A formal comparison between Marych-Moritz's series, Sansò's change of boundary method and a variational approach for solving some linear geodetic boundary value problems

J.Otero

Instituto de Astronomía y Geodesia (UCM-CSIC), Facultad de Matemáticas, Universidad Complutense, Plaza de Ciencias 3, Ciudad Universitaria, 28040 Madrid, Spain

A.Auz

GMV, Isaac Newton 11, P.T.M. Tres Cantos, 28760 Madrid, Spain

Abstract. In this paper we are concerned with the simple Molodensky problem and the linearized fixed-boundary gravimetric boundary-value problem in spherical approximation. We find a series solution for these problems from a variational approach using the Molodensky shrinking. These series are compared with the solution by analytical continuation and the change of boundary method.

1 Introduction

Let S_1 be the unit sphere in $\mathbb{R}^3$. If $h \in C^0(S_1)$ and $h \geq 0$, let Σ be the image of S_1 under the map $\phi_h : S_1 \mapsto \mathbb{R}^3$,

$$\phi_h(\sigma) = (R + h(\sigma))\sigma,$$

where $\sigma \in S_1$ and R is a positive constant.

Let f be a given function defined on the closed surface Σ . We are interested in the boundary-value problems

$$P_k(\Sigma, f) : \begin{cases} \Delta u = 0 & \text{in } \Omega, \\ -\frac{\partial u}{\partial r} - \frac{k}{r}u = f & \text{on } \Sigma, \\ u(\infty) = 0, \end{cases}$$

where Ω denotes the domain exterior to Σ and $k \in \{0, 2\}$. If S_R is the sphere of radius R then $\Sigma = S_R$ if, and only if, $h = 0$. $P_2(S_R, f)$ is Stokes' problem and $P_0(S_R, f)$ is Hotine's problem. The function f can be extended throughout the whole space $\mathbb{R}^3$ (with exception of the origin and infinity) so that it remains constant on rays passing through the origin. Below, by f we denote this extension.

At present, $P_2(\Sigma, f)$ is called in geodesy *the simple Molodensky problem* or simply (as some years ago) Molodensky's problem. $P_0(\Sigma, f)$ is the linearized fixed-boundary gravimetric boundary-value problem (or GPS-gravimetric boundary problem as recently

called by Moritz (2000)) in spherical approximation. The meaning of Σ and of the functions taking part in $P_k(\Sigma, f)$ is the following: R is a mean radius of the earth, u is the disturbing potential,

- if $k = 2$, Σ is the telluroid and f is the gravity anomaly Δg at ground level,
- if $k = 0$, Σ is the earth's surface and f is the gravity disturbance δg ,

and $h(\sigma)$ is the height of $\omega \in \Sigma$ above the sphere S_R where $\sigma = \omega/|\omega|$.

The problem $P_0(\Sigma, f)$ is particularly significant nowadays. Indeed, its solution gives the gravity potential on the earth surface and therefore geopotential numbers (the physical measure of height above sea level), as has been pointed out by Moritz (2000, p.85). By means of approximations such as, for example, the reduction of Poincaré and Prey (see Heiskanen and Moritz, 1967, § 4.4), from these geopotential numbers we can get orthometric heights, and hence the fixed-boundary gravimetric boundary-value problem could be a good alternative to the so called GPS-levelling which needs a very precise determination of the geoid.

From a mathematical point of view the problems $P_0(\Sigma, f)$ and $P_2(\Sigma, f)$ have a different treatment. Whereas $P_0(\Sigma, f)$ is unconditional solvable, the problem $P_2(\Sigma, f)$ may have no solution and if a solution exists it is not unique.

M.S.Molodensky (see Molodensky *et al.*, 1962, Chapter V, §15) solved $P_2(\Sigma, f)$ by means of Fredholm's integral method: he sought to find the solution of $P_2(\Sigma, f)$ in the form of a single layer potential and this leads to the solution of the following integral equation (Molodensky *et al.*, 1962, Eq. V.15.9)

$$2\pi\chi \cos^2 \alpha = (1 + \varepsilon)^2 f + \frac{R}{2}(1 + \varepsilon) \int_{S_1} M(\sigma, \sigma') \chi(\sigma') d\sigma', \quad (1)$$

where χ is an auxiliary unknown function defined on S_1 and $\varepsilon = h/R$. The kernel M is given by

$$M(\sigma, \sigma') = 3L_h^{-1}(\sigma, \sigma') + L_h^{-3}(\sigma, \sigma')(r^2(\sigma') - r^2(\sigma)),$$

