Unraveling the Landau's consistence criterion and the meaning of interpenetration in the "Two-Fluid" Model.

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In this letter we show that it is possible to unravel both the physical origin of the Landau's consistence criterion and the specific and subtle meaning of interpenetration of the "two fluids" if one takes into account that in the hydrodynamic regime one needs a coarse-graining in time to bring the system into local equilibrium. That is, the fuzziness in time is relevant for the phenomenological Landau's consistency criterion and the meaning of interpenetration. Note also that we are not questioning the validity of the "Two-Fluid" Model.

I. INTRODUCTION

A quantum fluid is a substance that remains fluid (i.e. gas or liquid) at such low temperatures that the effects of quantum mechanics play a dominant role and the laws of classical statistical mechanics do not serve. The two isotopes of Helium, ^{4}He and ^{3}He, are the only quantum fluids on the Earth. The ^{4}He atom is a zero-spin boson, which is determinant for its quantum properties. In fact, at temperatures and pressures below $T_\lambda \approx 2.17$ K and 2.5 MPa respectively, ^{4}He is at the so called He II superfluid phase where it displays among other properties [1, 2] that of flowing through narrow channels with no measurable viscosity.

In order to explain the behaviour of He II, L. Tisza and L.D. Landau [3–5] developed the Two-Fluid Model. They described the He II phase by two components, one viscous and another non-viscous, related to each other in a very sophisticated way, as we shall explain in Section II. The main differences between the Tisza and Landau proposals are discussed in detail in [6]. Here we will deal with the theory of Landau and Feynman

[4, 5, 7–9], the commonly accepted Two-Fluid Model.

A very complete summary of the Two-Fluid theory can be seen in [10]. In [11] the authors grasp some interesting subtleties of the Two-Fluid model. We only highlight here that one of the most important Feynman's contributions was to note [7, 8] the relevance of Bose-Einstein statistics to correctly account for the thermally excited states of compression, with a resultant change in the density: the phonon-roton density modes of the normal component (see also [12]).

II. THE LANDAU'S CONSISTENCY CRITERION AND THE MEANING OF INTERPENETRATION

In the Two-Fluid Model of the He II, the thermodynamical equilibrium states are described, at any temperature $T < T_\lambda$, [13], by two independent velocities, $\vec{v}_s(\vec{r})$, $\vec{v}_n(\vec{r})$, and associated densities, $\rho_s(\vec{r})$ and $\rho_n(\vec{r})$, such that the densities of mass current $\vec{J}(\vec{r})$ and of kinetic energy flow $Q(\vec{r})$ are given by,

$$\rho(\vec{r}) = \rho_s(\vec{r}) + \rho_n(\vec{r}), \tag{1}$$

$$J(\vec{r}) = \rho_s(\vec{r})\vec{v}_s(\vec{r}) + \rho_n(\vec{r})\vec{v}_n(\vec{r}), \tag{2}$$

$$Q(\vec{r}) = \frac{1}{2}\left[\rho_s(\vec{r})\vec{v}_s(\vec{r})^2 + \rho_n(\vec{r})v_n^2(\vec{r})\right], \tag{3}$$

where $\rho(\vec{r})$ is the total mass density and the superfluid component $\rho_s(\vec{r})$ is not viscous. The entropy of the liquid is entirely attributed to the normal fluid part ρ_n,

$$\rho(\vec{r})S = \rho_n(\vec{r})S_n,$$

where S is the total entropy per mass of the liquid and S_n the normal fluid entropy.

At $T = 0$, the entire liquid is supposed to be superfluid, i.e. $\rho = \rho_s$ while at $T = T_\lambda$ the superfluid component vanishes, i.e. $\rho = \rho_n$.

The model is completed by taking into account the following Landau's consistency criterion [4, 5]: "...there is no division of the real particles of the liquid into "superfluid and "normal" ones. In a certain sense one can speak of "superfluid" and "normal" masses of liquid as of masses connected with two simultaneously possible movements, but this by no means signifies the possibility of a real division of the liquid into two parts".

Here we shall discuss on the Landau's consistency criterion and on the meaning of interpenetration.

Firstly, $\vec{r}$ must be understood, as always in fluid mechanics [14], "as the position of the volume element corresponding to a "fluid particle", in the sense of a volume element containing many particles though regarded as a point". Then, when we speak of the displacement of a fluid particle we mean not the displacement of an individual atom of He, but that of a volume element containing many atoms, though still regarded as a point. Since both densities (1) are referred to the same $\vec{r}$, they "lend nicely to an intuitive picture of He II as a mixture of two independent, interpenetrating "fluids" or "components", the "superfluid" and the "normal" components" [13]. Of course, the interpenetration is not just that both components refer to the volume element of the fluid positioned in $\vec{r}$, this is common to fluid dynamics for a standard mixture of fluids, as e.g. water and wine, (see Chapter VI of [14]). In order to unravel the physical meaning of the interpenetration one can take into account a coarse-graining in time in correspondence with the mentioned volume element, or coarse-graining in space, positioned in $\vec{r}$.

