Delay system identification using permutation entropy and statistical complexity: resonance-like behavior in a noise environment

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Abstract. - In this Letter a novel approach to identify delay phenomena in noisy time series is introduced. We show that it is possible to perform a reliable time delay identification by using quantifiers derived from information theory, more precisely, permutation entropy and statistical complexity. These quantifiers show clear extrema when the embedding delay τ matches the characteristic time delay τ_S of the system. Numerical data originating from a time delay system based on the well-known Mackey-Glass equations operating in the chaotic regime were used as test beds. We demonstrate that our method is straightforward to apply and robust to additive observational and dynamical noise. In particular, we find that the identification of the time delay is even more efficient in a noise environment. We discuss the sources of this particular *noise-induced phenomenon.*

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Introduction. – When studying dynamical phenomena in nature the corresponding underlying equations are often not known. In fact the starting point to study many of these systems is a set of measurements of some representative variable of interest at discrete time intervals, i.e. a *black box* time series, given by the set $S = \{x_t, t = 1, \ldots, N\}$, with N being the number of observations. An important problem in the analysis of time series data is the identification of delay systems and the corresponding delay times since delay phenomena are intrinsic to many dynamical processes. The identified delay gives information about the interaction between the system components. It is then necessary to discriminate the presence of time delays in order to develope suitable models for simulation and forecasting purposes. Time delayed dynamics are naturally required and implemented to model real systems in different fields including biology [1–3], optics [4–6] and climatology [7] among others. Therefore, the identification from a time series of a possible delay present in the system has become one of the key problems in the study of nonlinear dynamical systems.

Numerous approaches were previously proposed to determine the unknown delay time τ_S from recorded time series. Without being exhaustive we can mention the *singular value fraction* measure [8], several methods from information theory [9–11], the *filling factor analysis* introduced by Bünner *et al.* [12], the practical criterion more recently proposed by Siefert [13], and the lagged detrended fluc-

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tuation analysis (DFA) very recently adapted by Alvarez-Ramirez *et al.* [14]. In this Letter we introduce a new approach by using quantifiers derived from information theory, more precisely entropy and statistical complexity. It should be stressed that, in order to evaluate these quantifiers, a particularly efficient symbolic technique, the Bandt and Pompe method [15], is used to estimate the probability distribution associated to the time series under study. As it is widely known, symbolic time series analysis methods that discretize the raw time series into a corresponding sequence of symbols have the potential of analyzing nonlinear data efficiently with low sensitivity to noise [16]. However, finding a meaningful symbolic representation of the original series is not an easy task [17]. To our knowledge, the Bandt and Pompe approach is the only symbolization technique among those in popular use that takes into account time causality of the system's dynamics. Then, important details concerning the ordinal structure of the time series are revealed [18–24].

As will be discussed in detail below, we have found that the permutation entropy is minimized and the statistical complexity maximized when the embedding delay τ of the symbolic reconstruction matches the intrinsic time delay τ_S of the system. The reliability of our methodology is tested using numerical time series obtained from the widely used Mackey-Glass equation subject to a time delay, operating in a chaotic regime. The main advantages of our quantifiers are their simplicity and robustness to noise. Most importantly, we have found a resonance-like behavior in the presence of observational and dynamical noise; i.e. the identification of the time delay is improved in a noise environment. The reasons behind this *noise-induced phenomenon* are discussed below.

Permutation entropy and statistical complexity measure. – The information content of a system is typically evaluated from a probability distribution function, P, describing the distribution of some measurable or observable quantity. An information measure can primarily be viewed as a quantity that characterizes this given probability distribution. Shannon entropy is very often used as a first natural approach. Given any arbitrary probability distribution $P = \{p_i : i = 1, \ldots, M\}$, the widely known Shannon's logarithmic information measure defined by

$$S[P] = -\sum_{i=1}^{M} p_i \ \ln p_i \tag{1}$$

is regarded as the measure of the uncertainty associated to the physical processes described by P. If $S[P] = 0$ we are in position to predict with complete certainty which of the possible outcomes i whose probabilities are given by p_i will actually take place. Our knowledge of the underlying process described by the probability distribution is maximal in this instance. In contrast, our knowledge is minimal for a uniform distribution.

