Hilbert transform is an isometry: A simple proof via real rational functions

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Abstract

The aim of this note is to give a simple proof of the fact that the Hilbert transform in $\mathcal{L}^2(\mathbb{R})$ is an isometry, using a basis for $\mathcal{L}^2(\mathbb{R})$ consisting orthonormal real rational functions.

1 Introduction

The Hilbert transform in $\mathbb{R}$ is the operator

$$(\mathcal{H}f)(x) = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(x-y)}{y} dy = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(u)}{x-u} du.$$

It is well known that $\mathcal{H}: \mathcal{L}^2(\mathbb{R}) \rightarrow \mathcal{L}^2(\mathbb{R})$ is a bounded linear operator (see e.g. Cotlar [2], Duoandikoetxea [3, 4], Grafakos [5], Laeng [7]). Moreover, it is known that

Theorem 1. *The Hilbert transform $\mathcal{H}: \mathcal{L}^2(\mathbb{R}) \rightarrow \mathcal{L}^2(\mathbb{R})$ is an isometry.*

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The usual proof of Theorem 1 uses Fourier transform and Plancherel equality (see e.g. Duoandikoetxea [3], Grafakos [5]).

Recently Duoandikoetxea [4] and Laeng [7] gave alternative proofs without using the Fourier transform. Duoandikoetxea [4] showed that the image of the orthonormal basis of $\mathcal{L}^2(\mathbb{R})$ consisting of the Hermite functions by the Hilbert transform is also an orthonormal basis of $\mathcal{L}^2(\mathbb{R})$ and therefore the Hilbert transform can be extended to $\mathcal{L}^2(\mathbb{R})$ isometrically. Whereas Laeng [7] used approximation of $\mathcal{L}^2(\mathbb{R})$ functions by simple functions and then used the norm of the Hilbert transform of the characteristic function of an interval.

In this work, we use the basis for $\mathcal{L}^2(\mathbb{R})$ $\mathcal{B}$ defined in [1] and, by calculating the Hilbert transform of the elements of $\mathcal{B}$, we obtain a simple proof of Theorem 1.

This note is organized as follows. In Section 2, we present the preliminaries needed to develop the result. In Section 3 we prove Theorem 1. The last section, Section 4, is devoted to address technical details of the proof and other auxiliary lemmas.

2 Preliminaries

We will denote by $\mathbb{N} = \{0, 1, 2, \dots\}$ and denote by $\mathbb{R}$ and $\mathbb{C}$ the set of the real and complex numbers respectively. Christov showed in [1] (see also [6, pp. 63]) that the family $\mathcal{B} = \{S_n, C_n\}_{n \in \mathbb{N}}$ where,

$$S_n(x) = \sqrt{\frac{2}{\pi}} \frac{(-1)^{n+1}}{(x^2 + 1)^{n+1}} \sum_{k=0}^n (-1)^k \binom{2n+1}{2k+1} x^{2k+1}$$

and

$$C_n(x) = \sqrt{\frac{2}{\pi}} \frac{(-1)^{n+1}}{(x^2 + 1)^{n+1}} \sum_{k=0}^n (-1)^{k+1} \binom{2n+1}{2k} x^{2k}$$

is a complete orthonormal basis for $\mathcal{L}^2(\mathbb{R})$. The Fourier series of $f \in \mathcal{L}^2(\mathbb{R})$, with respect to $\mathcal{B}$, is given by

$$f = \sum_{n \in \mathbb{N}} a_n S_n + b_n C_n$$

where the corresponding Fourier coefficients are determined by

$$a_n = \langle f, S_n \rangle, \quad b_n = \langle f, C_n \rangle.$$

With this representation of f , the $\mathcal{L}^2(\mathbb{R})$ norm of f is given by the Fourier coefficients and the Plancherel equality:

$$\|f\|_2^2 = \sum_{n \in \mathbb{N}} a_n^2 + b_n^2.$$

To end the section, we remind the reader that the Hilbert transform $\mathcal{H}$ is a bounded linear operator of $\mathcal{L}^2(\mathbb{R})$. Now we can proceed to the proof of the main Theorem.

3 Proof of Theorem 1

The proof of Theorem 1 is really simple using the complete orthonormal basis $\mathcal{B}$ of $\mathcal{L}^2(\mathbb{R})$ and the following lemma that states that the Hilbert transform acts as a kind of involution on $\mathcal{B}$. More precisely

Lemma 1. *For every $n \in \mathbb{N}$,*

$$\mathcal{H}S_n = C_n \quad \text{and} \quad \mathcal{H}C_n = -S_n.$$

We now turn to the proof of the principal result.

