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# On standard completeness and finite model property for a probabilistic logic on Łukasiewicz events

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ABSTRACT

The probabilistic logic FP(Ł, Ł) was axiomatized with the aim of presenting a formal setting for reasoning about the probability of infinite-valued Łukasiewicz events. Besides several attempts, proving that axiomatic system to be complete with respect to a class of *standard* models, remained an open problem since the first paper on FP(Ł, Ł) was published in 2007. In this article we give a solution to it. In particular we introduce two semantics for that probabilistic system: a first one based on Łukasiewicz states and a second one based on regular Borel measures and we prove that FP(Ł, Ł) is complete with respect to both these classes of models. Further, we will show that the finite model property holds for FP(Ł, Ł).


## 1. Introduction

Probability theory is a well-established branch of mathematics which has found, along the years, several applications in all those areas of pure and applied science that require a quantification of the uncertainty of "unknown statements about the world", i.e., *events*.

Beyond mathematics, the need of *reasoning* about uncertainty paved the way to several attempts aiming at capturing, in a formal way, the axioms and rules for correct deductions in that setting. It is interesting to recall that the perspective which look at probability theory as a branch of both mathematics and logic, surely was one of the groundbreaking ideas that George Boole reported in the introduction of his seminal work [4]:

> *[...] the subject of Probabilities belongs equally to the science of Number and to that of Logic. In recognizing the co-ordinate existence of both these elements, the present treatise differs from all previous ones.*

The best known formal systems for probabilistic reasoning are the logic $AX_{MEAS}$, introduced by Fagin, Halpern and Megiddo in [10] and the logic FP(Ł) defined in [21] by Hájek, Godo and Esteva. In the same papers [10] and [21] these logics have been proved to be sound and complete with respect to measurable probability structures. It is worth pointing out that $AX_{MEAS}$ and a variant of FP(Ł) have been shown to be syntactically interdefinable, and hence equivalent, in the recent article [2].

Nowadays models for uncertain quantification and reasoning do not relegate to probability theory only and they encompass several other possibilities. However, if we take probability theory as point of departure, one can imagine essentially

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two ways to proceed towards further generalizations: the first one is to consider alternative or more general uncertainty measures such as possibility and necessity measures [9], belief and plausibility functions [36], upper and lower probabilities [37]; the second consists in generalizing probability theory from classical to non-classical events [32,38]. Obviously, these two lines of research have no void intersection as one could consider general uncertainty measures on events that do not necessarily pertain to the classical logic world, see [16] for an overview.

Moving from classical to many-valued events presents non-trivial technical complications and poses intriguing philosophical questions [27]. However, there is a quite established conviction within the community of many-valued logicians that the realm of t-norm based fuzzy logics offers a suitable logical and algebraic setting for a reasonable generalization of probability theory on many-valued events. In particular, although generalizations of probability functions (called *states* by that community) have been introduced for Gödel logic [1], product logic [17] and more, the realm of Łukasiewicz logic is surely playing a pivotal role in that area, see [32,34] and [19, Ch. 8] for an overview. The interest in Łukasiewicz events is twofold: on the one hand these events capture properties of the world which are better described as *gradual* rather than yes-or-no; on the other hand, they also mimic bounded random variables. Indeed, any Łukasiewicz event $\psi$ can be regarded as a $[0,1]$-valued continuous function $m_\psi$ on a compact Hausdorff space (see [6, Theorem 9.1.5] and Example 2.5 below) and any state on $\psi$ coincides with the expected value of $m_\psi$, [25,35], and [18, Remark 2.8].

A generalization of Hájek, Godo and Esteva logic FP(Ł) was introduced in [14] as a formal framework for reasoning about the probabilities (i.e. the states) of Łukasiewicz infinite-valued events. For that logic, denoted by FP(Ł, Ł), the authors proposed an axiomatization which reflects the main properties of Łukasiewicz states, but they did not succeed in proving a completeness theorem with respect to its *natural* semantics. In [11] we presented only a partial solution and we showed that the logic FP(Ł, Ł) is sound and complete w.r.t. a class of models which make use of hyperreal-valued states (see Subsection 3.2 for more details). However, proving *standard* completeness for FP(Ł, Ł), that is completeness with respect to *real*-valued states,[1] was left open in [14]. Let us emphasize that solving that problem, besides its theoretical interest, actually answers to a question which can be formulated as follows: is FP(Ł, Ł) the logic of real-valued states on Łukasiewicz events? And, if not, is this latter logic axiomatizable?

In order to give an answer to the first of the above question a new technique to prove standard completeness was proposed in the recent [12]. Unfortunately, as we realized afterwords, that construction was grounded on the unsound claim [12, Lemma 1]. The present paper overcomes the problem of [12] and introduces an algebraic construction that allows to prove that FP(Ł, Ł) enjoys a standard completeness theorem. In particular, in this paper, we introduce two kinds of standard semantics: a first one based on real-valued Łukasiewicz states and a second one based on real-valued regular Borel measures on compact Hausdorff spaces. Our main results show completeness of FP(Ł, Ł) with respect to both classes of models. As a direct consequence of the main construction we will adopt to show the first completeness theorem, we also prove that FP(Ł, Ł) has the finite model property. Indeed, as our last result shows, models based on hyperreal-valued states, real-valued states, regular Borel measures and finite models, they all share the same tautologies.

The present paper is organized as follows: in Section 2 we introduce the algebraic semantics of Łukasiewicz logic, MV-algebras, we recall and prove some basic necessary results. In Section 3 we remind states and hyperstates of MV-algebras and we also prove a key lemma that collects some needed properties for hyperstates. In Section 4 we present standard semantics for FP(Ł, Ł) and prove standard completeness results, while in Section 5 we show that FP(Ł, Ł) has the finite model property. We conclude with Section 6 in which we discuss on our future work.

## 2. Logical, algebraic and geometric preliminaries

Among the wide family of t-norm based fuzzy logics, surely Łukasiewicz infinite-valued calculus is that one which received more attention along the last years. This logic, that following tradition we denote by Ł, is algebraizable in the sense of Blok and Pigozzi [3] and it finds in the variety of MV-algebras its equivalent algebraic semantics.

In the following subsections we first recall MV-algebras, MV-chains (i.e., totally ordered MV-algebras) and we give a glimpse on the methods usually adopted to prove standard and strong non-standard completeness for propositional Łukasiewicz logic; secondly, we focus on free MV-algebras and present their geometric representation in terms of McNaughton functions. Furthermore, we prove a first result that will have a later use.

### 2.1. MV-algebras, MV-chains and standard completeness of Łukasiewicz calculus

The language $\mathscr{L}$ of Łukasiewicz logic is made of a countable (finite or infinite) set $Var$ of propositional variables, a binary connective $\oplus$, a unary connective $\neg$ and a constant $\bot$. Formulas, that will be indicated by lowercase Greek letters, are defined by induction as usual. Further connectives and constants are defined as follows:

$\top := \neg\bot$; $\varphi \odot \psi := \neg(\neg\varphi \oplus \neg\psi)$; $\varphi \to \psi := \neg\varphi \oplus \psi$; $\varphi \vee \psi := (\varphi \to \psi) \to \psi$; $\varphi \wedge \psi := \neg(\neg\varphi \vee \neg\psi)$; $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$.

[1] We will clarify what "standard completeness" means in this context in Section 4.

Due to the algebraizability of Ł, we will focus on its equivalent algebraic semantics, the class of MV-algebras, rather than its axiomatization which, however, can be found in [6,22]. Nevertheless, it is convenient to recall that the unique deduction rule of Łukasiewicz logic is *modus ponens* and to introduce the following notation: if $\mathrm{T}\cup\{\varphi\}$ is a set of formulas in the language of Ł, we will write $\mathrm{T}\vdash\varphi$ to denote that $\varphi$ is a *provable* in Ł from the theory T, that is to say, there exists a *proof* of $\varphi$ from the axioms of Ł and the formulas in T, see [6,22] for further details.

Before formally defining MV-algebras, let us recall that every boolean algebra is an MV-algebra in which $\oplus=\vee$ and $\odot=\wedge$. Indeed, MV-algebras generalize boolean algebra in a way that can be easily understood recalling that the role, in classical propositional logic, of the two element boolean chain $\mathbf{2}=(\{0,1\},\vee,\neg,0)$ is played in Łukasiewicz logic by the so called *standard* MV-algebra $[0,1]_{MV}=([0,1],\oplus,\neg,0)$ where, for all $x,y\in[0,1]$, $x\oplus y=\min\{1,x+y\}$ and $\neg x=1-x$.

**Definition 2.1.** An *MV-algebra* is a system $\mathbf{A}=(A,\oplus,\neg,\bot)$ where $A$ is a nonempty set, the triple $(A,\oplus,\bot)$ is a commutative monoid with neutral element $\bot$ and the following equations hold for every $x,y\in A$:

(1) $\neg\neg x=x$;
(2) $\neg(\neg x\oplus y)\oplus y=\neg(\neg y\oplus x)\oplus x$.