It is widely known that an entropy measure does not quantify the degree of structure or patterns present in a process [25]. Moreover, it was recently shown that measures of statistical or structural complexity are necessary because they capture the property of organization [26]. This kind of information is not discriminated by randomness measures. The opposite extremes of perfect order and maximal randomness (a periodic sequence and a fair coin toss, for example) possess no complex structure, then these systems are too simple and should have zero statistical complexity. At a given distance from these extremes, a wide range of possible degrees of physical structure exists, that should be quantified by the statistical complexity measure. Lamberti *et al.* [27] introduced an effective statistical complexity measure (SCM) that is able to detect essential details of the dynamics and differentiate different degrees of periodicity and chaos. It provides important additional information regarding the peculiarities of the underlying probability distribution, not already detected by the entropy. This statistical complexity measure is defined, following the intuitive notion advanced by López-Ruiz *et al.* [28], through the product

$$\mathcal{C}_{JS}[P] = \mathcal{Q}_J[P, P_e] \ \mathcal{H}_S[P] \tag{2}$$

of the normalized Shannon entropy

$$\mathcal{H}_S[P] = S[P]/S_{\max} \tag{3}$$

with $S_{\max} = S[P_e] = \ln M$, $(0 \leq \mathcal{H}_S \leq 1)$ and $P_e = \{1/M, \ldots, 1/M\}$ the uniform distribution, and the disequilibrium $\mathcal{Q}_J$ defined in terms of the extensive (in the thermodynamical sense) Jensen-Shannon divergence. That is,

$$\mathcal{Q}_J[P, P_e] = \mathcal{Q}_0 \mathcal{J}[P, P_e] \tag{4}$$

with $\mathcal{J}[P, P_e] = \{S[(P + P_e)/2] - S[P]/2 - S[P_e]/2\}$ the above-mentioned Jensen-Shannon divergence and $\mathcal{Q}_0$ a normalization constant, equal to the inverse of the maximum possible value of $\mathcal{J}[P, P_e]$. This value is obtained when one of the component of P, say p_m, is equal to one and the remaining p_i are equal to zero. The complexity measure constructed in this way is intensive, as many thermodynamic quantities [27]. We stress the fact that the above SCM is not a trivial function of the entropy because it depends on two different probabilities distributions, the one associated to the system under analysis, P, and the uniform distribution, P_e. Furthermore, it was shown that for a given $\mathcal{H}_S$ value, there exists a range of possible SCM values [29]. Thus, it is clear that important additional information related to the correlational structure between the components of the physical system is provided by evaluating the statistical complexity.

In order to evaluate the two above-mentioned quantifiers, $\mathcal{H}_S$ and $\mathcal{C}_{JS}$, an associated probability distribution should be constructed beforehand. The adequate way of choosing the probability distribution associated to a time series is an open problem. Rosso *et al.* [18] have recently shown that improvements in the performance of information quantifiers, like entropy and statistical complexity measures, can be expected, if the time causality of the system dynamics is taken into account when computing the underlying probability distribution. Specifically, it was found that these information measures allow to distinguish between chaotic and stochastic dynamics when causal information is incorporated into the scheme to generate the associated probability distribution. Bandt and Pompe [15] introduced a successful method to evaluate the probability distribution considering this time causality. They suggested that the symbol sequence should arise naturally from the time series, without any model assumptions. Thus, they took partitions by comparing the order of neighboring values rather than partitioning the amplitude into different levels. That is, given a time series $\{x_t, t = 1, \ldots, N\}$, an embedding dimension $D > 1$, and an embedding delay time τ, the ordinal pattern of order D generated by

$$s \mapsto \left(x_{s-(D-1)\tau}, x_{s-(D-2)\tau}, \ldots, x_{s-\tau}, x_s\right) \qquad (5)$$

has to be considered. To each time s we assign a D-dimensional vector that results from the evaluation of the time series at times $s-(D-1)\tau, \ldots, s-\tau, s$. Clearly, the higher the value of D, the more information about the past is incorporated into the ensuing vectors. By the ordinal pattern of order D related to the time s we mean the permutation $\pi = (r_0, r_1, \cdots, r_{D-1})$ of $(0, 1, \cdots, D-1)$ defined by

$$x_{s-r_0\tau} \geq x_{s-r_1\tau} \geq \cdots \geq x_{s-r_{D-2}\tau} \geq x_{s-r_{D-1}\tau}. \qquad (6)$$