where

$$L_h(\sigma, \sigma') = |\phi_h(\sigma) - \phi_h(\sigma')|$$

and $r(\sigma) = |\phi_h(\sigma)| = R + h(\sigma)$. The terrain inclination $\alpha(\sigma)$ at $\phi_h(\sigma)$ is the angle between the outer normal to Σ and the radius vector at $\phi_h(\sigma)$, and it is given by

$$\tan \alpha = \frac{1}{r} |\nabla_s h| = \frac{1}{R} (1 + \varepsilon)^{-1} |\nabla_s h|,$$

where $\nabla_s h$ is the gradient of $h(\sigma)$. In terms of χ , the function u at $\phi_h(\sigma)$ is assumed to be

$$u(\phi_h(\sigma)) = R^2 \int_{S_1} L_h^{-1}(\sigma, \sigma') \chi(\sigma') d\sigma'.$$

For solving the integral equation (1), Molodensky replaces Σ by $\Sigma_t := \phi_{th}(S_1)$ with $t \in [0, 1]$ (*the Molodensky shrinking*). Then, expanding χ , α and the kernel $M(\sigma, \sigma')$ of the integral equation in series of powers of t , he obtains (in planar approximation ($\varepsilon = 0$))

$$u_t \circ \phi_{th} = \sum_{n=0}^{\infty} u_n t^n, \quad (2)$$

where $u_t(x)$ is the solution of $P_2(\Sigma_t, f)$ and u_n are functions defined on the unit sphere S_1 . From (2), setting $t = 1$ we get formally $u \circ \phi_h$.

The function u_0 in (2) is the restriction to S_R of the solution of $P_2(S_R, f)$ given by Stokes' formula

$$u_0 = \frac{R}{4\pi} \int_{S_1} G_0 [S(\psi) - 1] d\sigma,$$

where $S(\psi)$ is the Stokes function and $G_0 = f$. The other functions are also given in terms of Stokes' function and, in words of Molodensky, Ereemeev and Yurkina (Molodensky *et al.*, 1962, p. 123) *the topography is taken into account in the second equation (function) and the gradients appear clearly from the second equation (function) onwards*. For instance, the function u_1 is given by

$$u_1 = \frac{R}{4\pi} \int_{S_1} G_1 [S(\psi) - 1] d\sigma,$$

where

$$G_1(\sigma) = R^2 \int_{S_1} \frac{(h(\sigma') - h(\sigma))}{L_0^3(\sigma, \sigma')} \chi_0(\sigma') d\sigma',$$

and

$$\chi_0 = \frac{G_0}{2\pi} + \frac{3}{(4\pi)^2} \int_{S_1} G_0 [S(\psi) - 1] d\sigma.$$

Some refinements to this solution of Molodensky were given later by Brovar (see Brovar, 1964; Moritz, 1980). For the problem $P_0(\Sigma, f)$ a Molodensky-type solution has been derived by Stock (1983).

To our way of thinking, Molodensky's method has a clear connection with the *Poincaré's method of the small parameter* for solving a regularly perturbed Cauchy problem for ordinary differential equations (see, for example, Kuzmina, 2000, § 1 and § 9):

$$\frac{dx}{dt} = F(x, t, \varepsilon); \quad x(0, \varepsilon) = x^0(\varepsilon). \quad (3)$$

In fact, in Poincaré's method, the solution $x(t, \varepsilon)$ is expanded in powers of ε ($x(t, \varepsilon) = \sum_{k=0}^{\infty} x^k(t) \varepsilon^k$). Then $x(t, \varepsilon) = \sum_{k=0}^{\infty} x^k(t) \varepsilon^k$ is substituted in the equation (3) and both sides of (3) are expanded in powers of ε . Equating the coefficients in terms with equal powers of ε , we finally have the *linear* equations for the coefficients $x^k(t)$.

In 1969, Marych and Moritz independently found *an elementary solution by analytical continuation in the form of a Molodensky series*, using Moritz's own words (see Moritz (2000), where a brief history of the use of analytical continuation can also be found). These Marych-Moritz's series can be derived without needing integral equations and they are nowadays the basis for numerical computations (see, for example, Sünkel (1997)). More details about all these series solutions, their equivalence and the convergence point can be found in (Moritz, 1980). For the linearized fixed-boundary gravimetric boundary-value problem a similar solution has recently be derived by Moritz (2000).

In 1993, Sansò (1993) proposed an iterative method for solving a certain class of linear boundary value problems for Laplace's equation. This method is known as *the change of boundary method* and applied to solve $P_k(\Sigma, f)$ it leads to another series solution. This approach will be outlined in Section 3.

