Deformations Monitoring by Integrating Local and Global Reference Systems

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Abstract. Although the transformation between threedimensional reference systems has been widely studied in geodesy, in this paper, a different way of integrating, and not transforming, both systems is used. On one hand, a straightforward computational method without iterations nor “a priori” known values is shown. On the other hand, some applications have been made and presented. The method, recently recognised by Grafarend, is known as the Procrustes solution and it is based on the Singular Value Decomposition of a matrix from which we obtain the Euler Elements. Finally real network located in Gipuzkoa has been used to monitor a simulated deformation via the Procrustes solution, which is compared with the Helmert’s datum transformation classical approach.

Keywords: Procrustes, SVD, datum transformation, Euler elements, Orthogonal transformation.

1. Introduction

Still today, one of the most challenging tasks in geodesy is the datum transformation. We can admit that the datum transformation problem has never been as important as nowadays that a new global reference system has been adopted: ETRS89. Thus, obtaining correct and accurate transformation models from one system to other (global to global, global to local or local to local) is a main task in modern geodesy to warrant that there is no loose of accuracy in the data due to the transformation. Although the datum transformation has been widely studied by many authors (Krakiwsky and Thompson, 1974), this paper deals with the method known as the Procrustes problem (Schönemann, 1966, Sansò, 1973, Crosilla, 1983, Awange, 2003, Awange and Grafarend, 2005). The basic aim of the

Procrustes solution is to fit one given set of points into another known set of points and get a measure of the quality. In the next pages, the different solutions will be treated in detail and an example is provided where transformation models are compared, using real data.

2. Three Dimensional Positioning Systems

In order to understand the Procrustes solution, the local and the global three dimensional positioning systems must be defined. The main characteristics of such reference systems are pointed in the next lines.

2.1. Local Positioning System

The local positioning system (Figure 1) owes its name to the following characteristic: every measure is referred to a local origin which is, usually, the centre of the instrument. The local positioning system (f_1, f_2, f_3) can be described as follows:

- Origin: Centre of the instrument.
- f_3 is the direction of the plumb line.
- f_1 in the horizontal plane.
- f_2 form a right handed system with f_3 and f_1

Fig. 1: Local positioning system.

2.2. Global Positioning System

The main characteristics of the global positioning system (F_1, F_2, F_3) are (Figure 2):

- Origin: Center of mass of the Earth.
- F_3 is the ITRS pole direction.
- F_1 is the ITRS Origin of longitudes.
- F_2 forms aright handed system with F_1 and F_3 .

Fig. 2: Global positioning system.

3. Three Dimensional Datum Transformation Models

3.1. The Helmert Transformation

The classical Helmert datum transformation solution is given by a translation (vector $\mathbf{T}$), a scale factor k and a rotation (matrix $\mathbf{R}(\omega)$). The most important disadvantages of Helmert's solution are: they need "a priori" known values, they are iterative and need to be linearized. Commonly, the solution is presented by the equations

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{GPS} = \begin{bmatrix} T_U \\ T_V \\ T_W \end{bmatrix}_{Translation} + (1+k) \begin{bmatrix} 1 & \omega_W & -\omega_V \\ -\omega_W & 1 & -\omega_U \\ \omega_V & \omega_U & 1 \end{bmatrix} \begin{bmatrix} U \\ V \\ W \end{bmatrix}_{Local} \quad (1)$$

The parameters of the transformations are obtained by the least squares mixed conditional model

$$\mathbf{Ax} + \mathbf{Bv} - \mathbf{t} = \mathbf{0}, \quad (2)$$

were $\mathbf{A}$ and $\mathbf{B}$ are the design matrices, $\mathbf{x}$ the parameter vector, $\mathbf{t}$ the vector of observations and $\mathbf{v}$ the residuals.

3.2. The Procrustes Transformations

The weighted extended orthogonal Procrustes (WEOP) transformation is given by the equation (Awange and Grafarend 2005)

$$\mathbf{x} = \mathbf{u}\mathbf{T}^T + k\mathbf{X}\mathbf{R}^T + \mathbf{E} \quad (3)$$

were $\mathbf{E}$ represents the residuals, $\mathbf{x}$, $\mathbf{X}$, the coordinates of several points in each system respectively, k the scale factor, $\mathbf{R}$ the rotation matrix and $\mathbf{u}$ is a vector of as many ones as known points in both systems. $\mathbf{T}$ is the translation vector.