In this way the vector defined by Eq. (5) is converted into a unique symbol π. The procedure can be better illustrated with a simple example; let us assume that we start with the time series $\{1, 3, 5, 4, 2, 5, \ldots\}$, and we set the embedding dimension $D = 4$ and the embedding delay $\tau = 1$. In this case the state space is divided into 4! partitions and 24 mutually exclusive permutation symbols are considered. The first 4-dimensional vector is $(1, 3, 5, 4)$. According to Eq. (5) this vector corresponds with $(x_{s-3}, x_{s-2}, x_{s-1}, x_s)$. Following Eq. (6) we find that $x_{s-1} \geq x_s \geq x_{s-2} \geq x_{s-3}$. Then, the ordinal pattern which allows us to fulfill Eq. (6) will be $(1, 0, 2, 3)$. The second 4-dimensional vector is $(3, 5, 4, 2)$, and $(2, 1, 3, 0)$ will be its associated permutation, and so on. In order to get a unique result we consider that $r_i < r_{i-1}$ if $x_{s-r_i\tau} = x_{s-r_{i-1}\tau}$. This is justified if the values of x_t have a continuous distribution so that equal values are very unusual. Otherwise, it is possible to break these equalities by adding small random perturbations. Thus, for all the $D!$ possible permutations π_i of order D, their associated relative frequencies can be naturally computed by the number of times this particular order sequence is found in the time series divided by the total number of sequences. Thus, an ordinal pattern probability distribution $P = \{p(\pi_i), i = 1, \ldots, D!\}$ is obtained from the time series. To determine $p(\pi_i)$ exactly an infinite number of terms in the time series should be considered, i.e., $N \to \infty$ to determine the relative frequencies. This limit exists with probability 1 when the underlying stochastic process fulfills a very weak stationarity condition: for $k \leq D$, the probability for $x_t < x_{t+k}$ should not depend on t [15]. The probability distribution P is obtained once we fix the embedding dimension D and the embedding delay time τ. The former parameter plays an important role for the evaluation of the appropriate probability distribution, since D determines the number of accessible states, given by $D!$. Moreover, it was established [30] that the length N of the time series must satisfy the condition $N \gg D!$ in order to obtain a reliable statistics. In particular, Bandt & Pompe suggest for practical purposes to work in the range $3 \leq D \leq 7$. With respect to the selection of the other parameter, these authors specifically considered an embedding delay $\tau = 1$ in their cornerstone paper [15]. Nevertheless, it is clear that other values of τ could provide additional information. We consider that this way of symbolizing time series, based on a comparison of consecutive points, is more robust under noise allowing a more accurate empirical reconstruction of the underlying phase space.

In this work we evaluate the normalized Shannon entropy, $\mathcal{H}_S$ (Eq. (3)), and the SCM, $\mathcal{C}_{JS}$ (Eq. (2)), using the permutation probability distribution, $P = \{p(\pi_i), i = 1, \ldots, D!\}$. Defined in this way, the former quantifier is called permutation entropy and quantifies the diversity of possible ordering of the successively observed values of a time series just as Shannon entropy quantifies the diversity of the values themselves [31].

Numerical results and discussion. – To estimate the quantifiers, permutation entropy and statistical complexity, it is necessary to fix the embedding dimension and the embedding delay. It is clear that the condition $N \gg D!$ limits the possible values for the embedding dimension. However, a study about the influence of the embedding delay is still lacking. We hypothesize that this parameter could be strongly related, if it is relevant, with the intrinsic time delay of the system under analysis. In order to check this hypothesis we have estimated the permutation entropy and statistical complexity as a function of the embedding delay τ for the well-known Mackey-Glass equation, a paradigmatic time delay system. We consider the following model equation for the Mackey-Glass oscillator [1]:

$$\frac{dx}{dt} = -x + a\frac{x(t-\tau_S)}{1 + x^c(t-\tau_S)} \qquad (7)$$

with t being a dimensionless time, τ_S the time delay feedback, a the feedback strength and c the degree of nonlinearity. In particular, we choose the typical values $a = 2$, $c = 10$ and $\tau_S = 60$ for which the system operates in a chaotic regime. Time series were numerically integrated by using the Heun's method (also called the modified Euler's method) [32] with an integration step $\Delta t = 0.01$ and sampling step $\delta t = 0.2$ time units/sample. We analyzed time series with 10^6 data points (the total integration time was $2 \cdot 10^5$ time units).