For every MV-algebra $\mathbf{A}=(A,\oplus,\neg,\bot)$, an $\mathbf{A}$-*valuation* is a map $v$ from $Var$ to $A$ which extends to all formulas by truth-functionality, that is, $v(\bot)=\bot$, $v(\varphi\oplus\psi)=v(\varphi)\oplus v(\psi)$, $v(\neg\varphi)=\neg v(\varphi)$.[2]

Following a standard universal algebraic notation, if $t[x_1,\dots,x_k]$ is any MV-term on variables $x_1,\dots,x_k$, $\mathbf{A}$ an MV-algebra, and $v$ an $\mathbf{A}$-valuation we write $t^{\mathbf{A}}[v(x_1),\dots,v(x_k)]$ to denote the element of $A$ computed from $v(x_1),\dots,v(x_k)$, by the operations of $\mathbf{A}$. Thus, for instance, if $t[x_1,x_2]=(\neg x_1\oplus x_2)$, $\mathbf{A}=[0,1]_{MV}$ and $v(x_i)=a_i\in[0,1]$, $t^{[\mathbf{0},\mathbf{1}]_{\mathbf{MV}}}[v(x_1),v(x_2)]=\min\{1,1-a_1+a_2\}$.

Further operations and constants $\top$; $\odot$; $\to$; $\vee$; $\wedge$; $\leftrightarrow$ can be defined in every MV-algebras by the same syntactical definitions given above. These operations have the following semantics in the standard MV-algebra $[0,1]_{MV}$: for every $x,y\in[0,1]$,

$$\top=1;\ x\odot y=\max\{0,x+y-1\};\ x\to y=\min\{1,1-x+y\};\ x\vee y=\max\{x,y\},\ x\wedge y=\min\{x,y\};\ x\leftrightarrow y=1-|x-y|.$$

For every MV-algebra $\mathbf{A}$, every $n\in\mathbb{N}$ and every $a\in A$, we will abbreviate $a\oplus\dots\oplus a$ ($n$-times) by $na$.

A partial order $\le$ can be defined in every MV-algebra $\mathbf{A}$: for all $x,y\in A$

$$x\le y \text{ iff } x\to y=\top.$$

The partial order $\le$ coincides with the lattice order of the reduct $(A,\wedge,\vee)$ of $\mathbf{A}$. Whenever $\le$ is linear, we will say that $\mathbf{A}$ is an *MV-chain*.

Chang's completeness theorem [5] shows that $[0,1]_{MV}$ is *generic* for the variety $\mathbb{MV}$ of MV-algebras, meaning that an equation $\varphi=\top$ holds in $[0,1]_{MV}$ iff it holds in all MV-algebras. As a consequence, Łukasiewicz infinite valued logic is sound and *finitely strong standard* complete. This means that for every *finite* set of formulas $\mathrm{T}\cup\{\varphi\}$, $\mathrm{T}\vdash\varphi$ iff $v(\varphi)=1$ for every $[0,1]_{MV}$-valuation $v$ which maps to 1 all the formulas of T. In purely algebraic terms, the finite strong standard completeness of Łukasiewicz calculus can be obtained almost immediately from Lemma 2.2 below that will be also used in the proof of our main result in Section 4. Let us prepare. An *MV-homomorphism* between two MV-algebras $\mathbf{A}$ and $\mathbf{B}$ is a function $h:A\to B$ such that, adopting without danger of confusion the same symbols for the operations of both algebras: (1) $h(\bot)=\bot$; (2) $h(x\oplus y)=h(x)\oplus h(y)$; (3) $h(\neg x)=\neg h(x)$. If $X$ is a subset of $A$, a map $h:X\to B$ is a *partial homomorphism* provided that the above conditions (1-3) hold for the partial operations defined between the elements of $X$. Injective partial homomorphisms are called *partial embeddings*. A partial embedding of $X\subseteq\mathbf{A}$ to $\mathbf{B}$ will be denoted by $X\hookrightarrow_p\mathbf{B}$. Therefore, an MV-algebra $\mathbf{A}$ *partially embeds* into an MV-algebra $\mathbf{B}$ if for every finite subset $X$ of $A$, there exists a partial embedding $X\hookrightarrow_p\mathbf{B}$.

The proof of the following key lemma immediately follows from Gurevich-Kokorin theorem (see for instance [22, Theorem 1.6.17]).

**Lemma 2.2.** *Every MV-chain* $\mathbf{A}$ *partially embeds into the standard MV-algebra* $[0,1]_{MV}$.

Let us briefly see why the lemma above implies the finite strong standard completeness for Ł. First of all recall that Łukasiewicz logic is (obviously) sound and complete w.r.t $\mathbb{MV}$. Now, since every MV-algebra satisfies the *prelinearity equation*, $(x\to y)\vee(y\to x)=\top$, the subdirectly irreducible elements of $\mathbb{MV}$ are MV-chains (see [22]). This implies that, for every formula $\varphi$ of Łukasiewicz language, $v(\varphi)=\top$ in every MV-algebra $\mathbf{A}$ and for every $\mathbf{A}$-valuation $v$ iff $v'(\varphi)=\top$ in every MV-chain $\mathbf{C}$ and for every $\mathbf{C}$-valuation $v'$. Thus, Ł is sound and complete w.r.t. to the class of MV-chains. Therefore, for every

[2] We used here the same symbols to denote the connectives of Łukasiewicz calculus and the operations of $\mathbf{A}$.

finite set of formulas $\mathrm{T} \cup \{\varphi\}$ such that $\mathrm{T} \nvdash \varphi$, there exist an MV-chain $\mathbf{C}$ and a $\mathbf{C}$-valuation $v$ such that $v(\tau) = \top$ for all $\tau \in T$, but $v(\varphi) < \top$. Let

$$X = \{v(\psi) \mid \psi \text{ is a subformula of some formula in } \mathrm{T} \cup \{\varphi\}\}.$$

The set $X$ is a finite subset of $C$ which contains $v(\varphi) < \top$. By Lemma 2.2, there exists a partial embedding $\lambda : X \hookrightarrow_p [0,1]_{MV}$. Take $v'$ mapping each variable $q$ appearing in the formulas of $\mathrm{T} \cup \{\varphi\}$ to $\lambda(v(q))$. Notice that $v'$ is well defined because $v(q) \in X$ for each variable $q$. Since $\lambda$ is a partial embedding it preserves the $\top$ of $\mathbf{C}$, whence $v'$ determines a $[0,1]_{MV}$-valuation which, once extended to all formulas, maps each $\tau$ to 1, while $v'(\varphi) < 1$ as desired.

However, the *strong* standard completeness theorem, that is completeness w.r.t. $[0,1]_{MV}$ when deductions involve *infinite* theories, does not hold for Łukasiewicz logic [22]. Its failure can be seen as a consequence of the existence of countable MV-chains that cannot be (fully) embedded into the standard one $[0,1]_{MV}$ [7]. This problem can be overcome by moving from real-valued to hyperreal-valued models and hence by considering valuations in MV-chains which are ultrapowers of the standard algebra. Indeed, by a direct consequence of Di Nola's representation theorem [6], every MV-chain embeds into an ultrapower $[0,1]^*$ of the standard algebra. Hence, an easy adaptation of the argument explained above shows that Łukasiewicz logic enjoys strong completeness w.r.t. the class of non-standard MV-algebras, *strong non-standard completeness* in the notation of [11] (see also [13] for a more general treatment of ultrapowers and embedding properties). For a later use, let us summarize these facts in the following result whose proof can be found in [7,11,22].

**Theorem 2.3.** *Let $T \cup \{\varphi\}$ be a countable set of formulas in Łukasiewicz language and assume that $T \nvdash \varphi$. Then the following conditions hold:*

(1) *If T is finite, then there is a $[0,1]_{MV}$-valuation $v$ such that $v(\tau) = 1$ (for $\tau \in T$) and $v(\varphi) < 1$;*
(2) *If T is infinite, then there exists an ultrapower $[0,1]^*$ of $[0,1]_{MV}$ and a $[0,1]^*_{MV}$-valuation $v$ such that $v(\tau) = 1$ (for $\tau \in T$) and $v(\varphi) < 1$.*

It is hence clear that, in general terms, the problem of establishing completeness can be regarded as the (partial) embeddability problem for algebras belonging to a class of structures into algebras of particular kind (see for instance [7, Ch. 4] for further details). In the proof of our main result, Theorem 4.2, we will adopt a similar method to prove standard completeness for FP(Ł, Ł).

### 2.2. Free MV-algebras, McNaughton functions and Schauder hats

We now focus on a class of MV-algebras that will play a central role in this paper: finitely generated free MV-algebras. For $n$ being the number of free generators, we will denote them by $\mathbf{F}(n)$. These structures allow to represent the formulas of Łukasiewicz logic by a special kind of fuzzy sets and hence they provide us with a suitable algebraic setting for many-valued, fuzzy, events. Indeed, algebras like $\mathbf{F}(n)$ are, up to isomorphism, the Lindenbaum-Tarski algebras of Łukasiewicz logic over a language having $n$ propositional variables. A usual universal algebraic argument shows that $\mathbf{F}(n)$ is generic for the variety $\mathbb{MV}$ once we restrict to formulas containing at most $n$ propositional variables.

**Proposition 2.4.** *Let $\varphi$ be a formula of Ł containing n propositional variables. Then the equation $\varphi = \top$ holds in $\mathbf{F}(n)$ iff it holds in every MV-algebra. In other words, $\varphi = \top$ holds in $\mathbf{F}(n)$ iff $v(\varphi) = \top$ for every MV-algebra $\mathbf{A}$ and every $\mathbf{A}$-valuation $v$.*

The following example proposes a representation of finitely generated free MV-algebras.