From the basic idea of Molodensky of replacing Σ by Σ_t , in Section 2 we propose an alternative variational approach to solve $P_k(\Sigma, f)$. We show that the variational approach solution and the solution by analytical continuation are *formally* the same. The

essential difference is that no use of downward analytical continuation is made in the variational approach. Rather, upward harmonic continuation from S_R is what is needed to explain the terms of the derived series. In this way the Marych-Moritz's solution could be now correctly interpreted. In addition, this upward harmonic continuation allows that one can guess some linking between the change of boundary method and the variational approach, and this relation is also studied in this paper.

We have previously spent some time in describing the method of solution of $P_2(\Sigma, f)$ due to Molodensky since the idea of the variational approach is basically the same: using the Molodensky shrinking, to approximate the solution of $P_k(\Sigma, f)$ by solving boundary problems in the domain Ω_R exterior to S_R (integral equations in Molodensky's method) for which there exist explicit formulae for its solutions (in the case $k = 2$, Stokes' problem with **changing boundary values**). For this approximation to be effective we need that $\Omega \subset \Omega_R$, and for this reason $h \geq 0$ as we are assuming. The variational approach is more direct in the sense that the intermediate step of expressing u as a simple layer potential is avoided. For Dirichlet's problem on nearly circular domains (in the plane), Kautský (1962) follows a method similar to the variational approach described in this paper (see his Theorem 3.1 and compare (Kautský, 1962, formula (9)) with the formula (8) of this paper which gives the boundary values of the approximations of the solution of $P_k(\Sigma, f)$).

The origin of this paper is twofold: firstly, to understand the Marych-Moritz's solution by analytical continuation for Molodensky's problem; and in second place, to look for the connection between the series solutions obtained in the past for this problem and the solution given by the change of boundary method. Since this is the main motivation of this paper we have intentionally not considered solvability conditions for $P_2(\Sigma, f)$. Furthermore, neither regularity of solutions nor convergence problems have been borne in mind. The question of the convergence of the different series solutions presented here is still open.

2 A variational approach for solving $P_k(\Sigma, f)$

Another way of solving $P_k(\Sigma, f)$ is the following variational approach that we formally unfold in this section. We define the function $[0, 1] \times \bar{\Omega} \mapsto \mathbb{R}$, $u(t, x) := u_t(x)$ where u_t is the solution of $P_k(\Sigma_t, f)$. For any function $g(t, x)$ we shall use the

notation

$$g_n(t, x) = \frac{1}{n!} \frac{\partial^n g}{\partial t^n}(t, x),$$

and $g_n(x) := g_n(0, x)$. If $u_t(x)$ is smooth and depends smoothly on t , we may expand $u(t, x)$ in series of powers of t at $t = 0$ and we get

$$u(t, x) = \sum_{n=0}^{\infty} u_n(x) t^n \quad (x \in \bar{\Omega}).$$

Since $u(1, x)$ is the solution $u(x)$ of $P_k(\Sigma, f)$, then

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \quad (x \in \bar{\Omega}).$$

Note that

$$\Delta(u_n(t, x)) = (\Delta u(t, x))_n,$$

where Δ is the Laplace operator in $\mathbb{R}^3$ given by $\sum_{i=1}^3 \partial^2 / \partial x_i^2$. Hence the functions $u_n(t, x)$ are harmonic in the domain Ω_t exterior to Σ_t and regular at infinity. In particular, the functions $u_n(x)$ are harmonic outside S_R . Note in addition that $u_0(x)$ is the solution of $P_k(S_R, f)$.

In order to obtain a boundary condition for $u_n(x)$, we consider the function $U = r \partial u / \partial r + k u$ which is harmonic in Ω_t , and in Ω_t satisfies

$$U_n(t, x) = r \frac{\partial u_n(t, x)}{\partial r} + k u_n(t, x).$$

Let $F = -r^{-1}U$; then, in points of Ω_t we have

$$-\frac{\partial u_n(t, x)}{\partial r} - \frac{k}{r} u_n(t, x) = F_n(t, x). \quad (4)$$

If $F_n(t, \cdot) \in C^0(\bar{\Omega}_t)$, the functions $u_n(t, x)$ are then solutions of the problems $P_k(\Sigma_t, F_n(t, x))$. Hence, if $t = 0$, we conclude that $u_n(x)$ are solutions of $P_k(S_R, F_n(x))$.

