_o}{k_s} - \frac{k_1}{k_s} q'_j \right) \frac{r_E}{\omega_j + \epsilon_E} q_j - \frac{k_1}{k_s} r_j - \frac{k_o}{k_s} \right] B_j \epsilon \exp.(-i(\omega_j t + \beta_j)).$$

Then, substituting (5-8), and (5-10) into (4-1) it finally results for the polar motions

$$\begin{aligned} s_{\alpha}^{for} &= i \sum_j \frac{r_E}{\omega_j + \epsilon_E} q_j B_j \epsilon \exp.(-i(\omega_j t + \beta_j)), \\ p_z^{for} &= 0, \end{aligned}$$

$$p_{\text{P}}^{\text{free}} = i \sum_j \left[\frac{k_o}{k_s} - \frac{k_1}{k_s} q_j \right] \frac{r_E}{\omega_j + \epsilon_E} q_j - \frac{k_1}{k_s} r_j - \frac{k_o}{k_s}] B_j e \exp(-i(\omega_j + \beta_j)),$$

$$\begin{aligned} p_{\text{H}}^{\text{free}} = & i \sum_j \left[\frac{A}{C} + \frac{A_c}{C} q_j + \frac{C-A}{C} \left(\frac{k_o}{k_s} - \frac{k_1}{k_s} q_j \right) \right] \frac{r_E}{\omega_j + \epsilon_E} q_j + \frac{A}{C} r_j - \\ & - \frac{C-A}{C} \left(\frac{k_1}{k_s} r_j + \frac{k_o}{k_s} \right) B_j e \exp(-i(\omega_j t + \beta_j)). \end{aligned} \quad (5-11)$$

6 Nutritional motion.

The substitution of expressions 2-4) and (5-1) into relations (4-1) gives for the free nutational motion:

$$\begin{aligned} n_R^{\text{free}} &= \sum_{j=1}^z \frac{r_j a_j}{r_j + \Omega \Omega} e \exp(i \omega_j t), \\ n_Z^{\text{free}} &= - \sum_{j=1}^z \frac{a_j}{r_j + \Omega} e \exp(i \omega_j t), \\ n_P^{\text{free}} &= \sum_{j=1}^z \left(\frac{k_o}{k_s} \frac{a_j}{\Omega} - \frac{k_1}{k_s} \frac{b_j}{\Omega} - \frac{a_j}{r_j + \Omega} \right) e \exp(i \omega_j t), \\ n_H^{\text{free}} &= 0. \end{aligned} \quad (6-1)$$

This nutational motion refers to the Nutation frame (N). To obtain the nutation with respect to the inertial frame (I) we must consider the transformation (Moritz, 1980)

$$\Delta \theta + i \Delta \psi \sin \theta = - i n e \exp(i \Omega t).$$

Then, it results

$$\begin{aligned} (\Delta \theta + i \Delta \psi \sin \theta)_R^{\text{free}} &= -i \sum_{j=1}^z \frac{r_j a_j}{r_j + \Omega \Omega} e \exp(i(\omega_j + \Omega)t), \\ (\Delta \theta + i \Delta \psi \sin \theta)_Z^{\text{free}} &= i \sum_{j=1}^z \frac{a_j}{r_j + \Omega} e \exp(i(\omega_j + \Omega)t), \\ (\Delta \theta + i \Delta \psi \sin \theta)_P^{\text{free}} &= -i \sum_{j=1}^z \left(\frac{k_o}{k_s} \frac{a_j}{\Omega} - \frac{k_1}{k_s} \frac{b_j}{\Omega} - \frac{a_j}{r_j + \Omega} \right) e \exp(i(\omega_j + \Omega)t), \\ (\Delta \theta + i \Delta \psi \sin \theta)_{,,}^{\text{free}} &= 0. \end{aligned} \quad (6-2)$$

For the forced nutational motion, the frequency σ will be $-\omega_j$ corresponding to the lunisolar torque. The substitution of expressions (5-8) and (5-9) into relations (4-1) gives the nutational motion in the (N) reference system. Using the transformation equation (6-2) we obtain the nutational motion in the (I) reference system. As we have used for obtaining (5-8) and (5-9) the Nutation reference system (N), with this procedure the precession terms corresponding to the frequency $\sigma=0$ don't appear. But, if we derive the formulae obtained after integrating from t to t , it finally results

$$\begin{aligned} (\Delta\theta + i\Delta\psi \sin\theta)_{\text{R}}^{\text{for}} &= -i \frac{C-A}{C} B_0 \Omega(t-t_0) + \sum_{j \neq 0} \frac{\omega_j - \sigma_e}{\Delta\omega_j} q_j B_j e^{\exp.(-i(\Delta\omega_j t + \delta_j))}, \\ (\Delta\theta + i\Delta\psi \sin\theta)_{\text{Z}}^{\text{for}} &= -i \frac{C-A}{C} B_0 \Omega(t-t_0) + \sum_{j \neq 0} \frac{\Omega - \sigma_e}{\Delta\omega_j} q_j B_j e^{\exp.(-i(\Delta\omega_j t + \delta_j))}, \\ (\Delta\theta + i\Delta\psi \sin\theta)_{\text{F}}^{\text{for}} &= \sum_{j \neq 0} \frac{k_0 - k_j}{k_s} \left(q_j' + \frac{\Omega}{\Delta\omega_j} \right) \frac{\omega_j - \sigma_e}{\Delta\omega_j} q_j - \frac{k_1}{k_s} r_j - \frac{k_0}{k_s} \\ &\quad B_j e^{\exp.(-i(\Delta\omega_j t + \delta_j))}, \\ (\Delta\theta + i\Delta\psi \sin\theta)_{\text{H}}^{\text{for}} &= -i \frac{C-A}{C} B_0 \Omega(t-t_0) + \frac{A}{C} \sum_{j \neq 0} \frac{\sigma_e}{\Delta\omega_j} B_j e^{\exp.(-i(\Delta\omega_j t + \delta_j))}, \end{aligned} \quad (6-3)$$

with $\Delta\omega_j = \omega_j - \Omega$.

7 Celestial Ephemeris Pole.

For obtaining the Celestial Ephemeris Pole motion we must take

$$p_c = p_u^{\text{free}}$$

and

$$n_c = n_u^{\text{for}}.$$

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Rosero,P.,Sevilla,M.J.,Camacho,A.B.
Instituto de Astronomía y Geodesia
Facultad de Ciencias Matemáticas
Ciudad Universitaria
28040 Madrid , Spain.

DISCUSSION

Q: Which amplitude do you find for the annual nutation in longitude?. Is it different from Wahr's value?. (P. Melchior)

A: Yes, but it must be pointed out that the flattening of the core in the reference Earth model chosen plays a great role in the computed values of nutations. (P. Romero)

Variations of the inertia tensor for deformable models.

A.G. CAMACHO, M.J. SEVILLA, P. ROMERO

with 7 figures

Camacho, A.G., Sevilla, M.J., Romero, P., 1985: Variations of the inertia tensor for deformable models. Proceedings of the Tenth International Symposium on Earth Tides, pp. 347-352.

Abstract: General formulae to determine the infinitesimal variations δC of the inertia parameters produced by inner infinitesimal redistributions of matter are developed for a general stratified model.

Keywords: Earth's inertia tensor, deformations.

1. Introduction.

A lot of dynamic and kinematic properties of the Earth (polar motion, precession-nutation, rotation speed) are depending on the mass distribution and its time variations. The global configuration of masses can be described by the inertia tensor.

In a general case the inertia tensor is time dependent, and, just from the time variations of the inertia parameters, other dynamical effects can be deduced. We study the theoretical expressions of the inertia tensor components for a general model in function of the displacement components along the outer surface normal direction and the additional gravitative potential.