To see how (3) is obtained we need to know how a transformation of a point is performed. To transform the coordinates of a point from a known reference system into another known reference system a translation vector, a rotation matrix and a scale factor shall be considered. Therefore, for one single point we have:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} + k \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \quad (4)$$

In case of two points, (4) would be:

$$\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_2 \end{pmatrix} = \begin{pmatrix} T_1 & T_1 \\ T_2 & T_2 \\ T_3 & T_3 \end{pmatrix} + k \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \begin{pmatrix} X_1 & X_2 \\ Y_1 & Y_2 \\ Z_1 & Z_2 \end{pmatrix} \quad (5)$$

For n points, (4) would be written as:

$$\begin{pmatrix} x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \\ z_1 & z_2 & \dots & z_n \end{pmatrix} = \begin{pmatrix} T_1 & T_1 & \dots & T_1 \\ T_2 & T_2 & \dots & T_2 \\ T_3 & T_3 & \dots & T_3 \end{pmatrix} + k \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \begin{pmatrix} X_1 & X_2 & \dots & X_n \\ Y_1 & Y_2 & \dots & Y_n \\ Z_1 & Z_2 & \dots & Z_n \end{pmatrix} \quad (6)$$

We consider the matrices of the coordinates:

$$\mathbf{x} = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ \dots & \dots & \dots \\ x_n & y_n & z_n \end{pmatrix} \text{ and } \mathbf{X} = \begin{pmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ \dots & \dots & \dots \\ X_n & Y_n & Z_n \end{pmatrix} \quad (7)$$

Taking into account the notation used in (3), and the Cartesian coordinates in (7), the transformation equation for n points can be rewritten as:

$$\mathbf{x}^T = \mathbf{T}\mathbf{u}^T + k\mathbf{X}\mathbf{R}^T \quad (8)$$

By transposing (8) we get:

$$\mathbf{x} = \mathbf{u}\mathbf{T}^T + k\mathbf{X}\mathbf{R}^T \quad (9)$$

The data are nor error free and this fact is taken into account by adding the error matrix $\mathbf{E}$. Thus, the Procrustes final equation is:

$$\mathbf{x} = \mathbf{u}\mathbf{T}^T + k\mathbf{X}\mathbf{R}^T + \mathbf{E} \quad (10)$$

The conditions imposed to determine the parameters of the transformation are:

- Norm of $\mathbf{E}$ must be a minimum.
- $\mathbf{R}$ is orthogonal

Since we need to solve a conditioned minimum, among all the possible solutions, the Lagrange method is considered. The norm used will be the Frobenius Norm:

Thus, the equation to be solved is (3) with the conditions:

$$\|\mathbf{E}\|_F = \min. \quad \mathbf{R}^T \mathbf{R} = \mathbf{I} \text{ and } \det(\mathbf{R}) = +1 \quad (11)$$

The Frobenius norm (seminorm) is defined by means of the trace of matrix as follows

$$\|\mathbf{E}\|_F^2 = \text{tr}(\mathbf{E}^T \mathbf{E}) \quad (12)$$

The weight matrix $\mathbf{P}$ of the observations is semidefinite positive because it might occur that not all the elements are known. In this cases, the solution is obtained by using null values for such element. $\mathbf{P}$ is computed by taking into account the dispersion matrixes of both sets of coordinates (see sec. 3.3).

To solve (3), subject to conditions (11) the first step is to compute the translation vector $\mathbf{T}$. The norm to be minimized is

$$\begin{aligned} N(k, \mathbf{T}, \mathbf{R}) &= \text{tr}(\mathbf{x} - \mathbf{u}\mathbf{T}^T - k\mathbf{X}\mathbf{R}^T)^T \mathbf{P} (\mathbf{x} - \mathbf{u}\mathbf{T}^T - k\mathbf{X}\mathbf{R}^T) = \\ &= \text{tr}(\mathbf{x}^T \mathbf{P} \mathbf{x} - \mathbf{x}^T \mathbf{P} \mathbf{u} \mathbf{T}^T - k \mathbf{x}^T \mathbf{P} \mathbf{X} \mathbf{R}^T - \\ &\quad - \mathbf{T} \mathbf{u}^T \mathbf{P} \mathbf{x} + \mathbf{T} \mathbf{u}^T \mathbf{P} \mathbf{u} \mathbf{T}^T + k \mathbf{T} \mathbf{u}^T \mathbf{P} \mathbf{X} \mathbf{R}^T - \\ &\quad - k \mathbf{R} \mathbf{X}^T \mathbf{P} \mathbf{x} + k \mathbf{R} \mathbf{X}^T \mathbf{P} \mathbf{u} \mathbf{T}^T + k^2 \mathbf{R} \mathbf{X}^T \mathbf{P} \mathbf{X} \mathbf{R}^T) \end{aligned} \quad (13)$$

For (13) to be a minimum, its partial derivative with respect to the unknowns must be null:

$$\begin{aligned} \frac{\partial N}{\partial \mathbf{T}^T} &= -\mathbf{x}^T \mathbf{P} \mathbf{u} - \mathbf{x}^T \mathbf{P} \mathbf{u} + 2 \mathbf{T} \mathbf{u}^T \mathbf{P} \mathbf{u} + k \mathbf{R} \mathbf{X}^T \mathbf{P} \mathbf{u} + k \mathbf{R} \mathbf{X}^T \mathbf{P} \mathbf{u} = \\ &= -2 \mathbf{x}^T \mathbf{P} \mathbf{u} + 2 \mathbf{T} \mathbf{u}^T \mathbf{P} \mathbf{u} + 2 k \mathbf{R} \mathbf{X}^T \mathbf{P} \mathbf{u} = 0 \end{aligned} \quad (14)$$