**Example 2.5.** Finitely generated free MV-algebras $\mathbf{F}(n)$ are, up to isomorphism, algebras of functions $m : [0,1]^n \to [0,1]$ which are continuous, piecewise linear and such that each piece has an integer coefficient [28,31]. Operations on $\mathbf{F}(n)$ are the pointwise applications of those from $[0,1]_{MV}$, that is to say, for every $m_1, m_2 \in \mathbf{F}(n)$, $m_1 \oplus m_2 : \mathbf{x} \in [0,1]^n \mapsto \min\{1, m_1(\mathbf{x}) + m_2(\mathbf{x})\} \in [0,1]$ and $\neg m_1 : \mathbf{x} \in [0,1]^n \mapsto 1 - m_1(\mathbf{x}) \in [0,1]$.

The functions in the universe of $\mathbf{F}(n)$ are known in the literature as *McNaughton functions*. As we already recalled above, the free MV-algebra on $n$ free generators is, up to isomorphism, the Lindenbaum-Tarski algebra of Łukasiewicz logic over a language with $n$ propositional variables, and hence every equivalence class (modulo equi-provability) $[\psi]$ of a formula $\psi$ in this language corresponds to a unique McNaughton function that we will denote by $m_\psi$, see [31] for details.

The following observation introduces a notation and a recap on known results about McNaughton functions that will be often used along this paper.[3]

[3] A more exhaustive treatment on the ensuing topic can be found in [6, §3] (see also [33, §3]).

**Remark 2.6.** Let us fix a finite set of Łukasiewicz formulas, say $\psi_1, \ldots, \psi_k$, on $n$ propositional variables and let us denote by $m_{\psi_1}, \ldots, m_{\psi_k}$ the McNaughton functions on $[0,1]^n$ to $[0,1]$ corresponding to (the equivalence classes of) each $\psi_i$. The piecewise linearity of each $m_{\psi_i}$ ensures the existence of a unimodular triangulation $\Delta$ of the $n$-cube $[0,1]^n$ such that each $m_{\psi_i}$ is linear on each simplex of $\Delta$. In this case, we will also say that $\Delta$ *linearizes* $m_{\psi_1}, \ldots, m_{\psi_k}$.

Let us denote by $\mathbf{x}_1, \ldots, \mathbf{x}_t$ the vertices of $\Delta$ (i.e., the set of vertices of each simplex in $\Delta$). The unimodular triangulation $\Delta$ can be chosen in such a way that each vertex $\mathbf{x}_j$ is a *rational point*, i.e., it has rational coordinates: $\mathbf{x}_j = (q_{j_1}, \ldots, q_{j_n})$. For each $j = 1, \ldots, t$ let us denote by $\mathrm{den}\,\mathbf{x}_j$ the least common multiple of the set of denominators of the $q_{j_i}$'s:

$$\mathrm{den}\,\mathbf{x}_j = \mathrm{lcm}\{\mathrm{den}\,q_{j_i} \mid i = 1, \ldots, n\}.$$

For each $\mathbf{x}_j$, let us consider the continuous function $h_j : [0,1]^n \to [0,1]$ such that $h_j(\mathbf{x}_j) = 1/\mathrm{den}\,\mathbf{x}_j$, $h_j(\mathbf{x}_l) = 0$ for each $l \neq j$ and $h_j$ is linear on each simplex of $\Delta$. Further, let us denote by $\hat{h}_j$ the function $\mathrm{den}\,\mathbf{x}_j \cdot h_j$ which hence takes value 1 on $\mathbf{x}_j$ and 0 on any other vertex of $\Delta$. The maps $h_j$ and $\hat{h}_j$ are called respectively the *Schauder hat* and the *normalized Schauder hat at* $\mathbf{x}_j$. Each (normalized) Schauder hat is hence continuous, piecewise linear and the unimodularity of $\Delta$ also ensures that each piece of each $h_j$ (and $\hat{h}_j$) has an integer coefficient, that is to say, (normalized) Schauder hats are McNaughton function. Therefore, there exist Łukasiewicz propositional formulas $\gamma_1, \ldots, \gamma_t$ and $\hat{\gamma}_1, \ldots, \hat{\gamma}_t$ such that, for each $j = 1, \ldots, t$

$$h_j = m_{\gamma_j} \text{ and } \hat{h}_j = m_{\hat{\gamma}_j}.$$

In the rest of this paper we will often use the notation adopted in this observation and in particular the one concerning formulas $\psi_1, \ldots, \psi_k$, $\gamma_1, \ldots, \gamma_t$, $\hat{\gamma}_1, \ldots, \hat{\gamma}_t$ and their corresponding McNaughton functions.

It is well-known that MV-algebras with MV-homomorphisms form a category which is equivalent to that of lattice-ordered abelian groups with strong unit (unital $\ell$-groups for short) whose morphisms are group homomorphisms that respect the lattice structure [30]. Details on that equivalence fall out of the scope of the present paper, however, it is convenient to recall that for every MV-algebra $\mathbf{A}$ there exists a unique unital $\ell$-group $(\mathbf{G_A}, \top)$ (or simply by $\mathbf{G_A}$ when it is not needed to specify the strong unit) such that $A = \{g \in G \mid 0 \leq g \leq u\}$ and for all $x, y \in A$, $x \oplus y = \min\{u, x + y\}$ and $\neg x = u - x$.

In the statement and proof of the following result we will adopt the notation from Remark 2.6.

**Lemma 2.7.** *Let $\psi_1, \ldots, \psi_k$ be Łukasiewicz formulas and let $\Delta$ be a unimodular triangulation linearizing $m_{\psi_1}, \ldots, m_{\psi_k}$ and with vertices $\mathbf{x}_1, \ldots, \mathbf{x}_t$. Then, for every $i = 1, \ldots, k$ there are uniquely determined natural numbers $n^i_1, \ldots, n^i_t$ such that each $n^i_j \leq \mathrm{den}\,\mathbf{x}_j$ and for every MV-algebra $\mathbf{A}$, every $\mathbf{A}$-valuation $v$, and every $i = 1, \ldots, k$ the following properties hold (sums are taken in $\mathbf{G_A}$):*

1. $v(\bigoplus_{j=1}^t \hat{\gamma}_j) = \bigoplus_{j=1}^t v(\hat{\gamma}_j) = \sum_{j=1}^t v(\hat{\gamma}_j) = \top$;
2. *For all $1 \leq j \leq t$ and for all natural number $n < \mathrm{den}\,\mathbf{x}_j$, $v(n\gamma_j \odot \gamma_j) = \bot$;*
3. $v(\psi_i) = v(\bigoplus_{j=1}^t n^i_j \gamma_j) = \bigoplus_{j=1}^t n^i_j v(\gamma_j) = \sum_{j=1}^t n^i_j v(\gamma_j)$;
4. *For distinct $j_1, \ldots, j_r$, and for all natural numbers $n^i_{j_e} \leq \mathrm{den}\,\mathbf{x}_{j_e}$,*

$$v\left(\left(\bigoplus_{e=1}^{r-1} n^i_{j_e}\gamma_{j_e}\right) \odot n^i_{j_r}\gamma_{j_r}\right) = \bot;$$

*Therefore, in particular, $v((\bigoplus_{e=1}^{r-1} \hat{\gamma}_{j_e}) \odot \hat{\gamma}_{j_r}) = \bot$.*

**Proof.** Let $n$ be the number of propositional variables occurring in $\psi_1, \ldots, \psi_k$. Since free finitely generated MV-algebras are generic for the variety of MV-algebras (Proposition 2.4), in order to prove the claim, it is sufficient to show that the above equalities holds in $\mathbf{F}(n)$ under the $\mathbf{F}(n)$-valuation which maps every propositional formula $\varphi$ to $m_\varphi$.

(1) The claim follows from [33, Lemma 3.4] (ii).

(2) Let us prove that, for all $n < \mathrm{den}\,\mathbf{x}_j$, $nm_{\gamma_j} \odot m_{\gamma_j} = 0$ holds in $\mathbf{F}(n)$. To this end recall that every McNaughton function of the form $m_{\gamma_j}$ takes value 0 on each vertex $\mathbf{x}_l$ (with $j \neq l$), while $m_{\gamma_j}(\mathbf{x}_j) = 1/\mathrm{den}\,\mathbf{x}_j$. Therefore, if $n < \mathrm{den}\,\mathbf{x}_j$, $nm_{\gamma_j}(\mathbf{x}_j) = n/\mathrm{den}\,\mathbf{x}_j$ and $nm_{\gamma_j}(\mathbf{x}_j) \odot m_{\gamma_j}(\mathbf{x}_j) = \max\{0, nm_{\gamma_j}(\mathbf{x}_j) + m_{\gamma_j}(\mathbf{x}_j) - 1\} = \max\{0, n/\mathrm{den}\,\mathbf{x}_j + 1/\mathrm{den}\,\mathbf{x}_j - \mathrm{den}\,\mathbf{x}_j/\mathrm{den}\,\mathbf{x}_j\} = \max\{0, (n + 1 - \mathrm{den}\,\mathbf{x}_j)/\mathrm{den}\,\mathbf{x}_j\} = 0$ since $n + 1 - \mathrm{den}\,\mathbf{x}_j \leq \mathrm{den}\,\mathbf{x}_j - \mathrm{den}\,\mathbf{x}_j = 0$.

(3) is [33, Lemma 3.4] (iv).