Proof of Theorem 1. Given $f \in \mathcal{L}^2(\mathbb{R})$, and since $\mathcal{B}$ is a complete orthonormal basis and the Hilbert transform is a bounded linear operator, we have

$$\mathcal{H}f = \sum_{n \in \mathbb{N}} a_n \mathcal{H}S_n + b_n \mathcal{H}C_n,$$

where a_n y b_n are, as before, the Fourier coefficients of f with respect to $\mathcal{B}$. By Lemma 1 $\mathcal{H}S_n = C_n$ and $\mathcal{H}C_n = -S_n$, hence

$$\mathcal{H}f = \sum_{n \in \mathbb{N}} a_n C_n - b_n S_n.$$

Finally by the Plancherel equality we conclude

$$\|\mathcal{H}f\|_2^2 = \sum_{n \in \mathbb{N}} a_n^2 + b_n^2 = \|f\|_2^2.$$

□

4 Auxiliary results

To prove Lemma 1 we introduce the family $\mathcal{D} = \{\phi_n\}_{n \in \mathbb{N}}$ where

$$\phi_n := C_n + iS_n, \quad n \in \mathbb{N}.$$

Notice that ϕ_n takes complex values so that we are working in $\mathcal{L}^2(\mathbb{R}, \mathbb{C})$ which is a Hilbert space with inner product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx$$

for $f, g \in \mathcal{L}^2(\mathbb{R}, \mathbb{C})$. Also for $f \in \mathcal{L}^2(\mathbb{R}, \mathbb{C})$,

$$\mathcal{H}f = \mathcal{H}\operatorname{Re}(f) + i\mathcal{H}\operatorname{Im}(f).$$

Figure 1: The contour Γ_R

Proof of Lemma 1 A direct calculation shows that

$$\phi_n(x) = -i\sqrt{\frac{2}{\pi}} \frac{(x+i)^n}{(x-i)^{n+1}}. \quad (1)$$

Then

$$(\mathcal{H}\phi_n)(x) = \frac{-i}{\pi} \sqrt{\frac{2}{\pi}} P.V. \int_{-\infty}^{\infty} \frac{(u+i)^n}{(u-i)^{n+1}(x-u)} du.$$

Let

$$\Phi_n(w) := \frac{\phi_n(w)}{x-w} = -i\sqrt{\frac{2}{\pi}} \frac{(w+i)^n}{(w-i)^{n+1}(x-w)}$$

for $x \in \mathbb{R}$. The function $\Phi_n(w)$ is a meromorphic function in the upper half plane with a pole of order $n+1$ at $w=i$ and a simple pole at $w=x$. Consider the contour Γ_R shown in Figure 1, that contains the pole i in its interior and avoids the pole at $w=x$. Then on the one hand we have

$$\int_{\Gamma_R} \Phi_n(w) dw = 2\pi i \operatorname{Res}(\Phi_n(w), w=i)$$

and on the other

$$\int_{\Gamma_R} \Phi_n(w) dw = \left(\int_{\gamma_R} + \int_{\gamma_1} + \int_{\gamma_\varepsilon} + \int_{\gamma_2} \right) \Phi_n(w) dw. \quad (2)$$

Since on γ_R

$$|\Phi_n(w)| < \sqrt{\frac{2}{\pi}} \left(\frac{R+1}{R-1} \right)^n \frac{1}{(R-1)|R-|x||} \approx \sqrt{\frac{2}{\pi}} \frac{1}{R^2},$$

then

$$\lim_{R \rightarrow \infty} \int_{\gamma_R} \Phi_n(w) dw = 0. \quad (3)$$

We also have that

$$\lim_{\substack{\varepsilon \rightarrow 0 \\ R \rightarrow \infty}} \left(\int_{\gamma_1} \Phi_n(w) dw + \int_{\gamma_2} \Phi_n(w) dw \right) = P.V. \int_{-\infty}^{\infty} \Phi_n(w) dw.$$

To evaluate the remaining integral on the right hand side of equation (2), note that the function we are integrating is uniformly continuous near $x - i$ then

$$\lim_{\varepsilon \rightarrow 0} \int_{\gamma_\varepsilon} \Phi_n(w) dw = \sqrt{\frac{2}{\pi}} \int_0^\pi \lim_{\varepsilon \rightarrow 0} \frac{(x + i - \varepsilon e^{-it})^n}{(x - i - \varepsilon e^{-it})^{n+1}} dt = i\pi\phi_n(x).$$

Hence

$$\int_{\Gamma_R} \Phi_n(w) dw \rightarrow P.V. \int_{-\infty}^{\infty} \frac{\phi_n(w)}{x - w} dw + i\pi\phi_n(x), \quad (4)$$

as $R \rightarrow \infty$ and $\varepsilon \rightarrow 0$. The proof will be complete as soon as we compute the residue of Φ_n around $w = i$. The expansion of $\Phi_n(w)(w - i)^{n+1}$ around $w = i$ gives

$$\Phi_n(w)(w - i)^{n+1} = -i\sqrt{\frac{2}{\pi}} \sum_{k=0}^n \binom{n}{k} \frac{(2i)^{n-k}}{x - i} \sum_{j=0}^{\infty} \frac{(w - i)^{j+k}}{(x - i)^j}.$$

Hence

$$\text{Res}(\Phi_n(w), w = i) = -i\sqrt{\frac{2}{\pi}} \sum_{k=0}^n \binom{n}{k} \frac{(2i)^{n-k}}{(x - i)^{n-k+1}} = \phi_n(x). \quad (5)$$

From (3), (4) and (5) we get

$$\mathcal{H}\phi_n = 2i\phi_n - i\phi_n = i\phi_n.$$

Finally

$$\mathcal{H}C_n = \text{Re}(\mathcal{H}\phi_n) = -S_n, \quad \mathcal{H}S_n = \text{Im}(\mathcal{H}\phi_n) = C_n.$$

□

5 Final remarks

The proof we present not only is quite simple but has the advantage that it requires elementary results. On the other hand, the elements in the family are C^∞ rational functions that behave like $1/|x|$ at $\pm\infty$.

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