To compute $F_n(x)$ on S_R we note that on Σ_t we have $F = f$, that is to say

$$F(t, (R + th(\sigma))\sigma) = f(\sigma). \quad (5)$$

Since f does not depend on t , differentiation of (5) with respect to t gives on Σ_t ,

$$\frac{\partial F}{\partial t} = -h \frac{\partial F}{\partial r}. \quad (6)$$

Repeated differentiation of (6) with respect to t gives on Σ_t the following recursive formula ($n \geq 1$)

$$\begin{aligned} \frac{\partial^n F}{\partial t^n} &= -h^n \frac{\partial^n F}{\partial r^n} \\ &- \sum_{i=1}^{n-1} \binom{n}{i} h^{n-i} \frac{\partial^{n-i}}{\partial r^{n-i}} \left(\frac{\partial^i F}{\partial t^i} \right). \end{aligned} \quad (7)$$

Since

$$\frac{\partial^j}{\partial r^j} \left(\frac{\partial^i F}{\partial t^i} \right) (0, x) = \frac{\partial^j}{\partial r^j} \left(\frac{\partial^i F}{\partial t^i} (0, x) \right),$$

then from (7) we get on S_R ($t = 0$)

$$F_n = - \sum_{i=1}^n \frac{1}{i!} h^i \frac{\partial^i F_{n-i}}{\partial r^i}, \quad (8)$$

where, from (4), if $j = 0, \dots, n-1$

$$F_j = - \frac{\partial u_j}{\partial r} - \frac{k}{r} u_j.$$

The functions F_j are defined in $\bar{\Omega}_R$ and rF_j are harmonic in Ω_R .

Remark 2.1 Hitherto we have avoided questions of regularity. For instance, to compute u_1 the existence of the radial derivative $\partial F_0 / \partial r$ on S_R is required and this depends on the regularity of f . $\square$

Remark 2.2 As it has been pointed out by Sansò (1993, Remark A.1) in the context of the change of boundary method (see Section) *the only convergence we can expect from the above approach is at best on Σ , but not certainly in the layer between S_R and Σ ; in fact, such a convergence can happen only if the sought solution u has in fact a backward harmonic continuation and this can happen only under very restrictive conditions*, using his own words. $\square$

In order to compare the solution that we have obtained with the solution by analytical continuation we introduce the operators

$$L_n : C^\infty(\bar{\Omega}_R) \mapsto C^\infty(S_R),$$

defined by

$$L_n(g) = \frac{1}{n!} \gamma \left(\frac{\partial^n g}{\partial r^n} \right),$$

where $\gamma(v)$ denotes the trace (restriction) of $v \in C^\infty(\bar{\Omega}_R)$ on S_R . Then we may write (8) in the form ($n \geq 1$)

$$F_n = - \sum_{i=1}^n h^i L_i(F_{n-i}). \quad (9)$$

This expression, if compared with the formula given by Moritz (1980, § 45: *Solution by analytical continuation*, Eq. 45-44), shows the *formal* equivalence between the variational approach and the solution by analytical continuation. However, we stress that the equation (9) is meaningful (if f is smooth) and it has

been obtained without downward analytical continuation of $f(\omega)$ ($\omega \in \Sigma$).

Instead of (9) we write ($n \geq 1$)

$$F_n(\sigma) = - \sum_{i=1}^n \varepsilon(\sigma)^i H_{i,n-i}(R\sigma)$$

where

$$H_{i,j} = \frac{1}{i!} r^i \frac{\partial^i F_j}{\partial r^i},$$

and $\varepsilon = h/R$.

The functions rF_j are harmonic in Ω_R . Consequently the functions $rH_{i,j}$ are as well harmonic in Ω_R for all $i \geq 1$. In fact, we have

$$H_{i,j} = \frac{1}{i} \frac{\partial}{\partial r} (rH_{i-1,j}) - H_{i-1,j}. \quad (10)$$

Since $rH_{0,j} = rF_j$ is harmonic, multiplying both members in (10) by r , this relation shows recursively that $rH_{i,j}$ are harmonic for all $i \geq 1$.