2. General formulae.

With respect to a reference system (Ox_i , $i=1,2,3$), the components of the inertia tensor of a general body are given by (Moritz, 1980):

$$-C_{ij} = \iiint_V x_i x_j \rho \, dv , \quad i \neq j$$

$$C_{kk} = \iiint_V (x_1^2 + x_2^2) \rho \, dv , \quad i \neq j \neq k$$

where V is the body volume, x_i the Cartesian coordinates of a variable point, ρ the mass density at this point, and dv the corresponding element of volume.

To study the time variations of the inertia parameters C produced by infinitesimal redistributions of mass, let us consider in these formulae variations of volume and density at the point $P(x_1, x_2, x_3)$. Thus, we must study:

$$\frac{d}{dt} \iiint_{V(t)} x_i x_j \rho(t) \, dv .$$

According to the rules of the Integral Calculus (Apostol, 1960) we have:

$$\frac{d}{dt} \int_0^{f(t)} g(t,s) \, ds = g(t,f(t)) \cdot f'(t) + \int_0^{f(t)} \frac{\partial}{\partial t} g(t,s) \, ds ,$$

f, g being continuous functions with continuous first derivatives in V .

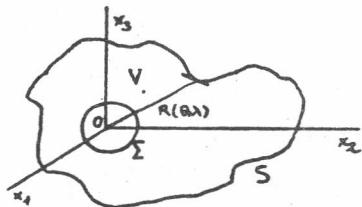


Fig 1. Volume of integration.

Let us now consider a triple integral for a volume V which contains the coordinates origin O , and which is bounded by a regular closed and orientable surface S (and whose section with a radial direction from O is a single point), then we can write:

$$\iiint_{V(t)} g(t, x) dv = \iint_{\Sigma} \left(\int_0^R R(\theta, \lambda, t) g(t; r, \theta, \lambda) r dr \right) \sin \theta d\theta d\lambda ,$$

Σ being the unit sphere. Thus, with $dr = \sin \theta d\theta d\lambda$, we have:

$$\begin{aligned} \frac{d}{dt} \iiint_{V(t)} g(t, x) dv &= \iint_{\Sigma} \left(\frac{d}{dt} \int_0^R R(\theta, \lambda, t) g(t; r, \theta, \lambda) r^2 dr \right) ds = \\ &= \iint_{\Sigma} g(t; R(\theta, \lambda, t), \theta, \lambda) R^2 d\theta d\lambda + \iint_{\Sigma} \int_0^R \frac{d}{dt} g(t; r, \theta, \lambda) r^2 dr ds . \end{aligned}$$

Now we have:



where α is the angle between the radial and outer normal directions at a point of the surface S .

Fig 2. Surface elements.

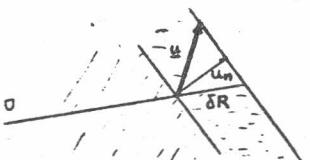


Fig 3. Differential elements.

Also by a simple geometrical reasoning and supposing that the displacement u is an infinitesimal and continuous function in the space, we have:

$$dR = u_n / \cos \alpha .$$

Here, u_n is the component of u along the surface normal.

Thus, we have the time variations:

$$\delta \iiint_{V(t)} g(t, x) dv = \iint_S g(t, x) u_n ds + \iiint_V \delta g(t, x) dv$$

In our case: $g(t, x) = x_i x_j \rho(x, t)$ and (supposing that the density is continuous in the space-time):

$$\delta C_{ij} = \iint_{S_0} x_i x_j u_n \rho_0 ds + \iiint_{V_0} x_i x_j (\rho - \rho_0) dv , \text{ if } j ,$$

and similarly:

$$\delta C_{ij} = \iint_{S_0} (x_i^2 + x_k^2) u_n \rho_0 dS + \iiint_{V_0} (x_i^2 + x_k^2) (\rho - \rho_0) dv, \quad i \neq j \neq k.$$

(The first term at the right hand side can be interpreted as an infinitesimal mass expansion or contraction across the initial boundary surface and the second term as the effect of the density variation inside the initial volume).

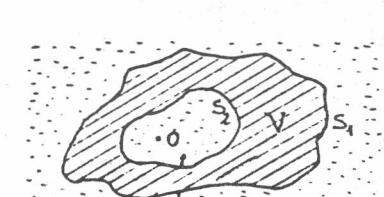


Fig 4. Continuous model.

If the volume of integration is bounded by two initial surfaces S_1, S_2 we have:

$$-\delta C_{ij} = \iint_{S_1} x_i x_j u_n \rho_0 dS - \iint_{S_2} x_i x_j u_n \rho_0 dS + \iiint_{V_0} x_i x_j (\rho - \rho_0) dv.$$

(The sign minus in the right hand side comes from the fact that the inside of V is outside of S).

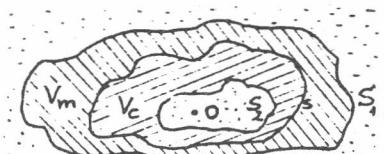


Fig 5. Model with inner density discontinuity surface.

Now, we suppose a body bounded by the initial surfaces S_1, S_2 but with an inner surface s of density discontinuity, so that we have the volume divided in two initial volumes V_c, V_m . Then we write:

$$\begin{aligned} -\delta C_{ij} &= -\delta C_{ij}^e - \delta C_{ij}^m \\ &= \iint_{S_1} x_i x_j u_n \rho_0 dS - \iint_s x_i x_j u_n \rho(e) dS + \iiint_{V_m} x_i x_j (\rho - \rho_0) dv + \\ &\quad + \iint_s x_i x_j u_n \rho(i) dS - \iint_{S_2} x_i x_j u_n \rho_0 dS + \iiint_{V_c} x_i x_j (\rho - \rho_0) dv = \\ &= \iiint_V x_i x_j (\rho - \rho_0) dv + \iint_{S_1} x_i x_j u_n \rho_0 dS - \iint_{S_2} x_i x_j u_n \rho_0 dS - \\ &\quad - \iint_s x_i x_j u_n (\rho(e) - \rho(i)) dS, \end{aligned}$$

$\rho(e)$ is the density in the surface as limit from the exterior of this surface, $\rho(i)$ from the inside.

If we have several inner initial surfaces s_k of discontinuity for the density, then:

$$\begin{aligned} -\delta C_{ij} &= \iiint_V x_i x_j (\rho - \rho_0) dv + \iint_{S_1} x_i x_j u_n \rho_0 dS - \iint_{S_2} x_i x_j u_n \rho_0 dS - \\ &\quad - \sum_{s_k} \iint_{s_k} x_i x_j u_n (\rho(e) - \rho(i)) dS_k, \end{aligned}$$

and similarly for δC_{ik} .

The function $\rho - \rho_0$ of density redistribution is not suitable for further developments, but it can be connected to the additional gravitational potential W_a created by the mass redistribution. By the Poisson formula (Heiskanen, Moritz, 1976):

$$\Delta W_a = -4\pi G (\rho - \rho_0)$$

Thus:

$$\iiint_V (\rho - \rho_0) x_i x_j dv = - (4\pi G)^2 \iiint_V \Delta W_a x_i x_j dv.$$

Here we can substitute safely W_a by any other function $W_a + \xi$, ξ being a harmonic function in V (for instance, the tidal potential).