Finally, the translation vector can be written as:

$$\mathbf{T} = (\mathbf{u}^T \mathbf{P} \mathbf{u})^{-1} (\mathbf{x}^T - k \mathbf{R} \mathbf{X}^T) \mathbf{P} \mathbf{u} \quad (15)$$

In the special case that $\mathbf{P} = \mathbf{I}$, the solution is :

$$\mathbf{T} = \frac{1}{n} (\mathbf{x}^T - k \mathbf{R} \mathbf{X}^T) \mathbf{u} = \frac{1}{n} (\mathbf{x} - k \mathbf{X} \mathbf{R}^T)^T \mathbf{u} \quad (16)$$

Once the translation vector has been computed, we do now proceed to obtain the scale factor.

Taking into account (15), the equation to be solved is now:

$$\begin{aligned} N(k, \mathbf{R}) &= \text{tr} \left[\mathbf{x} - \mathbf{u} \mathbf{u}^T \mathbf{P} (\mathbf{x}^T - k \mathbf{R} \mathbf{X}^T)^T (\mathbf{u}^T \mathbf{P} \mathbf{u})^{-1} - k \mathbf{X} \mathbf{R}^T \right]^T \mathbf{P} \\ &\quad \left[\mathbf{x} - \mathbf{u} \mathbf{u}^T \mathbf{P} (\mathbf{x}^T - k \mathbf{R} \mathbf{X}^T)^T (\mathbf{u}^T \mathbf{P} \mathbf{u})^{-1} - k \mathbf{X} \mathbf{R}^T \right] \end{aligned} \quad (17)$$

(17) is centred because the translation parameters have been included in the formula.

$$\begin{aligned} N(k, \mathbf{R}) &= \text{tr} \left[(\mathbf{x} - k \mathbf{X} \mathbf{R}^T) - (\mathbf{u}^T \mathbf{P} \mathbf{u})^{-1} \mathbf{u} \mathbf{u}^T \mathbf{P} (\mathbf{x} - k \mathbf{X} \mathbf{R}^T) \right]^T \mathbf{P} \\ &\quad \left[(\mathbf{x} - k \mathbf{X} \mathbf{R}^T) - (\mathbf{u}^T \mathbf{P} \mathbf{u})^{-1} \mathbf{u} \mathbf{u}^T \mathbf{P} (\mathbf{x} - k \mathbf{X} \mathbf{R}^T) \right] \\ &= \text{tr} \left\{ \left[\mathbf{I} - (\mathbf{u}^T \mathbf{P} \mathbf{u})^{-1} \mathbf{u} \mathbf{u}^T \mathbf{P} \right] (\mathbf{x} - k \mathbf{X} \mathbf{R}^T) \right\}^T \mathbf{P} \cdot \\ &\quad \cdot \left\{ \left[\mathbf{I} - (\mathbf{u}^T \mathbf{P} \mathbf{u})^{-1} \mathbf{u} \mathbf{u}^T \mathbf{P} \right] (\mathbf{x} - k \mathbf{X} \mathbf{R}^T) \right\} \end{aligned} \quad (18)$$

A new matrix shall be defined:

$$\mathbf{C} = \mathbf{I} - (\mathbf{u}^T \mathbf{P} \mathbf{u})^{-1} \mathbf{u} \mathbf{u}^T \mathbf{P} \quad (19)$$

With (19), (18) can now be written as:

$$\begin{aligned} N(k, \mathbf{R}) &= \text{tr} \left\{ \left[\mathbf{C} (\mathbf{x} - k \mathbf{X} \mathbf{R}^T) \right]^T \mathbf{P} \left[\mathbf{C} (\mathbf{x} - k \mathbf{X} \mathbf{R}^T) \right] \right\} \\ &= \text{tr} \left\{ (\mathbf{x}^T - k \mathbf{R} \mathbf{X}^T) \mathbf{C}^T \mathbf{P} \mathbf{C} (\mathbf{x} - k \mathbf{X} \mathbf{R}^T) \right\} \end{aligned} \quad (20)$$

or

$$\begin{aligned} N(k, \mathbf{R}) &= \text{tr} \left(\begin{aligned} &\mathbf{x}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{x} - k \mathbf{x}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{X} \mathbf{R}^T - \\ &- k \mathbf{R} \mathbf{X}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{x} + k^2 \mathbf{R} \mathbf{X}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{X} \mathbf{R}^T \end{aligned} \right) \end{aligned} \quad (21)$$