The first claim of (4) follows from [33, Lemma 3.4] (v) and (vi). The last claim of (4) follows from the first one plus the fact that, for all $j = 1, \ldots, t$, $m_{\hat{\gamma}_j} = \mathrm{den}\,\mathbf{x}_j \cdot m_{\gamma_j}$, recall Remark 2.6. □

## 3. States and hyperstates and their logic

Now that we have presented our algebraic setting, in the following subsections we recall states, hyperstates and an axiomatization for the logic FP(Ł, Ł). Furthermore, we present hyperreal-state models and review the main steps which allow to prove completeness of FP(Ł, Ł) with respect to that non-standard semantics.

### 3.1. States and hyperstates

Let us first focus on the uncertainty measures that play, in MV-setting, the role of probability functions in the boolean realm. Those are called *states* of MV-algebras and they are defined as follows.

**Definition 3.1** *([32]).* Let $\mathbf{A}$ be an MV-algebra. A *state* of $\mathbf{A}$ is a map $s : A \to [0, 1]$ which satisfies the following conditions:

- $s(\top) = 1$ (normalization);
- for all $a, b \in A$ such that $a \odot b = \bot$, $s(a \oplus b) = s(a) + s(b)$ (finite additivity).

Every finitely additive probability measure $P$ on a boolean algebra $\mathbf{B}$, regarded as an MV-algebra (recall Section 2), is a state of $\mathbf{B}$. Moreover, for every MV-algebra $\mathbf{A}$, its *boolean skeleton* $B(\mathbf{A})$ (i.e., the MV-subalgebra of $\mathbf{A}$ whose universe is $\{a \in A \mid a = a \oplus a\}$) determines the largest boolean subalgebra of $\mathbf{A}$ and hence the restriction of every state $s$ of $\mathbf{A}$ to $B(\mathbf{A})$ is a finitely additive probability measure.

Every homomorphism of an MV-algebra $\mathbf{A}$ to the standard algebra $[0, 1]_{MV}$ is a state and every state of $\mathbf{A}$ belongs to the topological closure of the convex hull of the set of all the homomorphisms of $\mathbf{A}$ to $[0, 1]_{MV}$ (see [32] for further details).

By Belluce-Chang theorem [6], an MV-algebra $\mathbf{A}$ is semisimple iff it is an MV-subalgebra of the MV-algebra $\mathscr{C}(\mathbf{X_A})$ of continuous $[0, 1]$-valued functions on a compact Hausdoff space $\mathbf{X_A} = (X_\mathbf{A}, \tau)$. For every element $a$ of a semisimple MV-algebra $\mathbf{A}$ we will denote by $c_a$ its representation as element of $\mathscr{C}(\mathbf{X_A})$. The theorem below can be proved in general for every state of any MV-algebra, see [25,35].

**Theorem 3.2.** *For every state $s$ of a semisimple MV-algebra $\mathbf{A}$ there exists a unique regular Borel measure $\mu$ on $\mathbf{X_A}$ such that, for every $a \in A$,*

$$s(a) = \int_{X_\mathbf{A}} c_a \, \mathrm{d}\mu.$$

If semisimple MV-algebras are representable as algebras of $[0, 1]$-valued continuous functions, non-semisimple algebras can be characterized by the presence of positive *infinitesimal* (and *co-infinitesimal*) elements. Those, as in non-standard analysis, are the elements $a$ of a non-semisimple algebra $\mathbf{A}$ such that $a > \bot$ and for every $n \in \mathbb{N}$, $na < \top$. It has been observed in [29] that every state $s$ of a non-semisimple algebra maps its infinitesimals to 0 and its co-infinitesimals to 1. This fact points out a limitation in the use of states in the general realm of MV-algebras. Also to overcome this behavior of states, *hyperstates* of semisimple and non-semisimple algebras have been introduced and quite largely employed (see for instance [11,15,29]).

**Definition 3.3.** Let $\mathbf{A}$ be an MV-algebra. A *hyperstate* of $\mathbf{A}$ is a map $s^* : A \to [0, 1]^*$ where $[0, 1]^*$ is an ultrapower of the real unit interval and $s^*$ is normalized and finitely additive in the sense of Definition 3.1.

In the definition above the ultrapower $[0, 1]^*$ is fixed with no restriction and it might be the case that $[0, 1]^* = [0, 1]$.[4] Therefore, every state is a hyperstate in the sense of Definition 3.3. Less trivial examples of hyperstates are the following: (1) every homomorphism of an MV-algebra $\mathbf{A}$ to an ultrapower $[0, 1]^*_{MV}$ of the standard algebra is a hyperstate; (2) every convex combination of finitely many homomorphisms of $\mathbf{A}$ to $[0, 1]^*_{MV}$ is a hyperstate as well.

The same proofs of [32, Proposition 2.2] and [19, Proposition 3.1.1] can be easily adapted to prove the following result which collects basic facts about hyperstates.

**Proposition 3.4.** *Every hyperstate $s^*$ of an MV-algebra $\mathbf{A}$ satisfies the following properties:*

(1) $s^*(a \oplus b) = s^*(a) + s^*(b) - s^*(a \odot b)$;
(2) if $a \leq b$, then $s^*(a) \leq s^*(b)$;
(3) $s^*(\neg a) = 1 - s^*(a)$;
(4) $s^*(\bot) = 0$.

[4] This is the case when the ultrafilter used to define the ultrapower is principal, see for instance the discussion in [7, Ch. 4].

The following key lemma, that can be regarded as a continuation of Lemma 2.7 in the setting of hyperstates, presents some basic facts that will find application in the proof of the main result of Section 4. In its statement and proof we will adopt the notation from Remark 2.6.

**Lemma 3.5.** *For every MV-algebra* $\mathbf{A}$*, every* $\mathbf{A}$*-valuation* $v$ *and every hyperstate* $s^*: \mathbf{A} \to [0,1]^*$*, the following properties hold (sums are taken in* $\mathbf{G}_{[0,1]^*}$*):*

(1) $s^*(v(n\gamma_j)) = ns^*(v(\gamma_j))$ *for all* $1 \le j \le t$ *and for all* $n \le \operatorname{den} \mathbf{x}_j$*;*
(2) $s^*\left(\bigoplus_{j=1}^t v(\hat{\gamma}_j)\right) = \sum_{j=1}^t s^*(v(\hat{\gamma}_j)) = 1$*;*
(3) $s^*(v(\psi_i)) = \sum_{j=1}^t n_j^i s^*(v(\gamma_j))$.

**Proof.** (1) Let us prove the claim by induction on $n$. If $n = 2$, by Proposition 3.4 (1), plus $v(\gamma_j \odot \gamma_j) = \bot$ (Lemma 2.7 (3)) and $s^*(\bot) = 0$ (Proposition 3.4 (4)), one has

$$\begin{aligned} s^*(2v(\gamma_j)) &= s^*(v(\gamma_j) \oplus v(\gamma_j)) \\ &= s^*(v(\gamma_j)) + s^*(v(\gamma_j)) - s^*(v(\gamma_j) \odot v(\gamma_j)) \\ &= 2s^*(v(\gamma_j)) - s^*(v(\gamma_j \odot \gamma_j)) \\ &= 2s^*(v(\gamma_j)). \end{aligned}$$

Now assume that $s^*((n-1)v(\gamma_j)) = (n-1)s^*(v(\gamma_j))$. The associativity of $\oplus$ and Proposition 3.4 (1) give

$$\begin{aligned} s^*(nv(\gamma_j)) &= s^*((n-1)v(\gamma_j) \oplus v(\gamma_j)) \\ &= (n-1)s^*(v(\gamma_j)) + s^*(v(\gamma_j)) - s^*((n-1)v(\gamma_j) \odot v(\gamma_j)). \end{aligned}$$

By Lemma 2.7 (3), since $n \le \operatorname{den} \mathbf{x}_j$ by hypothesis, one has $n - 1 < \operatorname{den} \mathbf{x}_j$ and hence $(n-1)v(\gamma_j) \odot v(\gamma_j) = v((n-1)\gamma_j \odot \gamma_j) = \bot$ which settles the claim.

(2) By Lemma 2.7 (1) and normalization property of $s^*$, $s^*(\bigoplus_{j=1}^t v(\hat{\gamma}_j)) = 1$. Let hence prove that

$$s^*\left(\bigoplus_{j=1}^t v(\hat{\gamma}_j)\right) = \sum_{j=1}^t s^*(v(\hat{\gamma}_j))$$

by induction on $t$. The case $t = 2$ is easy because of the additivity property of $s^*$ together with the fact that for distinct $\hat{\gamma}_1$ and $\hat{\gamma}_2$, $v(\hat{\gamma}_1) \odot v(\hat{\gamma}_2) = v(\hat{\gamma}_1 \odot \hat{\gamma}_2) = \bot$ (Lemma 2.7 (4)).