We now define

$$\hat{H}_{i,j}(x) = |x| H_{i,j}(Rx).$$

The function $\hat{H}_{i,j}$ is harmonic outside S_1 and on S_1 we have $\hat{H}_{i,j}(\sigma) = H_{i,j}(R\sigma)$. In addition

$$\begin{aligned} \frac{\partial \hat{H}_{i,j}}{\partial r}(\sigma) &= H_{i,j}(R\sigma) + R \frac{\partial H_{i,j}}{\partial r}(R\sigma) \\ &= \frac{\partial}{\partial r} (rH_{i,j})(R\sigma). \end{aligned}$$

From (10) we conclude that

$$H_{i,j}(R\sigma) = \frac{1}{i} \frac{\partial \hat{H}_{i-1,j}}{\partial r}(\sigma) - \hat{H}_{i-1,j}(\sigma). \quad (11)$$

Equation (11) has important consequences. It first shows that the function $H_{i,j}(R\sigma)$ ($\sigma \in S_1$) is the trace on S_1 of the harmonic function

$$\frac{1}{i} r \frac{\partial \hat{H}_{i-1,j}}{\partial r} - \hat{H}_{i-1,j}.$$

Since one can define the **trace** on S_1 of an **arbitrary** harmonic function in Ω_1 and regular at infinity (see Dautray and Lions, 1990, Chapter 2, § 6.2) (and therefore also the trace of its radial derivative), equation (11) could be the key to give a solid mathematical foundation to the variational approach that we have followed for solving $P_k(\Sigma, f)$.

From a computational point of view, we recall that if u is harmonic outside S_1 and regular at infinity, its radial derivative on S_1 is given by the well-known

formula (see, for example, Heiskanen and Moritz, 1967, Eq. 1-97)

$$\frac{\partial u}{\partial r}(\sigma) = -u(\sigma) + \frac{1}{2\pi} \int_S \frac{1}{|\sigma' - \sigma|^3} (u(\sigma') - u(\sigma)) d\sigma'.$$

Hence we have

$$H_{i,j}(R\sigma) = -\left(\frac{1+i}{i}\right) H_{i-1,j}(R\sigma) + \frac{1}{2i\pi} \int_S \frac{(H_{i-1,j}(R\sigma') - H_{i-1,j}(R\sigma))}{|\sigma' - \sigma|^3} d\sigma'. \quad (12)$$

Formula (12) allows a recursive computation of the functions $F_n(\sigma)$.

3 Sansò's change of boundary method

With the notation used in (Sansò, 1993), the series solution of $P_k(\Sigma, f)$ that we get by means of the change of boundary method is the following

$$u = u_0 + \sum_{n=1}^{\infty} \delta u_n \quad (\text{in } \bar{\Omega}),$$

where u_0 is the solution of $P_k(S_R, f)$. Let B be the operator

$$Bu = -\frac{\partial u}{\partial r} - \frac{k}{r} u.$$

The corrections δu_n are computed as follows: δu_1 is the solution of $P_k(S_R, \Delta_1)$ where

$$\Delta_1(\sigma) = f(\sigma) - Bu_0(\phi_h(\sigma)).$$

Let $v_1 := u_0 + \delta u_1$, then δu_2 is the solution of $P_k(S_R, \Delta_2)$ where

$$\Delta_2(\sigma) = f(\sigma) - Bv_1(\phi_h(\sigma)).$$

We define

$$v_n = \begin{cases} u_0 & \text{if } n = 0, \\ u_0 + \sum_{i=1}^n \delta u_i & \text{if } n \geq 1. \end{cases}$$

In general, δu_n ($n \geq 1$) is the solution of $P_k(S_R, \Delta_n)$ where

$$\Delta_n(\sigma) = f(\sigma) - Bv_{n-1}(\phi_h(\sigma)).$$

In other words, Δ_n is the pullback to S_1 of the function defined on Σ

$$f(\omega) - Bv_{n-1}(\omega) \quad (\omega \in \Sigma).$$

Note that if $n \geq 1$

$$\Delta_{n+1}(\sigma) = \Delta_n(\sigma) - B(\delta u_n)(\phi_h(\sigma)).$$

Hence we may conclude at least formally that on Σ

$$-\frac{\partial u}{\partial r} - \frac{k}{r} u = (f - \Delta_1) + \sum_{n=1}^{\infty} (\Delta_n - \Delta_{n+1}) = f.$$

In order to compare the variational approach with the change of boundary method for solving $P_k(\Sigma, f)$ we observe the following. We have seen that the correction u_n to the partial sum $v_{n-1} := \sum_{i=0}^{n-1} u_i$ in the variational approach is the solution of $P_k(S_R, F_n)$. Let

$$(f - Bv_{n-1})(\omega) = (f - \sum_{j=0}^{n-1} F_j)(\omega) \quad (\omega \in \Sigma).$$