Eq. (21) must be a minimum, so its derivative with respect the unknown must be null:

$$\begin{aligned} \frac{\partial N(k, \mathbf{R})}{\partial k} = & -2\text{tr}(\mathbf{x}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{X} \mathbf{R}^T) + \\ & + 2k \text{tr}(\mathbf{R} \mathbf{X}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{X} \mathbf{R}^T) = 0 \end{aligned} \quad (22)$$

Finally, we get the scale factor:

$$k = \frac{\text{tr}(\mathbf{x}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{X} \mathbf{R}^T)}{\text{tr}(\mathbf{X}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{X})} \quad (23)$$

In the particular case that the weight matrix is diagonal unitary, the scale factor obtained is:

$$\begin{aligned} \mathbf{C} &= \mathbf{I} - \frac{1}{n} \mathbf{u} \mathbf{u}^T \\ k &= \frac{\text{tr}(\mathbf{x}^T \mathbf{C} \mathbf{X} \mathbf{R}^T)}{\text{tr}(\mathbf{X}^T \mathbf{C} \mathbf{X})} \end{aligned} \quad (24)$$

Once the translation vector and the scale factor have been computed, the rotation matrix will be computed. To compute the rotations, (17) and (23) are taken into account. Thus, the new equation to be minimized is:

$$\begin{aligned} N(\mathbf{R}) &= \text{tr} \left(\mathbf{x}^T - \frac{\text{tr}(\mathbf{x}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{X} \mathbf{R}^T)}{\text{tr}(\mathbf{X}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{X})} \mathbf{R} \mathbf{X}^T \right) \mathbf{C}^T \mathbf{P} \mathbf{C} \left(\mathbf{x} - \frac{\text{tr}(\mathbf{x}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{X} \mathbf{R}^T)}{\text{tr}(\mathbf{X}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{X})} \mathbf{X} \mathbf{R}^T \right) \\ &= \text{tr}(\mathbf{x}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{x}) - \frac{\text{tr}(\mathbf{x}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{X} \mathbf{R}^T)}{\text{tr}(\mathbf{X}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{X})} \text{tr}(\mathbf{x}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{X} \mathbf{R}^T) - \\ &\quad - \frac{\text{tr}(\mathbf{x}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{X} \mathbf{R}^T)}{\text{tr}(\mathbf{X}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{X})} \text{tr}(\mathbf{R} \mathbf{X}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{x}) + \\ &\quad + \frac{\text{tr}(\mathbf{x}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{X} \mathbf{R}^T)}{\text{tr}(\mathbf{X}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{X})} \frac{\text{tr}(\mathbf{x}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{X} \mathbf{R}^T)}{\text{tr}(\mathbf{X}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{X})} \text{tr}(\mathbf{R} \mathbf{X}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{X} \mathbf{R}^T) \\ &= \text{tr}(\mathbf{x}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{x}) - 2 \frac{\text{tr}(\mathbf{x}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{X} \mathbf{R}^T)^2}{\text{tr}(\mathbf{X}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{X})} + \frac{\text{tr}(\mathbf{x}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{X} \mathbf{R}^T)}{\text{tr}(\mathbf{X}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{X})} \\ &= \text{tr}(\mathbf{x}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{x}) - \frac{\text{tr}(\mathbf{x}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{X} \mathbf{R}^T)^2}{\text{tr}(\mathbf{X}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{X})} \end{aligned} \quad (25)$$

The condition for (25) to be a minimum is:

$$\text{tr}(\mathbf{x}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{X} \mathbf{R}^T) = \max \quad (26)$$

Let $\mathbf{D} = \mathbf{x}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{X}$ and let $\mathbf{U} \mathbf{\Sigma} \mathbf{V}^T$ be its Singular Value Decomposition. Then,

$$\begin{aligned} \mathbf{U} \mathbf{U}^T &= \mathbf{U}^T \mathbf{U} = \mathbf{I} \\ \mathbf{V} \mathbf{V}^T &= \mathbf{V}^T \mathbf{V} = \mathbf{I} \\ \mathbf{\Sigma} &= \text{diagonal}(\sigma_1, \sigma_2, \dots, \sigma_n) \end{aligned} \quad (27)$$

We can now write:

$$\mathbf{x}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{X} \mathbf{R}^T = \mathbf{D} \mathbf{R}^T = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^T \mathbf{R}^T \quad (28)$$

Now:

$$\begin{aligned} \text{tr}(\mathbf{x}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{X} \mathbf{R}^T) &= \text{tr}(\mathbf{D} \mathbf{R}^T) \\ \text{tr}(\mathbf{U} \mathbf{\Sigma} \mathbf{V}^T \mathbf{R}^T) &= \text{tr}(\mathbf{\Sigma} \mathbf{V}^T \mathbf{R}^T \mathbf{U}) \end{aligned} \quad (29)$$

let $\mathbf{K} = \mathbf{V}^T \mathbf{R}^T \mathbf{U}$, which is an orthogonal matrix with diagonal entries such that $|k_{ii}| \leq 1$, then

$$\text{tr}(\mathbf{\Sigma} \mathbf{V}^T \mathbf{R}^T \mathbf{U}) = \text{tr}(\mathbf{\Sigma} \mathbf{K}) = \sum_{i=1}^n \sigma_i k_{ii} \leq \sum_{i=1}^n \sigma_i \quad (30)$$