Assume the claim true for $t-1$: $s^*(\bigoplus_{j=1}^{t-1} v(\hat{\gamma}_j)) = \sum_{j=1}^{t-1} s^*(v(\hat{\gamma}_j))$. Then, by the associativity of $\oplus$, the additivity of $s^*$, Lemma 2.7 (4) (used in the third equality below) and Proposition 3.4 (4),

$$\begin{aligned} s^*\left(\bigoplus_{j=1}^t v(\hat{\gamma}_j)\right) &= s^*\left(\bigoplus_{j=1}^{t-1} v(\hat{\gamma}_j) \oplus v(\hat{\gamma}_t)\right) \\ &= s^*\left(\bigoplus_{j=1}^{t-1} v(\hat{\gamma}_j)\right) + s^*(v(\hat{\gamma}_t)) - s^*\left(\bigoplus_{j=1}^{t-1} v(\hat{\gamma}_j) \odot v(\hat{\gamma}_t)\right) \\ &= \sum_{j=1}^{t-1} s^*(v(\hat{\gamma}_j)) + s^*(v(\hat{\gamma}_t)) - s^*(\bot) \\ &= \sum_{j=1}^{t} s^*(v(\hat{\gamma}_j)). \end{aligned}$$

(3) By Lemma 2.7 (2), $s^*(v(\psi_i)) = s^*(\sum_{j=1}^t n_j^i v(\gamma_j))$. By the additivity of $s^*$, the latter equals $\sum_{j=1}^t s^*(n_j^i v(\gamma_j))$. For every $j = 1, \ldots, t$, since $n_j^i \le \operatorname{den} \mathbf{x}_j$, $s^*(n_j^i v(\gamma_j)) = n_j^i s^*(v(\gamma_j))$ because of (1) above. Thus, $s^*(v(\psi_i)) = \sum_{j=1}^t n_j^i s^*(v(\gamma_j))$. □

### *3.2. The logic FP(Ł, Ł) and its hyperstate-based models*

The logic FP(Ł, Ł) was introduced in [14] as the generalization to fuzzy events of the fuzzy probabilistic logic FP(Ł) studied in [21,22]. Its language is obtained by expanding that of Łukasiewicz logic (recall Section 2) by a unary modality $P$. The set of formulas, denoted by **PFm** is made of the following two classes:

(EF): the set of *event* formulas which contains all formulas of Łukasiewicz language; these formulas will be denoted, as above, by lowercase Greek letter $\varphi, \psi, \ldots$ with possible subscripts;

(MF): the set of *modal* formulas which contains expressions of the form $P(\varphi)$ for every event formula $\varphi$, the constants $\top$ and $\bot$ and which is closed under the connectives of Łukasiewicz language. Modal formulas will be denoted by uppercase Gerek letters $\Phi, \Psi, \dots$ with possible subscripts.

A remark is in order: by its definition, modal formulas in **PFm** are just MV-terms written using atomic modal formulas as variables. In other words, every (compound) modal formula $\Phi$ is of the form $t[P(\psi_1), \dots, P(\psi_k)]$ for $P(\psi_1), \dots, P(\psi_k)$ atomic modal formulas (regarded as variables) and $t$ is an MV-term.

Axioms and rules of FP(Ł, Ł) are as follows:

(EŁ): all axioms and rules of Łukasiewicz calculus for event formulas;

(MŁ): all axioms and rules of Łukasiewicz calculus for modal formulas;

(P): the following axioms and rules specific for the modality $P$:

(P1) $\neg P(\varphi) \leftrightarrow P(\neg\varphi)$;
(P2) $P(\varphi \to \psi) \to (P(\varphi) \to P(\psi))$;
(P3) $P(\varphi \oplus \psi) \leftrightarrow [(P(\varphi) \to P(\varphi \odot \psi)) \to P(\psi)]$;
(N) The *necessitation rule*: from $\varphi$ derive $P(\varphi)$.

The notion of *proof* is defined as usual and for every modal formula $\Phi$, we will henceforth write $\vdash_{FP} \Phi$ to denote that $\Phi$ is a theorem of FP(Ł, Ł).

The axioms and rules above, are enough to syntactically prove that the modality $P$ satisfies the basic properties of (hyper)states. For instance, $P(\top)$ (normalization) can be derived with a step of the necessitation rule (N) from the Łukasiewicz theorem $\top$. Instantiating (P1) with $\varphi = \top$, one obtains $\neg P(\top) \leftrightarrow P(\neg\top)$. Since $P(\top) = \top$ (normalization) and $\neg\top = \bot$, $\bot \leftrightarrow P(\bot)$ which reads "the probability of a contradiction is zero". Finally, the finite additivity of P is proved as follows: assume that $\varphi \odot \psi \leftrightarrow \bot$ holds. Then, $P(\varphi \odot \psi) \leftrightarrow \bot$ and substituting $P(\varphi \odot \psi)$ by $\bot$ in (P3), one has $P(\varphi \oplus \psi) \leftrightarrow [(P(\varphi) \to \bot) \to P(\psi)]$. Now, $P(\varphi) \to \bot$ is equivalent, in Łukasiewicz logic, to $\neg P(\varphi)$, thus $(P(\varphi) \to \bot) \to P(\psi)$ is $\neg P(\varphi) \to P(\psi)$ that equals $P(\varphi) \oplus P(\psi)$. Hence, from $\varphi \odot \psi \leftrightarrow \bot$, we infer $P(\varphi \oplus \psi) \leftrightarrow P(\varphi) \oplus P(\psi)$.

The first class of models we consider for the logic FP(Ł, Ł) are based on hyperstates and they are defined in the following way.

**Definition 3.6.** A *hyperstate model* $\mathscr{S}^* = (W, e, s^*)$ consists of

- a nonempty set $W$;
- a map $e : W \times Var \to [0,1]_{MV}$ such that for every $w \in W$, $e(w, \cdot) : Var \to [0,1]_{MV}$ uniquely extends to a $[0,1]_{MV}$-valuation;
- a hyperstate $s^* : [0,1]^W \to [0,1]^*$.

If $\Phi$ is a formula in **PFm**, its truth-value in a hyperstate model $\mathscr{S}^* = (W, e, s^*)$, at the world $w \in W$, is computed in the following way where, for every propositional Łukasiewicz formula $\psi$, $f_\psi : W \to [0,1]$ is defined as $f_\psi(w) = e(w, \psi)$.

– If $\Phi = \psi$ is a propositional formula, $\|\psi\|_{\mathscr{S}^*,w} = f_\psi(w)$;
– If $\Phi = P(\psi)$ is an atomic modal formula, then $\|P(\psi)\|_{\mathscr{S}^*,w} = s^*(f_\psi)$;
– If $\Phi = t[P(\psi_1), \dots, P(\psi_k)]$ is a compound modal formula, $\|\Phi\|_{\mathscr{S}^*,w}$ is computed by first evaluating all the atomic modal formulas $P(\psi_i)$'s in $[0,1]^*$ by $s^*$, and then by interpreting the term $t$ in the MV-chain $[0,1]^*$ as

$$\|t[P(\psi_1), \dots, P(\psi_k)]\|_{\mathscr{S}^*,w} = t^{[0,1]^*}[s^*(f_{\psi_1}), \dots, s^*(f_{\psi_k})].$$

Notice that the truth value $\|\Phi\|_{\mathscr{S}^*,w}$ of any modal formula $\Phi$ does not depend on the chosen world $w$. For this reason, in these cases, we will omit the subscript $w$ without danger of confusion.

The usual strategy to prove completeness of probabilistic modal logics like FP(Ł, Ł) w.r.t. a class of models consists in the following steps (see [8,16] for more details):

(S1) First of all we define a syntactic translation $\circ$ from modal to a propositional formulas of Łukasiewicz logic by interpreting every atomic modal formula $P(\varphi)$ in a new propositional variable $p_\varphi$ and extending $\circ$ to compound modal formulas by truth functionality.

(S2) The translation of all instances of the axioms (P1)-(P3), together with the set $\{p_\varphi \mid \vdash \varphi\}$ which encodes the propositional translation of the rule (N), give rise to a propositional Łukasiewicz theory $\mathbf{P}^\circ$ such that, for every modal formula $\Phi$, $\vdash_{FP} \Phi$ iff $\mathbf{P}^\circ \vdash \Phi^\circ$ (see [14, Ch. 4] for instance).

(S3) Finally, assume that $\nvdash_{FP} \Phi$ and hence $\mathbf{P}^\circ \nvdash \Phi^\circ$. Modulo the completeness of Łukasiewicz logic, find an MV-algebra $\mathbf{A}$ which models $\mathbf{P}^\circ$ and $\mathbf{A}$ does not model $\Phi^\circ$.

As we recalled in Section 2, Łukasiewicz logic does not enjoy the strong standard completeness and therefore, if $\mathbf{P}^\circ$ turns out to be infinite, we must extend the scope of countermodels of $\Phi^\circ$ to include also MV-chains of the form $[0,1]^*$ as in Theorem 2.3 (2). On the other hand, if $\mathbf{P}^\circ$ is finite, by Theorem 2.3 (1) we can safely consider a $[0,1]_{MV}$-valuation which models $\mathbf{P}^\circ$ and which maps $\Phi^\circ$ to $\alpha < 1$. More precisely there exists a standard construction which does the following:

(1) if $\mathbf{P}^\circ$ turns out to be finite (this is the case of the logics FP(Ł$_k$, Ł) or FP(Ł), see [14,21]), it builds a model $\mathcal{M}$ based on a *real-valued* state (or a probability function if events are classical) which does not satisfy $\Phi$.

(2) if $\mathbf{P}^\circ$ turns out to be infinite (as for the case of FP(Ł, Ł)), the construction determines a model $\mathcal{M}^*$ based on a *hyperreal-valued* state which does not satisfy $\Phi$.

A detailed discussion about the above strategy to prove completeness for this class of modal logics can be found in [16] (see in particular Theorem 20 of that book chapter for a more general result). In particular, [16, Theorem 25] whose proof is obtained by the above ideas and constructions, reads as follows.