Now, for two functions of class two, f, g , the first Green's identity give us (Heiskanen, Moritz, 1967):

$$\iiint_V (f \cdot \Delta g - g \cdot \Delta f) dv = \iint_S (f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n}) ds,$$

and, for a volume bounded by two surfaces S_1, S_2 ,

$$\iiint_V (f \cdot \Delta g - g \cdot \Delta f) dv = \iint_{S_1} (f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n}) ds - \iint_{S_2} id.$$

Taking into account that $\Delta(x_i x_j) = 0$ and $\Delta(x_j^2 + x_k^2) = 4$, it is easy to see that, for each volume V_j , we have:



Fig 6. A layer volume.

$$\begin{aligned} & \iiint_{V_j} \Delta W_a x_i x_j dv = \\ & = \iint_{S_1(i)} \left(\frac{\partial W_a}{\partial n} x_i x_j - W_a \frac{\partial}{\partial n} (x_i x_j) \right) ds - \iint_{S_2(e)} id. \\ & = 4 \iiint_{V_j} W_a dv + \iint_{S_1(i)} \left(\frac{\partial}{\partial n} W_a \cdot (x_j^2 + x_k^2) + W_a \frac{\partial}{\partial n} (x_j^2 + x_k^2) \right) ds - \iint_{S_2(e)} id. \end{aligned}$$

Additioning for the several volumes between discontinuity surfaces, we get:

$$\begin{aligned} -\delta C_{ij} = & -\frac{1}{4\pi G} \iint_{S_1} \left(\frac{\partial W_a}{\partial n} - 4\pi G \rho_0 u_n \right) x_i x_j - W_a \frac{\partial}{\partial n} (x_i x_j) ds - \\ & - \frac{1}{4\pi G} \sum_{j=1}^k \iint_{S_j} (id) \frac{s_j(i)}{s_j(e)} + \frac{1}{4\pi G} \iint_{S_2} id. \end{aligned}$$

and similarly for δC_{ik} .

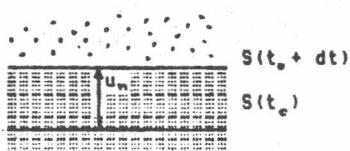
The additional gravitational potential ΔWa can be considered as a variation of:

$$\Delta \text{Wa} = \iiint_V \frac{1}{L} G (\rho - \rho_0) dv ,$$

L being the distance between the integration variable point and the point where ΔWa is calculated. And then, with a similar development that for the tensor components, we have:

$$\begin{aligned} \Delta \text{Wa} &= \delta \Delta \text{Wa} = \iiint_V \frac{1}{L} G (\rho - \rho_0) dv + \iint_{S_1} \frac{1}{L} G u_n \rho ds_1 - \iint_{S_2} \frac{1}{L} G u_n \rho ds_2 + \\ &\quad + \sum_i \iint_{S_i} \frac{1}{L} G u_n (\rho(i) - \rho_0) ds_i . \end{aligned}$$

If we study the normal derivative $\partial \Delta \text{Wa} / \partial n$, only the first term of the right hand side gives a continuous term in $\partial \Delta \text{Wa} / \partial n$. In fact, let us consider a general term :



$$\Delta \text{Wa} = \iint_S \frac{1}{L} G u_n (\rho(i) - \rho_0) ds$$

We can see this term as a single layer potential with surface density

Fig 7. Normal displacement.

$$r = u_n (\rho(i) - \rho_0)$$

From the theory of single layer potential we can write (Heiskanen, Moritz, 1976):

$$\left[\frac{\partial}{\partial n} \Delta \text{Wa} \right]_{S(i)} - \left[\frac{\partial}{\partial n} \Delta \text{Wa} \right]_{S(e)} = 4\pi G r = 4\pi G u_n(S) (\rho(i) - \rho_0)$$

Also, if we suppose that outside the surface S there is no mass ($\rho = 0$), we have:

$$\left[\frac{\partial}{\partial n} \Delta \text{Wa} \right]_{S(i)} - \left[\frac{\partial}{\partial n} \Delta \text{Wa} \right]_{S(e)} = 4\pi G \rho(i) u_n(S) .$$

Inserting this results in the last expression of δC_{ij} , and taking into account that $\Delta \text{Wa} \cdot \partial / \partial n (x_i x_j)$ and $\Delta \text{Wa} \cdot \partial / \partial n (x_i^2 + x_j^2)$ are continuous functions along the whole volume, we write finally (see Molodensky, 1981)

$$\delta C_{ij} = \frac{1}{4\pi G} \iint_S \left\{ \left(\frac{\partial}{\partial n} \Delta \text{Wa} - 4\pi G \rho u_n \right) x_i x_j - \Delta \text{Wa} \frac{\partial}{\partial n} (x_i x_j) \right\} ds \Big|_{S(e)}^{S(i)} ,$$

and similarly:

$$\delta C_{ii} = - \frac{1}{4\pi G} \iint_V 4 \Delta \text{Wa} dv - \frac{1}{4\pi G} \iint_S \left\{ \left(\frac{\partial}{\partial n} \Delta \text{Wa} - 4\pi G \rho u_n \right) (x_i^2 + x_j^2) - \right.$$

$$- \text{Wa} \frac{\partial}{\partial n} (x_j^2 + x_k^2) ds \Big]_{S(0)}^{S(1)}$$

If now we suppose, in particular, a body bounded only by an initial surface S_0 and so that there is an empty space ($\rho = 0$) outside of S we have:

$$\delta C_{ij} = \frac{1}{4\pi G} \iint_{S_0} \left(\frac{\partial}{\partial n} \text{Wa} \right)_{\text{ext}} x_i x_j - \text{Wa} \frac{\partial}{\partial n} (x_i x_j) ds_0 ,$$

$$\delta C_{ii} = - \frac{1}{4\pi G} \iiint_{V_0} \text{Wa} dv - \frac{1}{4\pi G} \iint_{S_0} \left(\frac{\partial}{\partial n} \text{Wa} \right)_{\text{ext}} (x_j^2 + x_k^2) - \text{Wa} \frac{\partial}{\partial n} (x_j^2 + x_k^2) ds_0 .$$

We conclude that we can obtain the time variations of the inertia parameters for a model composed by of layers of continuous density and subjected to infinitesimal continuous displacements, by calculating the values and the normal derivatives of the functions u and Wa at the inner and outer boundary surfaces of the model, whatever the inner surfaces discontinuity values are.

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Camacho, A.G., Sevilla, M.J., Romero, P.
 Instituto de Astronomía y Geodesia
 Facultad de Ciencias Matemáticas
 Ciudad Universitaria
 Madrid 28040 , SPAIN

Global and core inertia tensor components for a classical Earth model.

A.B. CAMACHO, M.J. SEVILLA, P. ROMERO

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Abstract: General formulae are applied to determine the several inertia tensor components for an elastic elliptical rotating Earth model with liquid core and tidal disturbing effects.

Keywords: inertia tensor, Earth models, liquid core.

1. Introduction.

In Camacho, Sevilla, Romero (1985) we present a general formulae to calculate the infinitesimal variations δC_{ij} of the inertia tensor components for a general model in function of the boundary values of the additional gravitational potential Wa and the displacement u_n along the boundary surface normal. In a reference system $\{dx_i\}$, $i=1,2,3$ connected with the body we have:

$$\begin{aligned}\delta C_{ij} &= (4\pi G)^{-1} \iint_S ((\frac{\partial}{\partial n} Wa - 4\pi G u_n \rho_0) x_i x_j - Wa \frac{\partial}{\partial n} (x_i x_j)) dS \Big|_{\substack{S(i) \\ S(e)}} \quad i \neq j \\ \delta C_{ii} &= - (4\pi G)^{-1} \iiint_{V_0} 4 Wa dv - (4\pi G)^{-1} \iint_S ((\frac{\partial}{\partial n} Wa - 4\pi G u_n \rho_0) (x_j^2 + x_k^2) - \\ &\quad - Wa \frac{\partial}{\partial n} (x_j^2 + x_k^2)) dS \Big|_{\substack{S(i) \\ S(e)}}, \quad i \neq j \neq k\end{aligned}\quad (1)$$

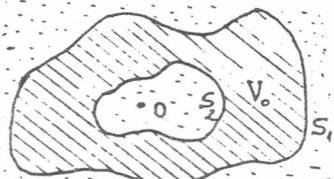


Fig 1. Model bounded by two initial surfaces.

for a body with initial desity $\rho(x)$ and bounded by the surfaces S_1 , S_2 (and with other inner density discontinuity surfaces).