The maximum value of (30) is given by:

$$\begin{aligned} \max(\text{tr}(\mathbf{x}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{X} \mathbf{R}^T)) &= \max(\text{tr}(\mathbf{\Sigma} \mathbf{K})) \\ &= \sum_{i=1}^n \sigma_i = \text{tr} \mathbf{\Sigma} \end{aligned} \quad (31)$$

But the maximum is:

$$\text{tr}(\mathbf{\Sigma} \mathbf{V}^T \mathbf{R}^T \mathbf{U}) = \text{tr} \mathbf{\Sigma}, \quad (32)$$

so that

$$\mathbf{V}^T \mathbf{R}^T \mathbf{U} = \mathbf{I} \quad (33)$$

Multiplying by $\mathbf{V}$ and $\mathbf{U}^T$, we get:

$$\begin{aligned} \mathbf{V} \mathbf{V}^T \mathbf{R}^T \mathbf{U} &= \mathbf{V} \\ \mathbf{R}^T \mathbf{U} &= \mathbf{V} \\ \mathbf{R}^T \mathbf{U} \mathbf{U}^T &= \mathbf{V} \mathbf{U}^T \end{aligned} \quad (34)$$

From (34) we can write:

$$\mathbf{R} = \mathbf{U} \mathbf{V}^T \quad (35)$$

For the special case that $\mathbf{P} = \mathbf{I}$, we get:

$$\begin{aligned} \mathbf{D} &= \mathbf{x}^T \mathbf{C} \mathbf{X} = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^T \\ \mathbf{R} &= \mathbf{U} \mathbf{V}^T \end{aligned} \quad (36)$$

We can now compute the residuals by taking into account all the parameters computed:

$$\mathbf{E} = \left(\mathbf{I} - (\mathbf{u}^T \mathbf{P} \mathbf{u})^{-1} \mathbf{u} \mathbf{u}^T \mathbf{P} \right) \left(\mathbf{x} - \frac{\text{tr}(\mathbf{x}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{X} \mathbf{V} \mathbf{U}^T)}{\text{tr}(\mathbf{X}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{X})} \mathbf{X} \mathbf{V} \mathbf{U}^T \right) \quad (37)$$

The Frobenius norm, which can be considered as a measure of the quality of the transformation equals:

$$\|E\|_F^2 = \text{tr} \left[\left(\mathbf{I} - (\mathbf{u}^T \mathbf{P} \mathbf{u})^{-1} \mathbf{u} \mathbf{u}^T \mathbf{P} \right) \left(\mathbf{x} - \frac{\text{tr}(\mathbf{x}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{X} \mathbf{V} \mathbf{U}^T)}{\text{tr}(\mathbf{X}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{X})} \mathbf{X} \mathbf{V} \mathbf{U}^T \right) \right]^T \mathbf{P} \left[\left(\mathbf{I} - (\mathbf{u}^T \mathbf{P} \mathbf{u})^{-1} \mathbf{u} \mathbf{u}^T \mathbf{P} \right) \left(\mathbf{x} - \frac{\text{tr}(\mathbf{x}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{X} \mathbf{V} \mathbf{U}^T)}{\text{tr}(\mathbf{X}^T \mathbf{C}^T \mathbf{P} \mathbf{C} \mathbf{X})} \mathbf{X} \mathbf{V} \mathbf{U}^T \right) \right] \quad (38)$$

The values for the residuals and the Frobenius Norm, in the special case $\mathbf{P}=\mathbf{I}$ are respectively:

$$\mathbf{E} = \left(\mathbf{I} - \frac{1}{n} \mathbf{u} \mathbf{u}^T \right) \left(\mathbf{x} - \frac{\text{tr}(\mathbf{x}^T \mathbf{C} \mathbf{X} \mathbf{V} \mathbf{U}^T)}{\text{tr}(\mathbf{X}^T \mathbf{C} \mathbf{X})} \mathbf{X} \mathbf{V} \mathbf{U}^T \right) \quad (39)$$

$$\|E\|_F^2 = \text{tr} \left(\mathbf{x} - \frac{\text{tr}(\mathbf{x}^T \mathbf{C} \mathbf{X} \mathbf{V} \mathbf{U}^T)}{\text{tr}(\mathbf{X}^T \mathbf{C} \mathbf{X})} \mathbf{X} \mathbf{V} \mathbf{U}^T \right)^T \left(\mathbf{I} - \frac{1}{n} \mathbf{u} \mathbf{u}^T \right) \left(\mathbf{x} - \frac{\text{tr}(\mathbf{x}^T \mathbf{C} \mathbf{X} \mathbf{V} \mathbf{U}^T)}{\text{tr}(\mathbf{X}^T \mathbf{C} \mathbf{X})} \mathbf{X} \mathbf{V} \mathbf{U}^T \right) \quad (40)$$

3.3. Computation of the Weight Matrix

The Weighted Procrustes solution takes into account the dispersions of both coordinate sets. Nevertheless, the weight matrix P , has not been defined yet. This section deals with the computation of the weight matrix.