**Theorem 3.7.** *For every formula $\Phi \in$ **PFm** the following conditions are equivalent:*

(1) $\vdash_{FP} \Phi$;
(2) *For every hyperstate model $\mathscr{S}^*$, $\|\Phi\|_{\mathscr{S}^*} = 1$.*

It is worth to point out that the theorem above does not claim that FP(Ł, Ł) fails to be complete with respect to a class of real-valued (i.e. *standard*) models. Rather it shows that the strategy sketched above, which in turn relies on the finite standard and strong non-standard completeness of Ł, does not bring to the desired conclusion. In the next section we will explain an alternative way to cope with probabilistic values on Łukasiewicz formulas allowing to prove that FP(Ł, Ł) *is* standard complete.

## 4. Standard probabilistic models and completeness

In this section we are going to prove a standard completeness theorem for the logic FP(Ł, Ł), that is completeness with respect to valuations which assign to every atomic modal formula $P(\varphi)$ a real number $s(\varphi)$ for some state $s$ on a suitably defined MV-algebra.

Afterwords, we will also introduce another class of models that, instead of states of MV-algebras, directly involves Borel (and hence $\sigma$-additive) regular probability measures. We will show that FP(Ł, Ł) is also complete w.r.t. to this type of models.

To begin with, let us introduce the class of state models.

**Definition 4.1.** A *state model* $\mathscr{S} = (W, e, s)$ consists of

- a nonempty set $W$;
- a map $e : W \times Var \to [0,1]_{MV}$ such that for every $w \in W$, $e(w, \cdot) : Var \to [0,1]_{MV}$ uniquely extends to a $[0,1]_{MV}$-valuation;
- a state $s : [0,1]^W \to [0,1]$.

If $\Phi$ is a formula from **PFm**, its truth-value in a state model $\mathscr{S} = (W, e, s)$, at the world $w \in W$, is computed in the following way where, for every propositional Łukasiewicz formula $\psi$, $f_\psi : W \to [0,1]$ is defined as $f_\psi(w) = e(w, \psi)$.

- If $\Phi = \psi$ is a propositional formula, $\|\psi\|_{\mathscr{S},w} = f_\psi(w)$;
- If $\Phi = P(\psi)$ is an atomic modal formula, then $\|P(\psi)\|_{\mathscr{S},w} = s(f_\psi)$;
- If $\Phi = t[P(\psi_1), \ldots, P(\psi_k)]$ is a compound modal formula, $\|\Phi\|_{\mathscr{S},w}$ is computed by first evaluating all the atomic modal formulas $P(\psi_i)$'s in $[0,1]$ by $s$, and then by interpreting the term $t$ in the standard MV-algebra $[0,1]_{MV}$:

$$\|t[P(\psi_1), \ldots, P(\psi_k)]\|_{\mathscr{S},w} = t^{[0,1]}[s(f_{\psi_1}), \ldots, s(f_{\psi_k})].$$

As in the previous case, if $\Phi$ is a modal formula, its truth value $\|\Phi\|_{\mathscr{S},w}$ does not depend on the chosen world $w$. For this reason, in these cases, we will omit the subscript $w$ without danger of confusion.

We are now in position of proving that the logic FP(Ł, Ł) is sound and complete with respect to the class of state models. In order to ease the reading of the following proof, let us anticipate that completeness will be obtained by showing that if a formula $\Phi$ from **PFm** is not valid in a hyperstate model $\mathscr{S}^*$, then one can find a state model $\mathscr{S}$ in which $\Phi$ does not hold either. In particular, the state model $\mathscr{S}$ will be defined from $\mathscr{S}^*$ thanks to the partial embeddability of every hyperreal MV-chain $[0,1]^*$ into the standard chain $[0,1]_{MV}$, Lemma 2.2. Then, we will apply Lemma 2.7 and Lemma 3.5 to prove that the composition of the hyperstate $s^*$ from $\mathscr{S}^*$ with a partial embedding $\lambda : X \subset [0,1]^* \hookrightarrow_p [0,1]$ defined on the *ad hoc* finite set $X$ can be extended to a state $s$ that preserves the truth of $\Phi$.

**Theorem 4.2.** *For every formula $\Phi \in$ **PFm** the following conditions are equivalent:*

(1) $\vdash_{FP} \Phi$;
(2) *For every state model $\mathscr{S}$, $\|\Phi\|_{\mathscr{S}} = 1$.*

**Proof.** (1)⇒(2), soundness, directly follows from Theorem 3.7 (1)⇒(2) and the easy observation that state models are a particular case of hyperstate modes (recall what we remarked after Definition 3.3).

As for (2)⇒(1) let $\Phi = t[P(\psi_1), \ldots, P(\psi_k)]$ and assume that $\nvdash_{FP} \Phi$. By Theorem 3.7 there exists a hyperstate model $\mathscr{S}^* = (W, e, s^*)$ such that

$$\|\Phi\|_{\mathscr{S}^*} = t^{[0,1]^*}[s^*(f_{\psi_1}), \ldots s^*(f_{\psi_k})] < 1.$$

We will now show that there exists a state model $\mathscr{S}$ such that $\|\Phi\|_{\mathscr{S}} < 1$.

To this end, let $n$ be the number of propositional variables occurring in $\psi_1, \ldots, \psi_k$ and let $\Delta$ be a unimodular triangulation of $[0,1]^n$ with vertices $\mathbf{x}_1, \ldots, \mathbf{x}_t$ that linearizes the McNautghton functions $m_{\psi_1}, \ldots, m_{\psi_k}$. Now, consider the propositional Łukasiewicz formulas $\gamma_1, \ldots, \gamma_t$ and $\hat{\gamma}_1, \ldots, \hat{\gamma}_t$ such that, for each vertex $\mathbf{x}_j$ of $\Delta$, $m_{\gamma_j}$ and $m_{\hat{\gamma}_j}$ respectively are the Schauder hat and the normalized Schauder hat at $\mathbf{x}_j$.

Let us consider the following subsets of the MV-chain $[0,1]^*$ (being the range of $s^*$):

$$\begin{aligned}
X_s &= \{d^{[0,1]^*} \mid d \text{ is a sub-term of } t\};\\
X_n &= \{ns^*(f_{\gamma_j}) \mid j = 1, \ldots, t; i = 1, \ldots, k; 1 \le n \le n^i_j\};\\
X_\gamma &= \{\textstyle\sum_{j=1}^t s^*(n^i_j f_{\gamma_j}) \mid i = 1, \ldots, k\};\\
X_{\hat{\gamma}} &= \{\textstyle\sum_{j\in J} s^*(f_{\hat{\gamma}_j}) \mid J \subseteq \{1, \ldots, t\}\}.
\end{aligned}$$

First of all notice that, although the sums are taken in $\mathbf{G}_{[0,1]^*}$, each expression of the form $\sum_{j=1}^t s^*(n^i_j f_{\gamma_j})$ (as in $X_\gamma$) and $\sum_{j\in J} s^*(f_{\hat{\gamma}_j})$ (as in $X_{\hat{\gamma}}$) denotes an element of $[0,1]^*$ because of Lemma 3.5 (3) and (2) respectively.

Therefore, each among $X_s$, $X_n$, $X_\gamma$ and $X_{\hat{\gamma}}$ is a finite subset of $[0,1]^*$. Hence, putting

$$X_\Phi = X_s \cup X_n \cup X_\gamma \cup X_{\hat{\gamma}},$$

by Lemma 2.2 there exists a partial embedding $\lambda : X_\Phi \hookrightarrow_p [0,1]_{MV}$. Let us notice that, since $X_{\hat{\gamma}} \subset X_\Phi$, $\lambda$ maps each $s^*(f_{\hat{\gamma}_j})$ into a real number. To ease the notation, let us write

$$\pi_j = \lambda(s^*(f_{\hat{\gamma}_j})). \tag{a}$$

**Claim 1.** *Taking sums in the additive group or real numbers* **R** *with strong unit 1, the following properties hold:*

(1) $\sum_{j=1}^t \pi_j = 1$;
(2) *For all $i = 1, \ldots, k$, $\lambda(s^*(f_{\psi_i})) = \sum_{j=1}^t n^i_j \lambda(s^*(f_{\gamma_j}))$.*

**Proof of Claim 1.** Let us first recall that, in the hyperstate model $(W, e, s^*)$, the domain of the hyperstate $s^*$ is the MV-algebra $[0,1]^W$ of all functions $f : W \to [0,1]$ and that the map which associates to each propositional formula $\varphi$ the function $f_\varphi : w \in W \mapsto e(w, \varphi) \in [0,1]$ is a Łukasiewicz valuation in the MV-algebra $[0,1]^W$.