If we suppose now a body bounded only by an initial surface S_0 and so that there is empty space ($\rho = 0$) outside S_0 we have:

$$\delta C_{ij} = (4\pi G)^{-1} \iint_{S_0} ((\frac{\partial}{\partial n} Wa)_{ext} x_i x_j - Wa \frac{\partial}{\partial n} (x_i x_j)) dS_0, \quad (2)$$

$$\delta C_{ii} = - (\pi G)^{-1} \iiint_{V_0} Wa dv - (4\pi G)^{-1} \iint_{S_0} ((\frac{\partial}{\partial n} Wa) (x_j^2 + x_k^2) - Wa \frac{\partial}{\partial n} (x_j^2 + x_k^2)) dS_0,$$

We will apply these formulae for a classical Earth's model.

2. Application for a classical Earth's model.

We suppose a model composed by an elastic mantle and an inner liquid core, subjected to variable (about a mean value) rotation and subjected to the differential tidal body force. For simplicity of further development we suppose the model studied from a suitable 'initial' spherical configuration.

If Ω is the mean value of the rotation speed, the rotation velocity vector can be written $\omega = \Omega (m_1, m_2, 1+m_3)$ for a mobile reference system connected to the elliptical mean figure of the model (m_i : infinitesimal parameters). The corresponding centrifugal potential U can be written (Munk & MacDonald, 1975):

$$U = 1/2 (\underline{\omega} \cdot \underline{x})^2 = 1/3 r^2 \Omega^2 + Ur$$

where:

$$Ur = -\frac{1}{3} r^2 \Omega^2 (1+2m_3) P_{20}(\cos \theta) - \frac{1}{3} r^2 \Omega^2 (m_1 \cos \lambda + m_2 \sin \lambda) P_{21}(\cos \theta)$$

is harmonic function of second degree.

The term $1/3 r^2 \Omega^2$ is just a radial one. Then, we consider as spherical reference configuration (of radius a and equal inertia moments A_m) the figure obtained from a spherical stratified non rotating earth in hydrostatic equilibrium applying the radial centrifugal deformation associated to $1/3 r^2 \Omega^2$. We suppose also that in this reference state the spherical surfaces of equal rigidity μ , compressibility α coincide with the equipotential surfaces (whole potential = gravitatory + radial centrifugal for $1/3 r^2 \Omega^2$).

The Hookean elasticity of the model connects the additional stress associated to the inner displacements \underline{u} with these same displacements in the form (Love, 1944):

$$\tau_{ij} = \lambda \cdot \operatorname{div} \underline{u} \cdot \delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),$$

for a rotating reference system connected to the mean elliptical configuration. We suppose an inner liquid ($\mu = 0$) core of initial radius c .

As final hypothesis we develop the differential tidal force as the gradient of the tidal potential, whose second degree term we write for a point (r, θ, λ) , in a general form, is (Melchior, 1978):

$$W_e = W_e | = -\frac{GM_E}{2r^3} r^2 \left[P_{20}(\cos \theta) \sum_j A_{20j} \cos \alpha_j + P_{21}(\cos \theta) \sum_j A_{21j} \sin(\alpha_j + \lambda) - P_{22}(\cos \theta) \sum_j A_{22j} \cos(\alpha_j + 2\lambda) \right],$$

where $\alpha_j = GMST + \pi + \Delta \alpha_j$, $\Delta \alpha_j$: astronomical long period argument.

From these hypothesis we can develop the general equations of displacements in a continuous medium and the Poisson equation. With a rather laborious work (see Shen & Mansinha, 1976) it is possible to obtain a general system of parcial (time-space) differential equations in \underline{u} and W_a .

For the resolution of this equations, the functions \underline{u} can be developed in the form of torsional and spheroidal components and in sinusoidal terms for the several frequencies: $\underline{u} = \underline{u}_T + \underline{u}_S$

The torsional terms are characterized by: $\operatorname{div} \underline{u}_T = 0$, $(\underline{u}_T)_r = 0$.

$$\underline{u} = \sum_{m,n,j} I_{nmj} \quad \text{with:}$$

$$(I_{nmj})_r = 0$$

$$(I_{nmj})_\theta = -a T_{nmj}(r) (\sin \theta)^{-1} P_{nm}(\cos \theta) \cos(\alpha_j t - n\lambda)$$

$$(I_{nmj})_\lambda = -T_{nmj}(r) \frac{d}{d\theta} P_{nm}(\cos \theta) \sin(\alpha_j t - n\lambda)$$

The spheroidal terms of the displacements contain the radial components

and the deformations

$$u_s = \sum_{n,m,j} S_{nmj} \quad \text{with:}$$

$$(S_{nmj})_r = U_{nmj}(r) P_{nm}(\cos \theta) \cos(r_j t - m\lambda)$$

$$(S_{nmj})_\theta = V_{nmj}(r) \frac{d}{d\theta} P_{nm}(\cos \theta) \cos(r_j t - m\lambda)$$

$$(S_{nmj})_\lambda = W_{nmj}(r) (\sin \theta)^l P_{nm}(\cos \theta) \sin(r_j t - m\lambda)$$

Also we can write W_a by a similar expression, but it is more suitable to write a development for $W_a + U_r + W_e$

$$W_a + U_r + W_e = \sum_{n,m,j} R_{nmj}(r) P_{nm}(\cos \theta) \cos(r_j t - m\lambda)$$

Sustituting these developments in the general equations and applying the relations between Legendre functions, the general equations for the functions U_n , V_n , R_n , T_n (it is not necessary to write the subscripts m,j) can be obtained.

The resolution of the resulting equations is different for the mantle and for the liquid core.

For the mantle the torsional terms can be neglected. The resulting system is of second order. But we can obtain a system of six linear ordinary differential equations definining the auxiliary functions $Y_n(r)$, $Z_n(r)$, $\Omega_n(r)$:

$$Y_n = \rho (\dot{V}_n + \frac{1}{r} U_n - \frac{1}{r} V_n)$$

$$Z_n = 2\rho \dot{U}_n + \frac{\alpha}{r} (\dot{U}_n + \frac{2}{r} U_n - \frac{n}{r} (n+1) V_n)$$

$$\Omega_n = \dot{R}_n - 4\pi G \rho_0 U_n$$

(Y_n is the coefficient for the radial stress τ_{rr} , Z_n for τ_{rz} and Ω_n for $\partial/\partial r W_a - 4\pi G \rho_0 U_r$).

With a rather laborious work and starting from the initial spherical configuration, the resulting system is (Shen & Mansinha, 1976):

$$\dot{U}_n = - \frac{2\alpha}{\alpha+2\mu} \frac{1}{r} U_n + \frac{1}{\alpha+2\mu} Z_n + \frac{\alpha}{\alpha+2\mu} \frac{1}{r} n(n+1) V_n$$

$$\dot{Z}_n = (-\frac{\rho_0 g^2}{r} - \frac{1}{r} 4\rho_0 g + 4\rho \frac{3\alpha+2\mu}{\alpha+2\mu} \frac{1}{r^2}) U_n - 4\rho \frac{1}{\alpha+2\mu} \frac{1}{r} Z_n - \rho_0 \Omega_n +$$

$$+ (\frac{1}{r} n(n+1) \rho_0 g - 2n(n+1) \rho \frac{3\alpha+2\mu}{r^2(\alpha+2\mu)} - 2\rho_0 g + \rho) V_n + n(n+1) \frac{1}{r} Y_n$$

$$\dot{V}_n = - \frac{1}{r} U_n + \frac{1}{r} V_n + \frac{1}{\mu} Y_n$$

$$\dot{Y}_n = (\frac{1}{r} \rho_0 g - 2\rho \frac{3\alpha+2\mu}{r^2(\alpha+2\mu)}) U_n - \frac{\alpha}{\alpha+2\mu} \frac{1}{r} Z_n - \rho_0 \frac{1}{r} R_n - \frac{1}{r} 3 Y_n +$$

$$+ (-\frac{\rho_0 g^2}{r} + \frac{1}{\alpha+2\mu} 2\rho [(2n^2 + 2n - 1) + 2(n^2 + n - 1)\rho]) \frac{1}{r^2} V_n$$

$$\dot{R}_n = 4\pi G \rho_0 U_n + \Omega_n \quad (3)$$

$$\dot{\Omega}_n = - 4\pi G \rho_0 \frac{1}{r} n(n+1) V_n + \frac{1}{r^2} n(n+1) R_n - \frac{1}{r} 2 \Omega_n$$