In Fig. 1 we plot the normalized permutation entropy,

$\mathcal{H}_S$, and the intensive SCM, $\mathcal{C}_{JS}$, as a function of the embedding delay τ for different embedding dimensions ($4 \leq D \leq 8$). It can be clearly observed that these quantifiers have sharp and well-defined minima and maxima, respectively, when the embedding delay τ of the symbolic reconstruction is very close to the intrinsic time delay τ_S of the system, i.e. for τ near 300 ($\tau_S/\delta t = 300$). The slight difference is attributed to the internal response time or inertia of the Mackey-Glass system. It can be seen from Figs. 1b) and c) that this result is independent of the embedding dimension value. The best discrimination is obtained for the largest value of D. By increasing the length and the number of symbols, i.e. by increasing the embedding dimension D, more information is being included when estimating any quantifier. Thus, it is reasonable that a better detection can be achieved with higher embedding dimensions. It is worth noting that there are other minima and maxima for the permutation entropy and statistical complexity, respectively, but being less pronounced. These other peaks correspond to *harmonics* and *subharmonics* of the system's time delay τ_S. Remarkably, in the case of the statistical complexity, the amplitude of the peak associated to the delay of the system is even larger than the amplitude for an embedding delay $\tau = 1$, as can be seen in Fig. 1a). We attribute this particular behavior to a reinforcement of the system delay effect associated with the special way of choosing the delay embedding sequence. According to the results shown in Fig. 1a), we conclude that the statistical complexity identifies the system delay better than the permutation entropy because the contrast with the base line is higher. It was recently shown that, in some cases [33, 34], the statistical complexity can be a particularly useful and efficient information theoretical quantifier. Based on these previous conclusions, from now on, we continue the analysis by considering that the statistical complexity $\mathcal{C}_{JS}$ with embedding dimension $D = 8$ is the best quantifier to reach the goal of identifying the system's time delay under study.

Our next goal is to quantify the effect an observational additive noise has on the proposed approach. Since experimental time series are naturally affected by a certain amount of observational noise it is important to check the performance of our approach in the case of noisy time series. For this purpose a Gaussian white noise [1] was added to the original Mackey-Glass simulated time series. Different noise levels (NL) from 0.05 to 1, defined by the standard deviation of the noise divided by the standard deviation of the original signal [35], were considered. Ten independent realizations were taken into account in order to have better statistics. Figure 2 shows the performance of $\mathcal{C}_{JS}$ for $D = 8$ in the region of interest, that is around $\tau = 300$. It can be clearly seen that our approach is very robust under the noise influence.

In order to better measure this effect, we have estimated the ratio between the amplitude at the peak and the mean