(1) By Lemma 2.7 (1), plus the facts just recalled,

$$1 = f_{\bigoplus_{j=1}^t \hat{\gamma}_j} = \bigoplus_{j=1}^t f_{\hat{\gamma}_j} = \sum_{j=1}^t f_{\hat{\gamma}_j}.$$

Therefore, since $s^*$ is normalized and additive, due to Lemma 3.5 (2),

$$1 = s^*(1) = s^*\left(\sum_{j=1}^t f_{\hat{\gamma}_j}\right) = \sum_{j=1}^t s^*(f_{\hat{\gamma}_j}).$$

Now, since $\lambda$ is a partial embedding and $X_{\hat{\gamma}} \subset X_\Phi$, it commutes with the + in all the expressions of the form $\sum_{j\in J} s^*(f_{\hat{\gamma}_j})$, for all $J \subseteq \{1, \ldots, t\}$. Thus,

$$1 = \lambda(1) = \lambda\left(\sum_{j=1}^t s^*(f_{\hat{\gamma}_j})\right) = \sum_{j=1}^t \lambda(s^*(f_{\hat{\gamma}_j})) = \sum_{j=1}^t \pi_j.$$

(2) By Lemma 3.5 (3), for every $i = 1, \ldots, k$,

$$\lambda(s^*(f_{\psi_i})) = \lambda\left(\sum_{j=1}^{t} n^i_j s^*(f_{\gamma_j})\right).$$

Since $X_\gamma \subset X_\Phi$, $\lambda(\sum_{j=1}^{t} n^i_j s^*(f_{\gamma_j})) = \sum_{j=1}^{t} \lambda(n^i_j s^*(f_{\gamma_j}))$, while $X_n \subset X_\Phi$ implies $\lambda(n^i_j s^*(f_{\gamma_j}) = n^i_j \lambda(s^*(f_{\gamma_j}))$. Therefore,

$$\lambda(s^*(f_{\psi_i})) = \sum_{j=1}^{t} n^i_j \lambda(s^*(f_{\gamma_j})). \quad \square$$

Let us now go back to the proof of our main result and consider a map $s : \mathbf{F}(n) \to [0,1]$ defined by the following stipulation: for every McNaughton function $m \in \mathbf{F}(n)$,

$$s(m) = \sum_{j=1}^{t} m(\mathbf{x}_j) \cdot \pi_j. \tag{b}$$

Claim 1 (1) and Theorem 3.2 immediately show that $s$ is a state of $\mathbf{F}(n)$.

Next, we need to prove that for every propositional subformula $\psi_i$ of our starting formula $\Phi = t[P(\psi_1), \ldots, P(\psi_k)]$, $s(m_{\psi_i})$ coincides with the image under $\lambda$ of $s^*(f_{\psi_i})$. That is to say, $s(m_{\psi_i}) = \lambda(s^*(f_{\psi_i}))$. To this end, let us fix a $\psi_i$. By the definitions of $s$ and $\pi_j$ (equations (b) and (a) respectively) one has

$$s(m_{\psi_i}) = \sum_{j=1}^{t} m_{\psi_i}(\mathbf{x}_j) \cdot \pi_j = \sum_{j=1}^{t} m_{\psi_i}(\mathbf{x}_j) \cdot \lambda(s^*(f_{\hat{\gamma}_j})). \tag{c}$$

Now, $m_{\hat{\gamma}_j}(\mathbf{x}_l) = 1$ if $l = j$ and $m_{\hat{\gamma}_j}(\mathbf{x}_l) = 0$ for all $l \neq j$. Therefore,

$$s(m_{\hat{\gamma}_j}) = \pi_j = \lambda(s^*(f_{\hat{\gamma}_j})).$$

Thus, we can substitute $\lambda(s^*(f_{\hat{\gamma}_j}))$ by $s(m_{\hat{\gamma}_j})$ in (c) obtaining

$$s(m_{\psi_i}) = \sum_{j=1}^{t} m_{\psi_i}(\mathbf{x}_j) \cdot s(m_{\hat{\gamma}_j}).$$

Since $s$ is a state and $m_{\hat{\gamma}_j} = \mathrm{den}\,\mathbf{x}_j \cdot m_{\gamma_j}$ (recall Remark 2.6), $s(m_{\hat{\gamma}_j}) = s(\mathrm{den}\,\mathbf{x}_j \cdot m_{\gamma_j}) = \mathrm{den}\,\mathbf{x}_j \cdot s(m_{\gamma_j})$. Therefore, the above expression becomes

$$s(m_{\psi_i}) = \sum_{j=1}^{t} m_{\psi_i}(\mathbf{x}_j) \cdot \mathrm{den}\,\mathbf{x}_j \cdot s(m_{\gamma_j}) = \sum_{j=1}^{t} m_{\psi_i}(\mathbf{x}_j) \cdot \mathrm{den}\,\mathbf{x}_j \cdot \lambda(s^*(f_{\gamma_j})).$$

Finally, recall that for every $i = 1, \ldots, k$, $m_{\psi_i}(\mathbf{x}_j) = n^i_j / \mathrm{den}\,\mathbf{x}_j$, whence $n^i_j = m_{\psi_i}(\mathbf{x}_j) \cdot \mathrm{den}\,\mathbf{x}_j$ from which, thanks to Claim 1 (2), we finally have that

$$s(m_{\psi_i}) = \sum_{j=1}^{t} n^i_j \cdot \lambda(s^*(f_{\gamma_j})) = \lambda(s^*(f_{\psi_i})). \tag{d}$$

Let us define a state model $\mathscr{S} = (W, e, s)$ such that:

- $W = [0,1]^n$ where, we recall, $n$ is the number of propositional variables occurring in $\psi_1, \ldots, \psi_k$;
- for every $\mathbf{x} \in W$ and for every formula $\varphi$ with $n$-variables, $e(\mathbf{x}, \varphi) = m_\varphi(\mathbf{x})$;
- $s$ is the state of $\mathbf{F}(n)$ defined as in (b).

By definition, $\|\Phi\|_{\mathscr{S}} = t^{[0,1]}[s(m_{\psi_1}), \ldots, s(m_{\psi_k})]$. From (d), let us substitute each $s(m_{\psi_i})$ by $\lambda(s^*(f_{\psi_i}))$ and hence we obtain

$$\|\Phi\|_{\mathscr{S}} = t^{[0,1]}[\lambda(s^*(f_{\psi_i})), \ldots, \lambda(s^*(f_{\psi_i}))].$$

Recall that $\lambda$ is a partial embedding and $X_s \subseteq X_\Phi$. Thus, $\lambda$ commutes with all the valuations, in $\mathscr{S}^*$, of all subformulas of $\Phi$. This implies that

$$\|\Phi\|_{\mathscr{S}} = \lambda(t^{[0,1]^*}[s^*(f_{\psi_i}), \ldots, s^*(f_{\psi_i})]) = \lambda(\|\Phi\|_{\mathscr{S}^*}) < 1.$$

Our claim is hence settled. $\square$

The following result is a direct consequence of Theorem 3.7 and Theorem 4.2.

**Corollary 4.3.** *Hyperstate models and state models share the same* **PFm** *tautologies.*

Let us now introduce a further class of models whose definition is inspired by the integral representation of states that we recalled in Subsection 3.1, namely, Theorem 3.2.

**Definition 4.4.** A *Borel model* $\mathscr{B}$ is a triple $(\mathbf{X}, v, \mu)$ where $\mathbf{X} = (X, \tau)$ is a compact Hausdorff space, $v$ is a Łukasiewicz valuation in the MV-algebra $\mathscr{C}(\mathbf{X})$ of continuous $[0,1]$-valued functions on $\mathbf{X}$ and $\mu$ is a regular Borel measure defined on the Borel subsets of $\mathbf{X}$.

For every formula $\Phi$ from **PFm**, its truth-value in a Borel model $\mathscr{B} = (\mathbf{X}, v, \mu)$, at the point $\mathbf{x} \in X$, is defined as follows where for every propositional formula $\psi$, $c_\psi : X \to [0,1]$ denotes the continuous function $v(\psi)$.

- If $\Phi = \psi$ is propositional, $\|\psi\|_{\mathscr{B},\mathbf{x}} = c_\psi(\mathbf{x})$;
- If $\Phi = P(\psi)$ is an atomic modal formula, then $\|P(\psi)\|_{\mathscr{B},\mathbf{x}} = \int_X c_\psi \, \mathrm{d}\mu$;
- If $\Phi = t[P(\psi_1), \ldots, P(\psi_k)]$ is a compound modal formula, the truth value $\|\Phi\|_{\mathscr{B},\mathbf{x}}$ is computed similarly to the case of state models:

$$\|t[P(\psi_1), \ldots, P(\psi_k)]\|_{\mathscr{B},\mathbf{x}} = t^{[0,1]}\left[\int_X c_{\psi_1} \, \mathrm{d}\mu, \ldots, \int_X c_{\psi_k} \, \mathrm{d}\mu\right].$$

Again, the truth-value, in $\mathscr{B}$, of a modal formula does not depend on the chosen $\mathbf{x}$, and hence we will omit such a subscript whenever it is not needed.

Let us recall the following result which has been proved in [24, Theorem 6] and provides an MV-analogous of the well-known Horn-Tarski extension theorem [23].

**Proposition 4.5.** *Let* **A** *and* **B** *be MV-algebras and let* **B** *be an MV-subalgebra of* **A**. *Every state* $s_B : B \to [0,1]$ *extends (not uniquely) to a state* $s_A : A \to [0,1]$.

Next result shows that the formulas from **PFm** that hold in all state models coincides with those holding in all Borel models.

**Proposition 4.6.** *State models and Borel models share the same* **PFm** *tautologies.*

**Proof.** Let $\Phi$ be a formula in **PFm** and $\mathscr{S} = (W, e, s)$ a state model such that $\|\Phi\|_{\mathscr{S}} < 1$. The MV-algebra $[0,1]^W$ is clearly semisimple. Therefore, from what we recalled in Section 3.1, there exists a compact Hausdorff space $\mathbf{X}$ such that $[0,1]^W$ is representable as a subalgebra of $\mathscr{C}(\mathbf{X})$. Let us call $\iota$ the embedding of $[0,1]^W$ into $\mathscr{C}(\mathbf{X})$. By Proposition 4.5 there exists a state $s' : \mathscr{C}(\mathbf{X}) \to [0,1]$ which extends $s$. Thus define $\mathscr{B} = (\mathbf{X}, v, \mu)$ where

- $v$ maps every propositional variable $q$ into the continuous function $\iota(f_q)$ of $\mathscr{C}(\mathbf{X})$;
- $\mu$ is the unique regular Borel measure on $\mathbf{X}$ as in Theorem 3.2.