The solution of this system also must verify a boundary conditions in the free outer surface (initially, $r=a$). So, the condition of null radial stress gives us: $\bar{W}_n(a)=0$, $\bar{V}_n(a)=0$. And the condition of known discontinuity for the radial derivate of the additional potential gives us:

$$[(\frac{\partial}{\partial r} \bar{W}_n)_{ext.} - (\frac{\partial}{\partial r} \bar{W}_n)_{int.}] = 4\pi G \rho(a) u_r(a) \quad r=a$$

Writing the aditional potential \bar{W}_n as an harmonic function outside of the boundary surface and using the expression of $\bar{W}_n + W_n + U_r$, we have, for $n=2$:

$$\frac{3}{a} R_{2m}(a) \begin{bmatrix} \cos st \\ \sin st \end{bmatrix} - \frac{5}{a} [2m]_{r=a} \begin{bmatrix} \cos st \\ \sin st \end{bmatrix} + Q_{2m}(a) \begin{bmatrix} \cos st \\ \sin st \end{bmatrix} = 0$$

where:

$$[20] = [-\frac{1}{3} r^2 \Omega^2 (1+2m_3) - \frac{GM_\alpha}{dr^3} r^2 A_{20j} \cos \alpha_j]$$

$$[21] = [-\frac{1}{3} r^2 \Omega^2 \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} - \frac{GM_\alpha}{dr^2} r^2 A_{21j} \begin{bmatrix} \sin \alpha_j \\ \cos \alpha_j \end{bmatrix}]$$

$$[22] = [\frac{GM_\alpha}{dr^3} r^2 A_{22j} \cos(\alpha_j + 2\lambda)]$$

For the liquid core the situation is something different. The main problem is that we must consider torsional terms in the solution, with another additional complications. The treatment of this problem is very different for several scientist as Poincare, Molodensky, Jobert, Shen & Mansinha, Sasaki, Okubo & Saito. Here, we make a simple treatment supposing that the only own torsional motion in the liquid core is a global rotation of velocity vector $\omega^c = \Omega (m_1^c, m_2^c, m_3^c)$. This hypothesis let us write, for the torsional displacements:

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{bmatrix} = \Omega \begin{bmatrix} 0 & -m_3^c & m_2^c \\ m_3^c & 0 & -m_1^c \\ -m_2^c & m_1^c & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \begin{bmatrix} \dot{u}_0 \\ \dot{u}_\lambda \\ \dot{u}_r \end{bmatrix} = \Omega \begin{bmatrix} m_2^c \cos \lambda - m_1^c \sin \lambda \\ m_3^c \sin \theta - m_1^c \cos \theta \cos \lambda - m_2^c \cos \theta \sin \lambda \\ 0 \end{bmatrix}$$

Additioning this torsional terms to the usual spheroidal terms and making a similar development than for the mantle (now, $\text{div } \vec{u} = 0$, $(\vec{u}_T)_r = 0$, $\vec{u}_\theta = 0$), we can obtain the corresponding system of differential equations. We observe that for $n=2$ the new equations can be obtained from the mantle equations (for this simplified hypothesis) if we take $\vec{u}_\theta = 0$ and substitute U_r by $\bar{U}_r - \bar{U}_r^c$ with:

$$\bar{U}_r = 1/3 r^2 \Omega^2 [2 m_3^c P_{20}(\cos \theta) + (m_1^c \cos \lambda + m_2^c \sin \lambda) P_{20}(\cos \theta)]$$

R_2 by R_2^c with:

$$W_n + W_n + U_r + \bar{U}_r^c = \sum_{m=0}^2 R_{2m}^c(r) \cos(m st - m \lambda) P_{2m}(\cos \theta)$$

and Q_2 by Q_2^c with:

$$Q_2^c = R_2^c - 4\pi G \rho \bar{u}_n$$

If we take the initial core-mantle boundary as a sphere of radius c we can write the following boundary conditions for $n=2$:

$$R_{2m}^c(c^-) \begin{bmatrix} \cos st \\ \sin st \end{bmatrix} + [2m]_{r=c}^c = R_{2m}^c(c^+) \begin{bmatrix} \cos st \\ \sin st \end{bmatrix} \quad (\text{continuity of } W_n)$$

$$Q_{2m}(c^+) \begin{bmatrix} \cos st \\ \sin st \end{bmatrix} = R_{2m}(c^-) \begin{bmatrix} \cos st \\ \sin st \end{bmatrix} + \frac{2}{c} [2m] \begin{bmatrix} c \\ r_{sc} \end{bmatrix} - 4\pi G \rho(c^-) U_2(c) \begin{bmatrix} \cos st \\ \sin st \end{bmatrix}$$

(known discontinuity for $\partial W_a / \partial r$) , where:

$$[20]^c = [-2/3 r^2 \Omega^2 n_3^c]$$

$$[21]^c = [-1/3 r^2 \Omega^2 \begin{bmatrix} n_1^c \\ n_2^c \end{bmatrix}]$$

$$[22]^c = 0 .$$

Applying ortogonality properties of spherical harmonics in (1), we have:

$$\delta C_{ij} = (4\pi G)^{-1} \left[\int_0^2 r^2 [Q_{2m}(r) - \frac{2}{r} R_{2m}(r)] (x_i x_j) \right] \frac{n_1}{r_2} P_{2m}(\cos \theta) \cos(st - \omega \lambda) d\omega$$

$$\delta C_{LL} = -\frac{4}{G} \int_0^2 R_o(r) r^2 dr -$$

$$- (4\pi G)^{-1} \left[\int_0^2 r^2 [Q_{2m}(r) - \frac{2}{r} R_{2m}(r)] (x_j^2 + x_k^2) \right] \frac{n_1}{r_2} P_{2m}(\cos \theta) \cos(st - \omega \lambda) d\omega$$
(4)

For the core, we can substitute Q_{2m} and R_{2m} for Q_{2m}^c and R_{2m}^c .

To have more suitable expressions we connect now the last coefficients with the relations for the Love numbers theory.

If we neglect the liquid core effects and study the displacements of our model, we get the general equations (3). We see that this equation connect the second degree terms of $u_r, u_\theta, u_\lambda, W_a$ with the same degree terms of the potentials W_a, U_r by an homogeneous linear system of differential equations. So, without lost of generality, we can write the solutions in the following form:

$$u_r = \sum_n \frac{1}{9} H_n(r) W_n(r, \theta, \lambda)$$

$$u_\theta = \sum_n \frac{1}{9} L_n(r) \frac{\partial}{\partial \theta} W_n(r, \theta, \lambda)$$

$$u_\lambda = \sum_n \frac{1}{9} L_n(r) (\sin \theta)^{-1} \frac{\partial}{\partial \lambda} W_n(r, \theta, \lambda) ,$$

where $W_a + U_r = \sum_n W_n$. These expressions are general enough to verify the boundary conditions.