The weight matrix must take into account the dispersion matrices of both sets of coordinates and, of course, the local dispersion matrix when it is transformed to the global frame. It has been proved (Awange and Grafarend., 2005) that the weight matrix equals to:

$$\Sigma_{\text{vec} \mathbf{E}^T} = \Sigma_{\text{vec} \mathbf{X}^T} + (\mathbf{I}_n \otimes k \mathbf{R}) \Sigma_{\text{vec} \mathbf{X}^T} (\mathbf{I}_n \otimes k \mathbf{R})^T - 2 \Sigma_{\text{vec} \mathbf{X}^T} (\mathbf{I}_n \otimes k \mathbf{R}) \text{vec} \mathbf{X}^T \quad (41)$$

In (41), Σ makes reference to the variance covariance matrix, vec is the vec operator and $\otimes$ is the Kronecker product.

3.4. Partial Procrustes Analysis

The partial Procrustes Analysis deals with the computation only of the rotation elements. No translation vector nor scale factor are computed (Awange and Grafarend, 2005). Thus, the solution is given by (35) and (36).

Once the rotations matrix is computed, the next step is to transform the matrix elements into the orientation elements: astronomic latitude (Φ) astronomic longitude (Λ) and the orientation unknown (Σ), as has been stated in section 3, Euler angles are obtained by the parameterization of the rotation matrix as follows:

$$\mathbf{R}(\Lambda, \Phi, \Sigma) = \mathbf{R}(\Sigma) \cdot \mathbf{R}\left(\frac{\pi}{2} - \Phi\right) \cdot \mathbf{R}(\Lambda) \quad (42)$$

Taking into account the general rotations matrix for each angle, substituting the orientation angles and multiplying in the order (42), we get:

$$\mathbf{R} = \mathbf{R}(\Sigma) \mathbf{R}\left(\frac{\pi}{2} - \Phi\right) \mathbf{R}(\Lambda) = \begin{pmatrix} \cos(\Sigma) \sin(\Phi) \cos(\Lambda) - \cos(\Sigma) \sin(\Phi) \sin(\Lambda) + & -\cos(\Sigma) \cos(\Phi) \\ -\sin(\Sigma) \sin(\Lambda) & +\sin(\Sigma) \cos(\Lambda) \\ -\sin(\Sigma) \sin(\Phi) \cos(\Lambda) - & -\sin(\Sigma) \sin(\Phi) \sin(\Lambda) + \\ -\cos(\Sigma) \sin(\Lambda) & +\cos(\Sigma) \cos(\Lambda) \\ \cos(\Phi) \cos(\Lambda) & \cos(\Phi) \sin(\Lambda) & \sin(\Phi) \end{pmatrix} \quad (43)$$

4. Case Study

The data sets (courtesy of the Gipuzkoako Foru Aldundia: b5m.gipuzkoa.net) correspond to the GPS network of Gipuzkoa, which is formed by a total of 21 stations which coordinates are known in the ITRF00 frame and, thus, in the ETRS89 Reference System by applying the Altamimi transformation parameters (Altamimi et al. 2002).

To verify the capability of the WEOP to detect deformations different movements have been simulated in 6 stations. The displacements vary from 1 to 6 cm in the X and in the Z components. The movements simulated have been: ITURREGI (1 cm), MENDIZORROTZ (2 cm), ONDDY (3 cm), BIANDITZ (4 cm), ZUBELTZU (5 cm) and SAN TELMO (6 cm). The analysis of the residuals should tell us if the WEOP is or not a valid model to detect deformations. Real data has been used.

The residuals will be compared with those obtained from the classical Helmert datum transformation.

Both systems are global. The network (Figure 3) was stabilised in 2002 and it covers completely the Territory of Gipuzkoa, which is located in the Basque Country (Northern Spain) and its surface is about 2000 km². The network consists in 21 station which belong to the “Active and Passive GPS Network of Gipuzkoa” and it was presented in the

“4^a Asamblea Hispanolusa de Geodesia y Geofísica”, February 2004, Figueira da Foz (Portugal) (Zurutuza et al., 2004). Each station has been occupied at least twice in separate sessions of not less than 8 hours of time span. A total of 7 dual frequency GPS receptors were used in each session. The 15 sessions were computed taking into account only the non trivial vectors. After the adjustment of the whole campaign, the global accuracy was estimated to be better than 2 cm with respect the active station. The coordinates of the Active Station were also computed and the accuracy estimated was better than 1,5 cm in the ITRF00 frame. Once the Reference Station was computed, the rest of the stations were adjusted constraining strongly the reference one.