The pullback to S_1 of this function is approximated in the following way. Let $\omega \in \Sigma$ and $\sigma = \omega/|\omega|$, then we approximate each F_j ($j = 0, \dots, n-1$) by the $(n-j)$ th partial sum of its Taylor series expansion with respect to r at $r = R$ ($h = 0$), getting in this way a decreasing order of approximation as j increases. Hence

$$F_0(\omega) \approx f(\omega) + \sum_{i=1}^n \frac{1}{i!} h^i(\sigma) \frac{\partial^i F_0}{\partial r^i}(R\sigma)$$

$$F_1(\omega) \approx F_1(R\sigma) + \sum_{i=1}^{n-1} \frac{1}{i!} h^i(\sigma) \frac{\partial^i F_1}{\partial r^i}(R\sigma)$$

⋮

$$F_{n-1}(\omega) \approx F_{n-1}(R\sigma) + h(\sigma) \frac{\partial F_{n-1}}{\partial r}(R\sigma).$$

Grouping terms we have

$$\begin{aligned} \sum_{j=0}^{n-1} F_j(\omega) &\approx f(\omega) + \underbrace{\sum_{j=1}^n \frac{1}{j!} h^j(\sigma) \frac{\partial^j F_{n-j}}{\partial r^j}(R\sigma)}_{= -F_n} \\ &+ \underbrace{\sum_{j=1}^{n-1} \left(F_j(R\sigma) + \sum_{i=1}^j \frac{1}{i!} h^i(\sigma) \frac{\partial^i F_{j-i}}{\partial r^i}(R\sigma) \right)}_{=0}. \end{aligned}$$

From equation (8) we finally find

$$\sum_{j=0}^{n-1} F_j(\omega) \approx f(\omega) - F_n(\sigma).$$

Thus we conclude that the pullback of $(f - Bv_{n-1})(\omega)$ to S_1 is approximately given by $F_n(\sigma)$.

4 Gradient solution

Following (Moritz, 1980), we call *gradient solution* of $P_k(\Sigma, f)$ to the function $v_1 := u_0 + u_1$ harmonic in Ω_R . This function can be seen as a first-order approximation to the solution of $P_k(\Sigma, f)$. The function u_1 is the solution of $P_k(S_R, F_1)$, where

$$F_1(\sigma) = -h(\sigma) \frac{\partial F_0}{\partial r}(R\sigma),$$

and

$$F_0 = -\frac{\partial u_0}{\partial r} - \frac{k}{r} u_0.$$

Therefore v_1 solves $P_k(S_R, f + F_1)$ and is given by

$$v_1 = \frac{R}{4\pi} \int_{S_1} \left(f - h \frac{\partial F_0}{\partial r} \right) N_k(r, \psi) d\sigma,$$

where the kernel $N_k(r, \psi)$ is the spatial Stokes function if $k = 2$ or the spatial Hotine function if $k = 0$ (for these functions see Hotine (1969, Eqs. 29.14, 29.17) or Grafarend *et al.* (1985)). Note that the function $(r/R)F_0$ is harmonic in Ω_R and f is its value on S_R (upward harmonic continuation). Defining momentarily $f(r\sigma) = F_0(r\sigma)$ ($r \geq R$) the gradient solution can be written in the form

$$v_1 = \frac{R}{4\pi} \int_{S_1} \left(f - h \frac{\partial f}{\partial r} \right) N_k(r, \psi) d\sigma.$$

In the geodetic case, if $k = 2$ then $f = \Delta g$ and we have

$$v_1 = \frac{R}{4\pi} \int_{S_1} \left(\Delta g - h \frac{\partial \Delta g}{\partial r} \right) S(r, \psi) d\sigma. \quad (13)$$

In this way we have obtained the Moritz gradient solution for Molodensky's problem (Moritz, 1980, Eq. (45-50)) but in an essentially different manner. In fact, in (13) the spatial function $(r/R)\Delta g(r\sigma)$ is the upward harmonic continuation of the surface function $\Delta g(R\sigma)$, whereas Moritz's solution is based in downward analytical continuation of $\Delta g(\omega)$ ($\omega \in \Sigma$). It should be remarked that in (13), $\Delta g(\sigma) = \Delta g(\phi_h(\sigma))$, that is to say $\Delta g(\sigma)$ is the pullback of Δg to the sphere S_1 .

Remark 4.1 Using a different method (based on a change of coordinates transforming the telluroid into an sphere), P.Holota (see Holota (1985) and Holota (1989, p.502)) also found an identical interpretation for the gradient solution of Molodensky's problem without resorting to downward analytical continuation. $\square$

Remark 4.2 Replacing in the equation (13) Δg by δg and $S(r, \psi)$ by $H(r, \psi)$, we have a similar formula for the gradient solution of $P_0(\Sigma, f)$ (see Auz and Otero, 2002). The problem $P_0(\Sigma, f)$ can also be solved by Molodensky's method (Stock, 1983, § 3). See in addition some of the pioneering works on the geodetic boundary value problem using the known surface of the earth, for instance (Koch and Pope, 1972). $\square$

Summing up, the following is the interpretation of the gradient solution as it is obtained by the variational approach.