For $n=2$ the connection of this Love coefficients with the spheroidal coefficients gives us:

$$U_{2m}(r) \begin{bmatrix} \cos st \\ \sin st \end{bmatrix} = 1/9 H_2(r) [2m]$$

$$V_{2m}(r) \begin{bmatrix} \cos st \\ \sin st \end{bmatrix} = 1/9 L_2(r) [2m]$$

$$R_{2m}(r) \begin{bmatrix} \cos st \\ \sin st \end{bmatrix} = (1 + K_2(r)) [2m]$$

And we can write the boundary conditions for this coefficients. So we have:

And then: $\Omega_{2m}(a) \begin{bmatrix} \cos st \\ \sin st \end{bmatrix} = 1/a (2 - 3 K_2(a)) [2m]_{r=a}$

$$[\Omega_{2m}(a) - 2/a R_{2m}(a)] \begin{bmatrix} \cos st \\ \sin st \end{bmatrix} = - 5/a K_2(a) [2m]_{r=a}$$

For a model with liquid core we have seen that the equations in the liquid core, for $n=2$, are similar to that for the mantle but taking $W_e + U_r + U_f$ instead of $W_e + U_r$. So, we could use an expression like:

$$U_{2m}(r) \begin{bmatrix} \cos st \\ \sin st \end{bmatrix} = 1/g H_2(r) ([2m] + [2m]^c),$$

but, though the differential equations can be satisfied with these expressions, in the boundary conditions the liquid core effects appear with a different role than the rest of disturbing potentials. So, it is more convenient to write the relation with two coefficients:

$$U_{2m}(r) \begin{bmatrix} \cos st \\ \sin st \end{bmatrix} = 1/g H_2^s(r) [2m] - 1/g H_2^d(r) [2m]^c$$

and similarly:

$$V_{2m}(r) \begin{bmatrix} \cos st \\ \sin st \end{bmatrix} = 1/g L_2^s(r) [2m] - 1/g L_2^d(r) [2m]^c$$

$$R_{2m}(r) \begin{bmatrix} \cos st \\ \sin st \end{bmatrix} = (1 + K_2^s(r)) [2m] - K_2^d(r) [2m]^c$$

$$R_{2m}^c(r) \begin{bmatrix} \cos st \\ \sin st \end{bmatrix} = (1 + K_2^s(r)) [2m] - (1 + K_2^d(r)) [2m]^c$$

This is for the core, but also for the mantle, because, though the liquid core effects do not appear in the differential equations, they appear in the core-mantle boundary conditions.

For this model with liquid core effects we have:

$$[\Omega_{2m}(a) - 2/a R_{2m}(a)] \begin{bmatrix} \cos st \\ \sin st \end{bmatrix} = - 5/a (k_o [2m]_{r=a} + k_d [2m]_{r=a}^c)$$

where:

$$k_o = K_2^s(a), \quad k_d = K_2^d(a)$$

Also:

$$\begin{aligned} [\Omega_{2m}^c(c^-) - \frac{2}{c} R_{2m}^c(c^-)] \begin{bmatrix} \cos st \\ \sin st \end{bmatrix} &= (1 + K_2^s(c) - \frac{2}{c} - 4\pi G \frac{\rho(c^-)}{g} H_2^s(c)) [2m]_{r=c} - \\ &- (1 + K_2^d(c) - 4\pi G \rho(c) \frac{1}{g} H_2^d(c)) [2m]_{r=c}^c = \\ &= - 15 G A c^6 g^{-2} (\gamma [2m] - \beta [2m]^c)_{r=c} \end{aligned}$$

for a suitable coefficients γ and β . (see Sasao et al., 1980)

With these relations we can substitute in the expressions of the inertia parameters:

(1) For the whole model:

$$\delta C_{ij} = a^2 (4\pi G)^{-1} \sum_{m=0}^2 [\Omega_{2m}(a) - \frac{2}{a} R_{2m}(a)] \iint_{\omega} x_i x_j |_{r=a} P_{2m}(\cos \theta) \cos(st - m\lambda) d\omega$$

$$\delta C_{11} = - a^2 (4\pi G)^{-1} \sum_{m=0}^2 [Q_{2m}(a) - \frac{2}{a} R_{2m}(a)] \iint_{\omega} \frac{(x_1^2 + x_2^2)}{r^2} P_{20}(\cos \theta) \cos(st-m\lambda) d\omega$$

$$\begin{bmatrix} \delta C_{13} \\ \delta C_{23} \end{bmatrix} = a^4 (4\pi G)^{-1} [Q_{24}(a) - \frac{2}{a} R_{24}(a)] \frac{1}{3} \iint_{\omega} P_{24}^2(\cos \theta) \begin{bmatrix} \cos^2 \lambda \cdot \cos st \\ \sin^2 \lambda \cdot \sin st \end{bmatrix} d\omega = \\ = \frac{4}{5} \frac{a^4}{6} [Q_{24}(a) - \frac{2}{a} R_{24}(a)] \begin{bmatrix} \cos st \\ \sin st \end{bmatrix} = - \frac{3}{5} \frac{a^3}{G} (k_0 [21]_{r=a} + k_1 [21]_{r=a}^c)$$

And similarly:

$$\delta C_{12} = - 2 \frac{a^3}{6} (k_0 [20]_{r=a} + k_1 [20]_{r=a}^c)$$

$$\begin{bmatrix} \delta C_{11} \\ \delta C_{22} \end{bmatrix} = - 2 \frac{a^3}{(36)} (k_0 [20]_{r=a} + k_1 [20]_{r=a}^c + k_0 6 [22]_{r=a})$$

$$\delta C_{33} = 4 \frac{a^3}{(36)} (k_0 [20]_{r=a} + k_1 [20]_{r=a}^c)$$

We observe that [20] has a secular part: $-1/3 r^2 \Omega^2$. So, we can expect a secular $(\delta C_{ij})_{sec}$

$$C_{ij} = (C_{ij})_0 + \delta C_{ij} = (C_{ij})_0 + (\delta C_{ij})_{sec} + c_{ij}$$

From the last relation we conclude:

$$(C_{ij})_0 = 0 \quad (\delta C_{ij})_{sec} = 0 \quad \text{for } i \neq j$$

$$(C_{ij})_0 = Ae \quad (\delta C_{11})_{sec} = (\delta C_{22})_{sec} = -1/2 (\delta C_{33})_{sec}$$

If we name:

$$Ae + (\delta C_{11})_{sec} = Ae + (\delta C_{22})_{sec} = A$$

then:

$$1/3 (A + A + C) = Ae$$

For this secular part we suppose that the Love number k_0 has a special value k_s . Thus:

$$C - A = k_s a^3 / (36) \Omega^2 a^2$$

and then:

$$k_s = 36 (C - A) / (a \Omega^2)$$

Sustituting this parameter k_s , $E = \frac{3GM_E}{d^3} \frac{C-A}{C}$ and the expressions of [2m], we get finally:

$$\begin{bmatrix} C_{13} \\ C_{23} \end{bmatrix} = \frac{k_0}{k_s} (C-A) \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} + \frac{k_0}{k_s} E C I_j A_{21j} \begin{bmatrix} \sin \alpha_j \\ \cos \alpha_j \end{bmatrix} - \frac{k_1}{k_s} (C-A) \begin{bmatrix} n_1^c \\ n_2^c \end{bmatrix}$$

$$C_{12} = - k_0 / k_s E C I_j A_{22j} \sin \alpha_j$$

$$\begin{bmatrix} C_{14} \\ C_{22} \end{bmatrix} = - \frac{2}{3} (C-A) \frac{k_0}{k_s} n_3 - \frac{1}{3} \frac{k_0}{k_s} E C I_j (A_{20j} \pm 6 A_{22j}) \cos \alpha_j + \frac{2}{3} \frac{k_1}{k_s} m_3^c (C-A)$$

$$c_{33} = 4/3 (C-A) k_s / k_3 \cdot a_3 + \frac{2}{3} \frac{k_0}{k_3} E C \sum_j A_{20j} \cos \alpha_j - \frac{4}{3} \frac{k_1}{k_3} (C-A) a_3^2$$