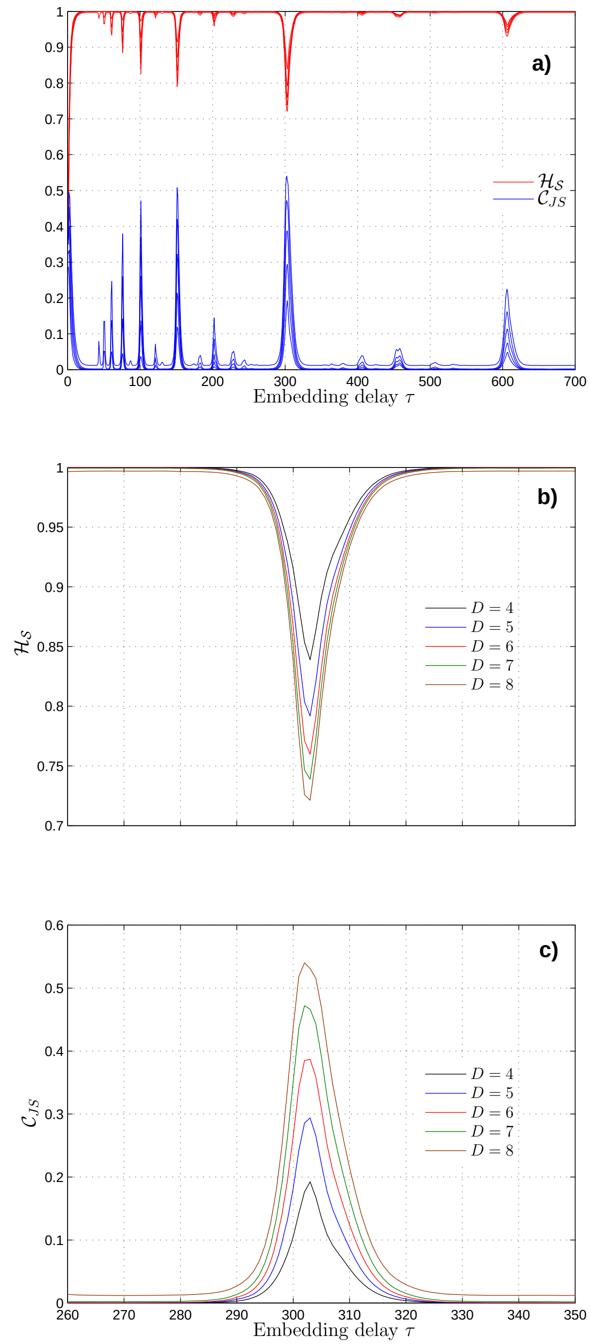

Fig. 1: a) Permutation entropy $\mathcal{H}_S$ and statistical complexity $\mathcal{C}_{JS}$ as a function of the embedding delay τ for embedding dimensions $4 \leq D \leq 8$. Enlargement near the time delay τ_S of the system in order to see more clearly the effect of the embedding dimension on the $\mathcal{H}_S$ (b)) and $\mathcal{C}_{JS}$ (c)) estimations.

value of the background (the usual signal-to-noise ratio). The results are shown in Fig. 3. The resulting plot displays a clear maximum of the ratio $\rho = \mathcal{C}_{JS}^{\text{peak}}/\mathcal{C}_{JS}^{\text{back}}$ at an intermediate noise level near 0.2. This value can be considered as the optimal amount of observational noise

[1] Time series were generated by using the Matlab function *randn*.

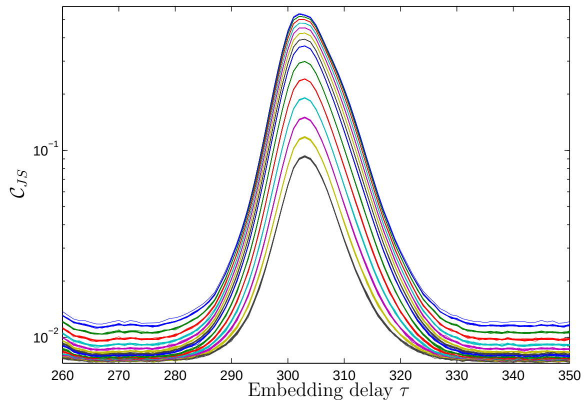

Fig. 2: Semi-logarithmic plot of the statistical complexity $\mathcal{C}_{JS}$ as a function of the embedding delay τ for different levels of observational noise. The noise level associated with the different curves increases from top to bottom. The embedding dimension $D = 8$. Ten independent realizations for each noise level are plotted. Since the dispersion is very small, the differences between these ten lines are hardly distinguishable.

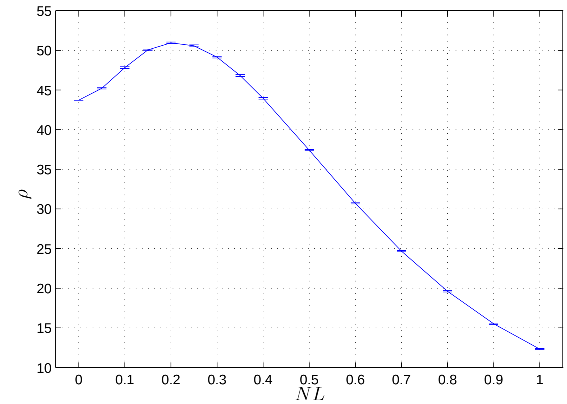

Fig. 3: Ratio $\rho = \mathcal{C}_{JS}^{\mathrm{peak}}/\mathcal{C}_{JS}^{\mathrm{back}}$ as a function of the static noise level. The embedding dimension was fixed equal to 8. Note that the maximum ratio is obtained for a value of NL close to 0.2. Error bars indicate standard deviations from 10 independent realizations.