Then, for every Łukasiewicz formula $\varphi$,

$$s(f_\varphi) = s'(\iota(f_\varphi)) = \int_{\mathbf{X}} \iota(f_\varphi) \mathrm{d}\mu.$$

Thus, $\|\Phi\|_{\mathscr{B}} = \|\Phi\|_{\mathscr{S}} < 1$.

Conversely, let $\Phi \in$ **PFm** and let $\mathscr{B} = (\mathbf{X}, v, \mu)$ be a Borel model such that $\|\Phi\|_{\mathscr{B}} < 1$. Then, consider the triple $\mathscr{S} = (W, e, s)$ where: $W = \mathbf{X}$, $e : W \times Var \to [0,1]_{MV}$ is such that for all $\mathbf{x} \in W$ and $q \in Var$, $e(\mathbf{x}, q) = c_q(\mathbf{x})$, and $s : \mathscr{C}(W) \to [0,1]$ is the state mapping each $c \in \mathscr{C}(W)$ to $\int_W c \mathrm{d}\mu$. Then, $\mathscr{S}$ is a state model. Moreover, from Theorem 3.2, for every Łukasiewicz formula $\psi$, $\|P(\psi)\|_{\mathscr{S}} = \|P(\psi)\|_{\mathscr{B}}$. Therefore, by truth functionality, $\|\Phi\|_{\mathscr{B}} = \|\Phi\|_{\mathscr{S}} < 1$. □

The following is hence a direct consequence of Proposition 4.6 and Theorem 4.2 above.

**Corollary 4.7.** *For every formula* $\Phi \in$ **PFm** *the following conditions are equivalent:*

(1) $\vdash_{FP} \Phi$;
(2) *For every Borel model* $\mathscr{B}$, $\|\Phi\|_{\mathscr{B}} = 1$.

## 5. On finite standard models and finite model property

In this final section we prove that the logic FP(Ł, Ł) has the finite model property. As the following definition points out, for *finite model* we mean a system which actually has a finite encoding.

**Definition 5.1.** A Borel model $\mathscr{B} = (\mathbf{X}, v, \mu)$ is said to be *finite* if:

(1) X is a finite set;
(2) $v$ evaluates propositional Łukasiewicz formulas as (necessarily continuous) functions of $X$ in a finite MV-chain $Ł_r$ for some natural number $r$;
(3) $\mu$ is a finite density function on $X$ which only takes value in a finite MV-chain $Ł_q$ for some natural number $q$.

Notice that every finite Borel model is a Borel model. Indeed, $X$ endowed with the discrete topology makes it a compact, Hausdorff and totally disconnected space. Observe that every finite Borel model has a finite encoding made of the cardinality of $X$, the natural numbers $r$ and $q$, and the finite set of rational numbers $\{\mu(x) \mid x \in X\}$.

Before proving that FP(Ł, Ł) enjoys the finite model property, let us recall a strengthening of Lemma 2.2 which deals with partial embeddings of any MV-chain into the rational MV-algebra $[0,1]_{MV}^{\mathbb{Q}} = \mathbb{Q} \cap [0,1]_{MV}$, the restriction of the standard MV-algebra to only rational numbers.

**Lemma 5.2.** *Every MV-chain* **A** *partially embeds into the rational MV-algebra* $[0,1]_{MV}^{\mathbb{Q}}$, *that is to say, for every MV-chain* **A** *and for every finite subset $X$ of $A$, there exists a map $\lambda^{\mathbb{Q}} : X \hookrightarrow_p [0,1]_{MV}^{\mathbb{Q}}$ which is injective and preserves all the partial operations of $X$.*

**Proof.** An immediate consequence of [7, Theorem 3.8 and Remark 4.12]. □

**Theorem 5.3.** *The logic FP(Ł, Ł) has the finite model property. That is to say, for every* **PFm**-*formula $\Phi$ such that $\nvdash_{FP} \Phi$, there exists a finite Borel model $\mathscr{B}$ such that $\|\Phi\|_{\mathscr{B}} < 1$.*

**Proof.** The claim can be proved following almost the same lines of the proof of Theorem 4.2. We will point out the main necessary modifications.

Let $\Phi = t[P(\psi_1), \ldots, P(\psi_k)] \in \mathbf{PFm}$ and let $\Delta, \mathbf{x}_1, \ldots, \mathbf{x}_t, \gamma_1, \ldots, \gamma_t, \hat{\gamma}_1, \ldots, \hat{\gamma}_t$ be as in the aforementioned proof. Assume that $\nvdash_{FP} \Phi$ and let $\mathscr{S}^* = (W, e, s^*)$ a hyperstate model such that $\|\Phi\|_{\mathscr{S}^*} < 1$. Thus, define $X_\Phi$, the subset of the MV-chain $[0,1]^*$, as in the proof of Theorem 4.2 and partially embed it into $[0,1]_{MV}^{\mathbb{Q}}$ via a map $\lambda^{\mathbb{Q}}$ as ensured by Lemma 5.2. Then, the values $\pi_j^{\mathbb{Q}} = \lambda^{\mathbb{Q}}(s^*(f_{\hat{\gamma}_j}))$ are rational numbers.

Let us define $\mathscr{B} = (X, e, \mu)$ as follows:

- $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_t\}$;
- For every propositional formula $\varphi$, let $m_\varphi^X$ be the restriction of the McNaughton function $m_\varphi$ to $X$. Then, for every $\mathbf{x}_j \in X$, $e(\mathbf{x}_j, \varphi) = m_\varphi^X(\mathbf{x}_j)$;
- For every $\mathbf{x}_j \in X$, $\mu(\mathbf{x}_j) = \pi_j^{\mathbb{Q}}$.

Notice that, with respect to evaluation map $e$, since $m_\varphi$ is piecewise linear and every $\mathbf{x}_j$ is has rational coordinates, each $m_\varphi^X(\mathbf{x}_j)$ is a rational number. Thus, for every $\mathbf{x}_j$, $e(\mathbf{x}_j, \cdot)$ is a Łukasiewicz evaluation into the finite MV-chain $Ł_r$ where

$$r = \mathrm{lcd}\{\mathrm{den}\, m_z(\mathbf{x}_j) \mid j = 1, \ldots, t,\ z \in Var(\psi_1, \ldots, \psi_k)\}.$$

Every $\pi_j^{\mathbb{Q}}$ is a rational number, thus put

$$q = \mathrm{lcd}(\mathrm{den}\, \pi_1^{\mathbb{Q}}, \ldots, \mathrm{den}\, \pi_t^{\mathbb{Q}}).$$

Then $\mu : X \to Ł_q$ and hence $\mathscr{B}$ is a finite Borel model. Further, for every $\psi_i$,

$$\|P(\psi_i)\|_{\mathscr{B}} = \sum_{j=1}^{t} m_{\psi_i}^X(\mathbf{x}_j) \cdot \pi_j^{\mathbb{Q}} = \lambda^{\mathbb{Q}}(s^*(f_{\psi_i}))$$

the latter equality being proved as in the proof of Theorem 4.2. Therefore, we conclude that

$$\|\Phi\|_{\mathscr{B}} = \lambda^{\mathbb{Q}}(\|\Phi\|_{\mathscr{S}^*}) < 1$$

showing that $\mathscr{B}$ is a finite countermodel of $\Phi$. □

We close this section with the following corollary which collects the main results obtained so far and, in particular, those from Corollary 4.3, Proposition 4.6 and Theorem 5.3.

**Corollary 5.4.** *Hyperstate models, state models, Borel models and finite Borel models share the same* **PFm** *tautologies.*

## 6. Conclusion

Proving standard completeness for the logic FP(Ł, Ł) has been a quite long-standing problem within the community of many-valued/fuzzy logicians and some ways to solve it have been proposed in the last years, [14, Ch. 6], [11,26]. A quite promising construction which could eventually lead to the desired conclusion, was presented in [12] but unfortunately it was grounded on an unsound claim.

In this paper we have presented a construction which does not make use of the false assertion of [12] and we proved that, indeed, FP(Ł, Ł) is sound and complete with respect to two classes of models: a first one based on Łukasiewicz states; a second one which considers regular Borel measures on compact Hausdorff spaces.

As welcome side-effect, we also proved that FP(Ł, Ł) has the finite model property.

Our future work in this direction will be mainly dedicated to apply and extend the construction presented here to prove a standard completeness theorem for a generalization of the logic FP(Ł, Ł), introduced in [20], whose language allows nested occurrences of the modality $P$ and *mixed* formulas like $\varphi \to P(\psi)$ where $\varphi$ and $\psi$ are Łukasiewicz events. That logic, denoted by SFP(Ł, Ł) in [20], in contrast to the case of FP(Ł, Ł), is algebraizable and its equivalent algebraic semantics is the variety $\mathbb{SMV}$ of *MV-algebras with internal state*, or SMV-algebras for short.

## Declaration of competing interest

We wish to confirm that there are no known conflicts of interest associated with this publication and there has been no significant financial support for this work that could have influenced its outcome.

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