(2) Inertia parameters for the liquid cores:

$$\delta c_{ij}^c = c^2 (4\pi G)^{-1} \sum_0^2 [Q_{2m}^c(c) - \frac{2}{c} R_{2m}^c(c)] \prod_{\omega} (x_i x_j)_{r=c} P_{2m}(\cos \theta) \cos(st-m\lambda) dw$$

$$\delta c_{ii}^c = -c^2 (4\pi G)^{-1} \sum_0^2 [Q_{2m}^c(c) - \frac{2}{c} R_{2m}^c(c)] \prod_{\omega} (x_i^2 + x_k^2)_{r=c} P_{2m}(\cos \theta) \cos(st-m\lambda) dw$$

Also we write:

$$c_{ij}^c = (c_{ij}^c)_o + (\delta c_{ij}^c)_{sec} + c_{ij}^c$$

where $(c_{ij}^c)_o = \bar{A}$

Thus:

$$\bar{A} + (\delta c_{11}^c)_{sec} = \bar{A} + (\delta c_{22}^c)_{sec} = Ac$$

$$\bar{A} + (\delta c_{33}^c)_{sec} = Cc$$

And, with a similar method than for the whole model, we have:

$$\begin{bmatrix} c_{13}^c \\ c_{23}^c \end{bmatrix} = A_c \int \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + A_c \int \frac{3GM_e}{dc^3 \Omega^2} \sum_j A_{20j} \begin{bmatrix} \sin \alpha_j \\ \cos \alpha_j \end{bmatrix} - A_c B \begin{bmatrix} a_1^c \\ a_2^c \end{bmatrix}$$

$$c_{12}^c = -A_c \int \frac{3GM_e}{dc^3 \Omega^2} \sum_j A_{20j} \sin \alpha_j$$

$$\begin{bmatrix} c_{11}^c \\ c_{22}^c \end{bmatrix} = -2/3 A_c \int a_3 - 1/3 A \int \frac{3GM_e}{dc^3 \Omega^2} \sum_j A_{20j} \begin{bmatrix} \sin \alpha_j \\ \cos \alpha_j \end{bmatrix} + 2/3 A_c B a_3^c$$

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Camacho A.G.; Sevilla M.J.; Romero P.
Instituto de Astronomía y Geodesia
Facultad de Ciencias Matemáticas
Madrid 28040 , SPAIN.

A simple representation of liquid core motion .

A.B. CAMACHO, M.J. SEVILLA, P. ROMERO

with 1 figure

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Abstract: We present a simple model of the motion of an elliptical homogeneous liquid core envelopped by an elastic mantle.

Keywords: Core motion, elastic deformations, Earth Tides.

1. Introduction.

In this paper we present a simple, rather geometrical, description of the inner motion in an elliptical homogeneous liquid core envelopped by an elastic mantle.

We shall describe the total displacements of the particles in the deformed core as:

$$\underline{u} = \underline{t} + \underline{s}$$

where \underline{s} represents the rotational and tidal deformations and \underline{t} represents a particular free motion in the core that we suppose simple in the sense of Poincare (Poincare, 1910).

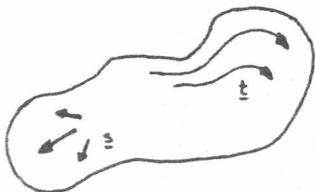


Fig 1. Core inner displacements.

For \underline{t} we use the Poincare representation by means of the equivalent sphere. The terms \underline{s} of deformation will be treated in function of Love numbers in the usual form for the global tidal and centrifugal deformations \underline{q} , and making a special study for the centrifugal deformations \underline{p} associated with the particular core rotation.

$$\underline{s} = \underline{q} + \underline{p}$$

This representation has an interesting application for the polar motion study. In fact, we can study the effects of this displacements in the inertia parameters and in the angular momentum; after we must substitute in the Liouville equations. Complementary equations (hidrodynamical equations) for the simple motion parameters will be necessary.

2. Simple additional motion \underline{s} .

If we suppose that the displacements \underline{s} are taken from the mean elliptical configuration, we use, to develop the simple additional motion, the following approach.

We suppose the usual reference axes Ox connected to the mean elliptical configuration. The components of the \underline{t} , \underline{s} , \underline{u} vectors will be t_i, s_i, u_i , $i=1,2,3$.

If \underline{x} is the position vector for the deformed configuration then $\underline{x} - \underline{s} = \underline{x}_0$ describes an elliptical configuration of semiaxes a, a, c for the liquid core. Now the coordinate transformation:

$$\underline{x}' = P \underline{x} = P (\underline{x} - \underline{s})$$

with:

$$P = \begin{bmatrix} 1/a & & \\ & 1/a & \\ & & 1/c \end{bmatrix}$$

give us a position vector \underline{x}' for a spherical configuration. The resulting simple motion for this sphere can be expressed, for the time derivates $\dot{\underline{x}}' = d\underline{x}'/dt$, in form of a global differential rotations:

$$\dot{\underline{x}}' = M(\omega^c) \underline{x}'$$

with angular velocity vector ω^c , where:

$$M(\omega^c) = \begin{bmatrix} 0 & -\omega_3^c & \omega_2^c \\ \omega_3^c & 0 & -\omega_1^c \\ -\omega_2^c & \omega_1^c & 0 \end{bmatrix}$$

Collecting this steps, the components of the velocity vector for a particle of the deformed core for the t displacements are

$$\dot{\underline{t}} = \dot{\underline{x}} = P^{-1} \cdot M(\omega^c) \cdot P \cdot (\underline{x} - \underline{s}) + \dot{\underline{s}} = P^{-1} \cdot M(\omega^c) \cdot P \cdot \underline{x} + \dot{\underline{s}}$$

neglecting the products of the infinitesimal magnitudes ω_1^c, ω_2^c .

3. General tidal and centrifugal deformations g .

If we suppose an elastic rotating earth model subjected to the tidal effects and also we suppose a disturbing effect created by the variations of the rotation velocity vector $\omega = \Omega(m_1, m_2, l+m_3)$ (m_i : components of the pole motion), the liquid core, as a part of the elastic model, is subjected to the tidal and centrifugal global deformations.

These effects can be studied from the disturbing potentials. The second degree terms of these potentials in a point (r, θ, λ) are (Melchior, 1978):

$$\begin{aligned} W_e = W_e |_2 = & - \frac{GM_e}{d_e^3} r^2 \left[\sum_j A_{21j} \sin(\alpha_j + \lambda) P_{21}(\cos \theta) + \right. \\ & \left. + \sum_j A_{20j} \cos \alpha_j P_{20}(\cos \theta) - \sum_j A_{22j} \cos(\alpha_j + 2\lambda) P_{22}(\cos \theta) \right] \end{aligned}$$

where $\alpha_j = BMST + \pi + \Delta \alpha_j$ and $\Delta \alpha_j$ is a long period astronomical argument; and (Munk & MacDonald, 1975):

$$Ur = - \frac{1}{3} r^2 \Omega^2 \left[(1+2m_3) P_{20}(\cos \theta) + (m_1 \cos \lambda + m_2 \sin \lambda) P_{21}(\cos \theta) \right]$$

For this kind of deformations it is more suitable to use a reference system $O x_r x_\theta x_\lambda$ associated to the mean (initial) position (r, θ, λ) of the particle:

$$\begin{bmatrix} x_\theta \\ x_\lambda \\ x_r \end{bmatrix} = R_2(\theta) R_3(\lambda) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$R_2(\cdot), R_3(\cdot)$ being rotation matrices.