for the time delay identification purpose. It is worth noting that according to these results the identification of the time delay is more reliable in the presence of observational noise in the range $0 < NL < 0.4$. A similar resonance-like behavior was recently found by Staniek and Lehnertz [20]. These authors analyzed the influence of a static (observational) noise in the detectability of directional coupling by estimating a symbolic transfer entropy. The ratio of the directionality indices for noisy and noise-free time series in a numerical example displays an analogous behavior (see for instance Fig. 3 of Ref. [20]). More importantly, the same symbolic technique, namely the Bandt and Pompe methodology, was adopted to estimate this quantifier.

With the aim of studying also the effect of a dynamical noise, we have simulated the Mackey-Glass system (Eq. 7) including an additive Gaussian white noise term of zero mean and correlation $\mathcal{D}$. Langevin forces of different strengths $\mathcal{D}$ were considered. The results obtained for the ratio $\rho = \mathcal{C}_{JS}^{\mathrm{peak}}/\mathcal{C}_{JS}^{\mathrm{back}}$ as a function of different noise strengths are shown in Fig. 4. A resonance-like behavior is also observed which indicates a better performance of the quantifier in the presence of noise. A significant maximum for ρ is found when $\mathcal{D}$ is near 0.15.

We speculate that the interplay between the deterministic noise associated to the chaotic dynamic of the system and the added, observational or dynamical, stochastic noise is the source of this *noise-induced phenomenon*. The added stochastic noise has a stronger influence on the chaotic background than on the peak height. More precisely, the decrease of the noisy background is more important that the reduction of the peak height. This kind of *noise-induced chaos reduction* has been introduced before by Revelli [36] but in a totally different context.

Conclusions. – Delay phenomena are of considerable practical importance. Thus, time delay identification from experimental time series within an inherent noise environment is, nowadays, an important challenge. In this Letter we introduce a new reliable and simple approach to perform this task. Two different information theory quantifiers, estimated by using an efficient symbolic technique, namely the normalized permutation entropy and the statistical complexity are able to reveal the presence of a time delay in the standard well-known Mackey-Glass system. Moreover, it has been shown that the latter quantifier is more sensitive than the entropy quantifier. By analyzing the influence of additive observational and dynamical noise we found a *noise-induced phenomenon*: the time delay identification can be enhanced by the presence of noise. A more in depth analysis for gaining insights into the nature of such mechanisms together with *real* experimental testing will be the goals of a next study.

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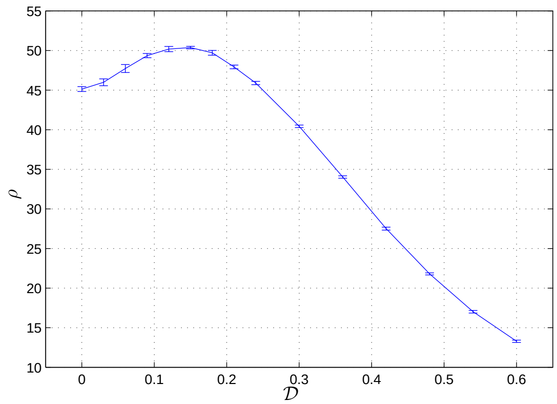

Fig. 4: Ratio $\rho = \mathcal{C}_{JS}^{\mathrm{peak}}/\mathcal{C}_{JS}^{\mathrm{back}}$ as a function of the dynamical noise level. The embedding dimension was fixed equal to 8. Observe the maximum obtained for a Langevin force $\mathcal{D}$ of intensity near 0.15. Error bars indicate standard deviations from 10 independent realizations.