1. As a zero-order approximation of u we take the solution u_0 of $P_k(S_R, f)$.
2. Now we consider the difference $f(\omega) - F_0(\omega)$ ($\omega \in \Sigma$) where

$$F_0 = -\frac{\partial u_0}{\partial r} - \frac{k}{r} u_0.$$

(If u_0 would solve $P_k(\Sigma, f)$ then this difference would be zero.) Since

$$\begin{aligned} F_0(\omega) &\approx F_0(R\sigma) + h(\sigma) \frac{\partial F_0}{\partial r}(R\sigma) \\ &= f(\omega) + h(\sigma) \frac{\partial F_0}{\partial r}(R\sigma), \end{aligned}$$

hence

$$\begin{aligned} f(\omega) - F_0(\omega) &\approx -h(\sigma) \frac{\partial F_0}{\partial r}(R\sigma) \\ &= F_1(R\sigma). \end{aligned}$$

3. Writing $f(\omega) = F_0(\omega) + (f(\omega) - F_0(\omega))$ we then add to u_0 a correction given by the solution of $P_k(S_R, \cdot)$ with boundary data the pullback of $(f - F_0)(\omega)$ ($\omega \in \Sigma$) to S_1 . According to step 2 this pullback is approximately given by F_1 . The correction is then the function u_1 .

Note that in *the change of boundary method* the approximation made in step 2 is avoided. As we have seen in Section the correction δu_1 is the solution of $P_k(S_R, \Delta_1)$, where Δ_1 is the pullback of

$(f - F_0)(\omega)$ ($\omega \in \Sigma$) to the sphere S_R . The advantage of this procedure is that $F_0(\omega)$ always exists since $u_0 \in C^\infty(\Omega_R)$. From a computational point of view the disadvantage may lie in that we have to compute u_0 and $\partial u_0 / \partial r$ on a non-spherical surface like the telluroid ($k = 2$) or the earth ($k = 0$).

Another interpretation is possible (with $k = 2$, for example), this time in accordance with Moritz's line of thought (see Heiskanen and Moritz, 1967, § 8-10). On the basis of the zero-order approximation u_0 to u we define the gravity anomalies outside S_R by means of

$$\Delta g_0(r\sigma) = -\frac{\partial u_0}{\partial r} - \frac{2}{r}u_0.$$

Let Δg^* be a fictitious field of gravity anomalies on S_R which generate on Σ the measured free-air gravity anomalies Δg (see Heiskanen and Moritz, 1967, p. 317),

$$\Delta g^*(\phi_h(\sigma)) = \Delta g(\sigma)$$

where $\Delta g^* = Bu^*$ and u^* solves $P_2(S_R, \Delta g^*)$. Since

$$\Delta g^*(\phi_h(\sigma)) = \Delta g^*(R\sigma) + h(\sigma) \frac{\partial \Delta g^*}{\partial r}(R\sigma) + \dots,$$

as a first approximation we set

$$\Delta g^*(R\sigma) = \Delta g(\sigma) - h(\sigma) \frac{\partial \Delta g_0}{\partial r}(R\sigma).$$

A first-order approximation to $P_2(\Sigma, \Delta g)$ is then the solution of $P_2\left(S_R, \Delta g - h \frac{\partial \Delta g_0}{\partial r}\right)$. Compare this approach with (Rummel, 1988, § 3).

5 Conclusion

We conclude this paper noting that the variational method that we have followed to solve $P_k(\Sigma, f)$ can also be applied to solve as well other boundary problems such as the *ellipsoidal Stokes boundary-value problem* (see Martinec and Grafarend, 1997) where a small parameter characterizing the difference between the ellipsoid of revolution and the sphere appears in a natural way (for example, the first eccentricity of the ellipsoid).

More complex problems to which one can apply the methods described in this paper are:

- The *linear gravimetric boundary problem* (see Holota, 1997) which generalizes $P_0(\Sigma, f)$. In this problem the boundary condition is

$$\langle \nabla T, s \rangle = -\delta g,$$

where $s = -(1/\gamma)\nabla U$, U is the normal potential, $\gamma = |\nabla U|$ is the normal gravity and T the disturbing potential.