Indeed, following the fundamental basis of the hydrodynamic regime [15–19], one should introduce a coarse-graining in time. In fact, Landau and Lifshitz thought that the fuzziness in time was somewhat irrelevant. Then, assuming this irrelevance, they extended the Navier-Stokes equation to a relativistic covariant form (see Chapter XV of [14]). However, this equation is known to be unstable, and we now know its origin: any volume element requires a finite relaxation time, or coarse-graining in time, to reach a thermodynamical equilibrium and the Landau and Lifshitz naive covariant extension of the Navier-Stokes equation, obtained neglecting this necessary finite relaxation time, contains acausal modes which are the origin of the instability. Let us recall Section 26 of [14]: "Not every solution of the equations of motion, even if it is exact, can actually occur in Nature. The flows that occur in Nature must not only obey the equations of fluid dynamics, but also be stable". That is, the consistency of the theory advises us to consider a coarse-graining in time. Indeed, in the hydrodynamic regime one needs a coarse-graining in time larger than the relaxation time that brings the system into local equilibrium.

It is true that for the most of non-relativistic cases we do not need explicitly a coarse-graining in time, as in [14]. Nevertheless, now we will see that taking into account this

coarse-graining we can unravel the physical origin of the Landau's consistency criterion and the meaning of interpenetration.

In fact, when fluid mechanics describes the evolution of a fluid with time what is being described is what it would be observed in successive snapshots corresponding to successive coarse-graining times. In our case what it will be observed is that the He atoms participating in the phonon-roton density modes [7, 8, 12] of the "normal component" will be different at each given coarse-graining time (as illustration one can think in the thermodynamic description of vapor-liquid equilibrium: molecules that participate in the vapor and in the liquid change with time). Then, "...there is no division of the real particles of the liquid into "superfluid" and "normal" ones..." or "...there is no chance of a real division of the liquid into two parts", as Landau emphasizes. This is a crucial difference with the water/wine mixture case mentioned above, in which there are water molecules different from wine molecules. To sum up, in the two fluid model the He atoms can not be labeled according to their contribution to the superfluid, $\rho_s(\vec{r})$, or to the normal component, $\rho_n(\vec{r})$: they are interpenetrating fluids, as Leggett says.

III. THE LOCAL EQUILIBRIUM HYDRODYNAMIC REGIME (LEHR)

The most extraordinary fact of the superfluid phase of He II is the stability of its flow. Since the discovery of this dramatic feature, it was described (almost always) within the framework of the LEHR, at the beginning implementing in this frame the Tisza's two fluid model and later the Landau's model. In section II we have taken advantage of this fact, in particular of the physical finite relaxation time, fundamental ingredient of the LEHR. Superflow stability has also been studied by starting from the hypothesis that steady supercurrents are metastable states. Therefore, in this case, it must be discussed in the language of statistical mechanics of irreversible processes [2, 20].

Now, on the one hand, we have the physical stability of the superflow and the physically meaningful relaxation time already discussed in section II. On the other hand we can ask about the absence of numerical instabilities when one is working with a system of nonlinear time-dependent equations, where also a mathematical Δt, without physical meaning, appears. The comment on this is the reason for this section.

Let us recall that in the two-fluid models of helium II, while at low enough flow veloc-

ities the normal and superfluid components of helium II move independently, at higher velocities (see section "Two-Fluid Models" of the second article of [10]), "quantized vortex lines appear in the superfluid component and the two fluids become coupled by a force called mutual friction". Phonon-rotons density modes continue being the normal component. To be specific, let's consider the Landau's Two Fluid Model (LTFM) in this framework.

Two points to keep in mind are:

First, the LEHR is also the physical frame suitable for raising generalizations of the LTFM (see the first three sections of [21]).

Second, the concrete physical properties which characterize a system in the LEHR must be searched using physical arguments as, for example, in the case considered, what is the physical time scale for the vortex lines to equilibrate? A vortex line is a hole with quantised circulation, so the answer must have to do with the speed of sound (240 m/s in helium), the size of the hole (diameter about 10^{-10} m) or the distance between vortices. Although much is known about quantized vortices in Helium II [22], as far as we know, that knowledge is not sufficient to answer this question. This lack of knowledge or certainty does not prevent us the construction of models living thanks to the physical support given by the LEHR: these models are supported on their success [4, 5, 7–9, 14].

We conclude this section by summarizing the situation from the mathematical side. To prove mathematically the absence of numerical instabilities when one is working with a system of nonlinear time-dependent partial differential equations in two or three spatial dimensions, as the LTFM equations, is a very difficult problem. In fact, as can be seen in [23–25], in order to achieve numerical stability, one monitors the power spectrum of the solution to make sure that it decays at large spatial frequency.The power-law decay means that spectral convergence has been achieved. Here the only point we want to clarify is that, in these studies, the time step Δt, which sets the boundary between stable and unstable numerical solutions (see section 5 of [23]), means nothing physically, because it depends on the precise numerical method: this Δt is not a physically meaningful relaxation time (see Section II).

IV. CONCLUSION

The discussion in Section II is supported on the physical finite relaxation time required by the fundamental basis of the LEHR, in which the LTFM and its generalization live. As we have seen, neglecting it can lead to serious inconsistencies and sometimes to forget the possible physical origin of important ingredients of a phenomenological theory such as, in our case, the Landau's consistency criterion and the meaning of interpenetration in the Two-fluid model.

Finally, note that one can conclude that the name of the model is in fact a bit misleading and it should contain a small footprint of the Landau's consistency criterion by typing in quotes "two fluid", as we have done in the title.

Author contribution: all the authors have contributed equally to the work.

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