REFERENCES

[1] Mackey M. C. and Glass L., *Science* , **197** (1977) 287.
[2] May R. M., Beddington J. R., Clark C. W., Holt S. J. and La R. M., *Science* , **205** (1979) 267.
[3] Gurney W. S. C., Blythe S. P. and Nisbet R. M., *Nature* , **287** (1980) 17.
[4] Ikeda K., *Opt. Commun.* , **30** (1979) 257.
[5] Lang R. and Kobayashi K., *IEEE J. Quantum Electron.* , **16** (1980) 347.
[6] Ikeda K. and Matsumoto K., *Physica D* , **29** (1987) 223.
[7] Tziperman E., Stone L., Cane M. A. and Jarosh H., *Science* , **264** (1994) 72.
[8] Fowler A. C. and Kember G., *Phys. Lett. A* , **175** (1993) 402.
[9] Tian Y.-C. and Gao F., *Physica D* , **108** (1997) 113.
[10] Lehrman M., Rechester A. B. and White R. B., *Phys. Rev. Lett.* , **78** (1997) 54.
[11] Azad R. K., Rao J. S. and Ramaswamy R., *Chaos, Solitons & Fractals* , **14** (2002) 633.
[12] Bünner M. J., Kittel A., Parisi J., Fischer I. and Elsässer W., *Europhys. Lett.* , **42** (1998) 353.
[13] Siefert M., *Phys. Rev. E* , **76** (2007) 026215.
[14] Alvarez-Ramirez J., Rodriguez E. and Echeverria J. C., *Phys. Rev. E* , **79** (2009) 057202.
[15] Bandt C. and Pompe B., *Phys. Rev. Lett.* , **88** (2002) 174102.
[16] Finn J. M., Goettee J. D., Toroczkai Z., Anghel M. and Wood B. P., *Chaos* , **13** (2003) 444.
[17] Daw C. S., Finney C. E. A. and Tracy E. R., *Rev. Sci. Instrum.* , **74** (2003) 915.
[18] Rosso O. A., Larrondo H. A., Martín M. T., Plastino A. and Fuentes M. A., *Phys. Rev. Lett.* , **99** (2007) 154102.
[19] Rosso O. A., Zunino L., Pérez D. G., Figliola A., Larrondo H. A., Garavaglia M., Martín M. T. and Plastino A., *Phys. Rev. E* , **76** (2007) 061114.
[20] Staniek M. and Lehnertz K., *Phys. Rev. Lett.* , **100** (2008) 158101.
[21] Bahraminasab A., Ghasemi F., Stefanovska A., McClintock P. V. E. and Kantz H., *Phys. Rev. Lett.* , **100** (2008) 084101.
[22] Zunino L., Pérez D. G., Martín M. T., Garavaglia M., Plastino A. and Rosso O. A., *Phys. Lett. A* , **372** (2008) 4768.
[23] Rosso O. A. and Masoller C., *Phys. Rev. E* , **79** (2009) 040106(R).
[24] Matilla-García M., Rodríguez J. M. and Ruiz Marín M., *J. Time Ser. Anal.* , **31** (2010) 76.
[25] Feldman D. P. and Crutchfield J. P., *Phys. Lett. A* , **238** (1998) 244.
[26] Feldman D. P., McTague C. S. and Crutchfield J. P., *Chaos* , **18** (2008) 043106.
[27] Lamberti P. W., Martín M. T., Plastino A. and Rosso O. A., *Physica A* , **334** (2004) 119.
[28] López-Ruiz R., Mancini H. L. and Calbet X., *Phys. Lett. A* , **209** (1995) 321.
[29] Martín M. T., Plastino A. and Rosso O. A., *Physica A* , **369** (2006) 439.
[30] Staniek M. and Lehnertz K., *Int. J. of Bifurcation and Chaos* , **17** (2007) 3729.
[31] Amigó J. M. and Kennel M. B., *Physica D* , **231** (2007) 137.
[32] Gard T. C., *Introduction to stochastic differential equations* (Marcel Dekker) 1988.
[33] Sañudo J. and López-Ruiz R., *Phys. Lett. A* , **372** (2008) 5283.
[34] Moustakidis C. C., Psonis V. P., Chatzisavvas K. C., Panos C. P. and Massen S. E., *Phys. Rev. E* , **81** (2010) 011104.
[35] Ramdani S., Bouchara F. and Lagarde J., *Chaos* , **19** (2009) 013123.
[36] Revelli J. A., Rodriguez M. A. and Wio H. S., *Physica A* , **387** (2008) 3103.