Using the Love numbers theory for this elastic model, the deformations q^e from an initial sphere (we use the spherical configuration obtained with the radial centrifugal deformation associated to the term $1/3 \cdot r^2 \Omega^2$ of the centrifugal potential from a non rotating sphere; this is the mean spherical configuration) ($q^e = q_{sec} + q$) can be written, for second degree (Melchior, 1978):

$$q_r^e = 1/g \quad H_2(r) \quad (Ur + We)$$

$$q_\theta^e = 1/g \quad L_2(r) \quad \frac{\partial}{\partial \theta} \quad (Ur + We)$$

$$q_\lambda^e = 1/g \quad L_2(r) \quad (\sin \theta)^4 \quad \frac{\partial}{\partial \lambda} \quad (Ur + We)$$

The functions H_2, L_2 represent the elastical properties of the body.

From the resulting expressions we can separate the secular terms (from spherical to ellipsoidal configuration) from the periodical infinitesimal terms.

4. Additional centrifugal deformations q from the particular core motion.

The simple motion of the liquid core (a global rotation for the equivalent sphere) can create an additional centrifugal deformation in the liquid core which load upon the elastic mantle.

For this kind of deformations in the core we make a special study. In fact, the simple motion and its possible associated centrifugal deformations depend on the shape of the core-mantle boundary surface. We can not use the same expressions starting from the initial spherical configuration (necesary for Love numbers theory), from the mean ellipsoidal configuration and from the deformed configuration, as it is possible for the global tidal and rotational deformations. Even, this deformations are of difficult treatment by disturbing potentials from the mean ellipsoide.

We try to solve the problem making the change to the equivalent sphere. For that, the simple motion is a global rotation and we can use the usual development of rotational deformations. So, the problem of the time variations in the simple motion and its associated deformations with the core shape variations is transformed into the time variations of the coordinate transformation $Ox'_1 \rightarrow Ox'_1$ for the core shape variations.

For the equivalent sphere (x'_1 coordinates) the simple motion is an additional infinitesimal global rotation of velocity vector $\omega^c = \Omega (m_1^c, m_2^c, m_3^c)$. The corresponding additional terms of the rotational potential in the equivalent sphere will be:

$$- 2/3 r'^2 \Omega^2 m_3^c P_2(\cos \theta') - 1/3 r'^2 \Omega^2 (m_1^c \cos \lambda' + m_2^c \sin \lambda') P_2(\cos \theta') ,$$

where r', θ', λ' are the polar coordinates in the equivalent sphere for

the initial state without additional rotational deformation.

We have the relations:

$$\begin{aligned}x_1 &= r \sin \theta \cos = a x'_1 = a r' \sin \theta' \cos \lambda' \\x_2 &= r \sin \theta \sin = a x'_2 = a r' \sin \theta' \sin \lambda' \\x_3 &= r \cos \theta = c x'_3 = c r' \cos \theta'\end{aligned}$$

And thus:

$$\begin{aligned}\lambda' &= \lambda \\r' &= r (1/a \sin^2 \theta + 1/c \cos^2 \theta) \approx \frac{r}{c} (1 - \frac{1}{2} \epsilon^2 \sin^2 \theta) \\\cos \theta' &= \frac{r}{cr'} \cos \theta \approx \cos \theta (1 + \frac{1}{2} \epsilon^2 \sin^2 \theta) \\\sin \theta' &\approx \sin \theta c/a (1 + \frac{1}{2} \epsilon^2 \sin^2 \theta)\end{aligned}$$

where we use the core geometrical eccentricity $\epsilon^2 = (a^2 - c^2)/a^2$ as first order infinitesimal term.

Now, we can apply the Love number theory for elastic deformations and to write the corresponding displacement as follows:

$$\begin{aligned}p'_r &= \frac{H'_2(r')}{g'} r'^2 \Omega^2 [-m_3^2 \frac{1}{3} (3 \cos^2 \theta' - 1) - (m_1^2 \cos \lambda' + m_2^2 \sin \lambda') \sin \theta' \cos \theta'] \\p'_\theta &= \frac{L'_2(r')}{g'} r'^2 \Omega^2 [2 m_3^2 \sin \theta' \cos \theta' - (m_1^2 \cos \lambda' + m_2^2 \sin \lambda') (\cos^2 \theta' - \sin^2 \theta')] \\p'_\lambda &= \frac{L'_2(r')}{g'} r'^2 \Omega^2 (m_1^2 \sin \lambda' - m_2^2 \cos \lambda') \cos \theta'\end{aligned}$$

where $H'_2(r')$, $L'_2(r')$ are different from the usual Love numbers $H_2(r)$, $L_2(r)$ for the model. In fact, the usual Love numbers characterize the elastic deformations for a free spherical configuration subjected to harmonic disturbing potentials of second degree; whilst for the equivalent spherical core, the supposed deformation is also an elastic one and upon a spherical configuration subjected to harmonic potential, but it has less magnitude because the core is not free but envelopped by the elastic mantle, which is not subjected to that direct disturbing effect and it reduce the deformation magnitude.

Finally, we look for the expression of this deformation in terms of the reference system Ox_L . So, we apply the corresponding transformation:

$$\begin{aligned}\begin{bmatrix} p_\theta \\ p_\lambda \\ p_r \end{bmatrix} &= R_2(\theta) R_3(\lambda) \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = R_2(\theta) R_3(\lambda) \begin{bmatrix} a \\ a \\ c \end{bmatrix} \begin{bmatrix} p'_1 \\ p'_2 \\ p'_3 \end{bmatrix} = \\ &= R_2(\theta) R_3(\lambda) \begin{bmatrix} a \\ a \\ c \end{bmatrix} R_3(-\lambda') R_2(-\theta') \begin{bmatrix} p'_1 \\ p'_2 \\ p'_3 \end{bmatrix} =\end{aligned}$$

$$= \begin{bmatrix} a \cos\theta \cos\theta' + c \sin\theta \sin\theta' & 0 & a \cos\theta \sin\theta' - c \sin\theta \cos\theta' \\ 0 & a & 0 \\ a \sin\theta \cos\theta' - c \cos\theta \sin\theta' & 0 & a \sin\theta \sin\theta' + c \cos\theta \cos\theta' \end{bmatrix} \begin{bmatrix} p'_\theta \\ p'_\lambda \\ p'_r \end{bmatrix}$$

Also, using the relation between r, θ, λ and r', θ', λ' we have:

$$H'_2(r') = H'_2(\frac{r}{c}) + c \frac{dH'_2(\frac{r}{c})}{dr} (r' - r/c) + \dots \approx H'_2(\frac{r}{c}) - \frac{dH'_2(\frac{r}{c})}{dr} 1/2 r \epsilon^2 \sin^2\theta$$

$$L'_2(r') = \dots \approx L'_2(\frac{r}{c}) - \frac{dL'_2(\frac{r}{c})}{dr} 1/2 r \epsilon^2 \sin^2\theta ,$$

$$g' = g(r') = g(r) + O(\epsilon^2) .$$

Collecting these several results and making some calculi, it is possible to get, in a first order approximation for the core geometrical excentricity :

$$p_\theta \approx \frac{\alpha}{c^2} L'_2(\frac{r}{c}) \frac{1}{9} r^2 \Omega^2 [2 \frac{c}{\alpha} m_3^c \sin\theta \cos\theta - (m_1^c \cos\lambda + m_2^c \sin\lambda)(\cos^2\theta - \sin^2\theta)]$$

$$p_\lambda \approx \frac{\alpha}{c^2} L'_2(\frac{r}{c}) \frac{1}{9} r^2 \Omega^2 (m_1^c \sin\lambda - m_2^c \cos\lambda) \cos\theta ,$$

$$p_r \approx \frac{\alpha}{c^2} H'_2(\frac{r}{c}) \frac{1}{9} r^2 \Omega^2 [- \frac{1}{3} \frac{\alpha}{c} m_3^c (3 \cos^2\theta - 1) - (m_1^c \cos\lambda + m_2^c \sin\lambda) \sin\theta \cos\theta].$$

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