- The *linearized scalar boundary problem* which generalizes $P_2(\Sigma, f)$ (see Sacerdote and Sansò, 1986), where the boundary condition on the telluroid is

$$\langle \nabla T, s \rangle - \frac{1}{\gamma \langle s, e_h \rangle} \frac{\partial \gamma}{\partial h} T = -\Delta g,$$

where e_h is the unit outer vector normal to the reference ellipsoid and $\partial \gamma / \partial h = \langle \nabla \gamma, e_h \rangle$.

Acknowledgements. J.Otero was partially supported by the Ministry of Science and Technology through Project REN2000-0766/CLI. We thank Professor F. Sansò for having made the presentation of this paper.

References

- Auz A, Otero J (2002) Gradient solution of the GPS-gravimetric boundary problem (in Spanish). In: Proc III Spanish-Portuguese Assembly on Geodesy and Geophysics, in print
- Brovar VV (1964) On the solutions of Molodensky's boundary value problem. Bulletin Géodésique 72: 167-173
- Dautrey R, Lions JL (1990) Mathematical analysis and numerical methods for science and technology. Volume 1: Physical origins and classical methods. Springer, Berlin Heidelberg New York
- Grafarend EW, Heck B, Knickmeyer EH (1985) The free versus fixed geodetic boundary value problem for different combinations of geodetic observables. Bulletin Géodésique 59: 11-32
- Heiskanen WA, Moritz H (1967) Physical Geodesy. W.H. Freeman and Co., San Francisco Londres
- Holota P (1985) Boundary value problems of physical geodesy: present state, boundary perturbation and the Green-Stokes representation. In: Proc 1st Hotine-Marussi Symposium on Mathematical Geodesy (Rome), Politecnico di Milano, pp 529-558
- Holota P (1989) Higher order theories in the solution of boundary value problems of physical geodesy by means of successive approximations. In: Proc 2nd Hotine-Marussi Symposium on Mathematical Geodesy (Pisa), Politecnico di Milano, pp 471-505
- Holota P (1997) Coerciveness of the linear gravimetric boundary-value problem. Journal of Geodesy 71: 640-651

- Hotine M (1969) Mathematical geodesy. ESSA Monograph 2, U.S. Department of Commerce, Washington
- Kautský J (1962) Approximations of solutions of Dirichlet's problem on nearly circular domains and their application in numerical methods. *Aplikace Matematiky* 7: 186–200
- Koch KR, Pope AJ (1972) Uniqueness and existence for the geodetic boundary value problem using the known surface of the earth. *Bulletin Géodésique* 106: 467–476
- Kuzmina RP (2000) Asymptotic methods for ordinary differential equations. Kluwer Academic Publishers, Dordrecht
- Martinec Z, Grafarend EW (1997) Solution of the Stokes boundary-value problem on an ellipsoid of revolution. *Studia Geoph. et Geod.* 41: 103–129
- Molodensky MS, Eremeev VF, Yurkina MI (1962) Methods for study of the external gravitational field and figure of the earth. Israel Program for Scientific Translations, Jerusalem
- Moritz H (1980) Advanced physical geodesy. Herbert Wichmann and Abacus Press, Karlsruhe Tunbridge
- Moritz H (2000) Molodensky's theory and GPS. In: Moritz H, Yurkina MI (eds) M.S.Molodensky: In memoriam. *Mitteilungen der geodätischen Institute der Technischen Universität Graz*, Folge 88, Graz, pp 69–85
- Rummel R (1988) Zur iterativen Lösung der Geodätischen Randwertaufgabe. In: *Deutsche Geodätische Kommission Reihe B*, Nr. 287, Munich, pp 175–181
- Sacerdote F, Sansò F (1986) The scalar boundary value problem of physical geodesy. *Manuscripta Geodaetica* 11: 15–28
- Sansò F (1993) Theory of geodetic B.V.P.s applied to the analysis of altimetric data. In: Rummel R, Sansò F (eds) *Satellite altimetry in geodesy and oceanography. Lecture notes in Earth Sciences* 50: 317–371, Springer, Berlin Heidelberg New York
- Stock B (1983) A Molodenskii-type solution of the geodetic boundary value problem using the known surface of the earth. *Manuscripta Geodaetica* 8: 273–288
- Sünkel H (1997) GBVP - Classical solutions and implementation. In: Sansò F, Rummel R (eds) *Geodetic Boundary Value Problems in View of the One Centimeter Geoid. Lecture notes in Earth Sciences* 65: 219–237, Springer, Berlin Heidelberg New York