The main processing parameters were:

- a. Elevation Mask: 10°.
- b. Precise IGS ephemeris.
- c. NO ocean loading model..
- d. Tropospheric model: Niell model (Niell, 1996)
- e. Ambiguities: QIF strategy (Hugentobler et al., 2001).
- f. Processing frequency: L₃.
- g. Software package: Bernese V 4.2.

The first part of the problem shows the differences between the classical and the WEOP transformations, while the second part deals with the deformation detection.

In table 1 the Cartesian coordinates of both sets are presented. Table 2 shows the computed coordinates after the transformation

Fig 3: Test network: Active and Passive GPS network of Gipuzkoa.

Station.	ITRF00 Coordinates (m)			ETRS89 Coordinates (m)		
	X	Y	Z	X	Y	Z
AITZORROTZ	4667813.0044	-207091.2377	4328176.0110	4667813.1903	-207091.4595	4328175.8077
AMETZMENDI	4648581.5707	-176745.6283	4349215.1022	4648581.7589	-176745.8491	4349214.8997
ANUGANA	4656972.4871	-181645.8861	4340292.5540	4656972.6747	-181646.1073	4340292.3512
ARRATE	4652438.4412	-199058.1997	4344726.7660	4652438.6280	-199058.4207	4344726.5633
ARTXUETA	4673825.5758	-160365.1493	4324588.3343	4673825.7640	-160365.3714	4324588.1311
BELAR	4668121.4190	-197465.2942	4328524.5982	4668121.6054	-197465.5160	4328524.3950
BIANDITZ	4651583.8997	-145430.0779	4348133.1014	4651584.0495	-145430.2989	4348132.8590
GARAGARTZA	4664840.1331	-169153.4995	4332432.9138	4664840.3211	-169153.7211	4332432.7108
GFA	4644833.0652	-160817.6784	4353575.0764	4644833.2544	-160817.8991	4353574.8741
INTXORTA TXIKI	4658356.6606	-202888.2927	4338445.1462	4658356.8470	-202888.5140	4338444.9433
ITURREGI	4660182.6604	-164249.4372	4338251.9550	4660182.8389	-164249.6586	4338251.7422
ITXUMENDI	4658830.9201	-193130.1434	4338396.6208	4658831.1070	-193130.3647	4338396.4179
LARREZKONDO	4672777.6644	-182116.7758	4323931.8770	4672777.8515	-182116.9978	4323931.6737
MENDIZORROTZ	4646636.3737	-167697.9967	4351962.2560	4646636.5425	-167698.2175	4351962.0336
OITZ	4650697.7378	-210481.1412	4346696.1158	4650697.9241	-210481.3621	4346695.9131
ONDDY	4652015.6655	-158468.8661	4346786.2290	4652015.8246	-158469.0871	4346785.9965
PAGOETAGANA	4646548.0436	-185892.9832	4351053.8710	4646548.2314	-185893.2039	4351053.6686
SAN TELMO	4640529.6294	-145676.2974	4358781.6667	4640529.7596	-145676.5179	4358781.4047
URBINA	4671792.8232	-213717.2423	4323528.7964	4671793.0086	-213717.4642	4323528.5930
USURBE	4664072.7258	-179486.9185	4333355.0737	4664072.9133	-179487.1401	4333354.8707
ZUBELTZU	4645449.6443	-146437.3727	4353693.2581	4645449.7842	-146437.5934	4353693.0059

Table 1: Test network: Active and Passive GPS network of Gipuzkoa.

Station	Error (cm):	WEOP detected error (m)			Helmert detected error (m)		
		X	Y	Z	X	Y	Z
AITZORROTZ		-0.0108	0.0003	-0.0104	-0.0107	0.0003	-0.0105
AMETZMENDI		0.0153	-0.0001	0.0156	0.0153	-0.0001	0.0156
ANUGANA		0.0082	0.0001	0.0084	0.0082	0.0001	0.0084
ARRATE		0.0019	-0.0002	0.0022	0.0018	-0.0002	0.0022
ARTXUETA		0.0107	0.0002	0.0097	0.0107	0.0001	0.0097
BELAR		-0.0058	0.0002	-0.0058	-0.0057	0.0002	-0.0058
BIANDITZ	4	-0.0097	-0.0003	-0.0103	-0.0097	-0.0003	-0.0103
GARAGARTZA		0.0104	0.0004	0.0103	0.0104	0.0005	0.0103
GFA		0.0254	0.0000	0.0257	0.0254	0.0000	0.0257
INTXORTA TXIKI		-0.0033	-0.0001	-0.0031	-0.0033	-0.0001	-0.0031
ITURREGI	1	0.0058	0.0000	0.0054	0.0058	0.0000	0.0054
ITXUMENDI		0.0016	-0.0001	0.0016	0.0015	-0.0001	0.0016
LARREZKONDO		-0.0005	0.0005	-0.0007	-0.0005	0.0005	-0.0007
MENDIZORROTZ	2	0.0012	-0.0002	0.0013	0.0012	-0.0003	0.0013
OITZ		-0.0028	-0.0006	-0.0026	-0.0028	-0.0006	-0.0026
ONDDY	3	-0.0069	-0.0002	-0.0071	-0.0069	-0.0001	-0.0071
PAGOETAGANA		0.0116	-0.0002	0.0121	0.0117	-0.0002	0.0121
SAN TELMO	6	-0.0243	-0.0001	-0.0243	-0.0243	-0.0002	-0.0242
URBINA		-0.0165	0.0005	-0.0161	-0.0164	0.0005	-0.0161
USURBE		0.0057	0.0002	0.0055	0.0057	0.0001	0.0055
ZUBELTZU	5	-0.0173	-0.0001	-0.0174	-0.0173	-0.0001	-0.0174

Table 2: Residuals: simulated values – displaced values.

As can be seen in tables 1 and 2, the results are the same to the mm level. This is very important because it means that the WEOP solution is as good

as the classical datum transformation. This paper deals with the datum transformation problem though the global to global solution has been tested

it is also a solution for the local to global or local to local datum transformation.

4.1. Results

A very important part of the results has already been shown in table 2. As well as the residuals, it is also interesting to check the parameters obtained:

▪ Helmert:

Variance: 0.000101895

Tx	-2.2890 m	0.8479
Ty	4.5539 m	0.7480
Tz	2.4625 m	0.8717
ω_x	-5.08316E-07	1.01058E-07
ω_y	-5.7154E-07	1.7058E-07
ω_z	5.50716E-07	1.04616E-07
k.	2.23824E-08	8.57186E-08

▪ WEOP:

Matrix Norm: 0.075588

Rotation matrix:

0.99999999999969	-0.00000055053971	-0.00000057193875
0.00000055054000	0.99999999999972	0.00000050783095
0.00000057193847	-0.00000050783126	0.99999999999971

k: 1.00000002191551

Tx: -2.2886 m

Ty: 4.55087 m

Tz: 2.46627 m

5. Conclusions

A different method, using real data, for the classical datum transformation has been used (Grafarend and Awange, 2003).

The main advantages are:

- non iterative method, no "a priori" results must be known and it is a straight forward computational method.
- Method capable of integrating the local and the global reference systems by computing the vertical deflections.
- No need of linearization and easy to compute.
- Promising results obtainable in the ITRF determination and densification.

From the authors point of view, the method worth deeper investigation, as well as the inclusion of the stochastic model, which has been already proposed (Awange and Grafarend, 2005)

Helmert vs Procrustes: difficulties comparing rotation results due to the small values obtained in global-to-global transformation.

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References

- Altamimi, Z., P. Sillard and C. Boucher, 2002: ITRF2000: A new release of the International Terrestrial Reference Frame for Earth science applications. *J. Geophys. Res.* 107, No B10, 2214, 10.1029/2001JB000056.
- Awange, J. L., 2003: Partial Procrustes solution of the three dimensional orientation problem from LPS/GPS observations. *Geodesy – the Challenge of the 3rd Millenium*. Springer, Heidelberg.
- Awange, J. L. and E. W. Grafarend, 2005. *Solving Algebraic Computational Problems in Geodesy and Geoinformatics*. Springer.
- Crosilla, F (1983): Procrustes transformation as a tool for the construction of a criterion matrix for control networks. *Manuscripta Geodaetica* 8, pp. 343-370.
- Grafarend, E. W. and J. L. Awange (2003): Nonlinear analysis of the three-dimensional datum transformation [conformal group $\mathbb{C}_7(3)$]. *Journal of Geodesy*, 77, pp. 66-76
- Hugentobler, U., S. Schaer & P. Fridez (Eds.), 2001: *Bernese GPS Software Version 4.2*. Astronomical Institute. University of Berne.
- Krakiwsky, E. J and Thompson, D. B., 1974: Mathematical models of combination of terrestrial and Satellite Networks. *The Canadian Surveyor* 28 (1974) 606-615.
- Niell, A.E., 1996: Global mapping functions for the atmosphere delay at radio wavelengths. *Journal of Geophysical Research*, Vol. 101, No. B2, pp 3227-3246.
- Sanso, F. (1973): An exact solution of the rototranslation problem. *Photogrammetria* n 29, pp. 203-216, Elsevier.
- Schönemann, P. H., 1996: Generalized solution of the Orthogonal Procrustes Problem, *Psychometrika* 31 No.1 pp. 1-10.
- Zurutuza, J., García, L. and Sevilla, M. J., 2004. Red de estaciones GPS y estación permanente en Gipuzkoa. 4ª Asamblea hispanolusa de Geodesia y Geofísica, Figueira da Foz, Portugal.