Versatile Atomic Magnetometry Assisted by Bayesian Inference

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Quantum sensors typically translate external fields into a periodic response whose frequency is then determined by analyses performed in Fourier space. This allows for a linear inference of the parameters that characterize external signals. In practice, however, quantum sensors are able to detect fields only in a narrow range of amplitudes and frequencies. A departure from this range, as well as the presence of significant noise sources and short detection times, lead to a loss of the linear relationship between the response of the sensor and the target field, thus limiting the working regime of the sensor. Here we address these challenges by means of a Bayesian inference approach that is tolerant to strong deviations from desired periodic responses of the sensor and is able to provide reliable estimates even with a very limited number of measurements. We demonstrate our method for an $^{171}\text{Yb}^+$ trapped-ion quantum sensor but stress the general applicability of this approach to different systems.

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I. INTRODUCTION

Achieving efficient magnetometry is of considerable importance in a broad range of areas of fundamental and applied science [1,2]. Nuclear magnetic resonance (NMR) techniques [3,4], which led to important applications such as NMR spectroscopy [5], magnetic resonance imaging [6], and their recent extensions to the nanoscale [7–9], are specific examples that depend crucially on accurate and efficient magnetometry techniques. Other remarkable applications include magnetic force microscopy [10], which allows the scanning of thin materials for—to throw an example—magnetic recording [11] and may achieve a spatial resolution of the order of tens of nanometers. A new generation of devices that exploit quantum properties to characterize weak electromagnetic signals are superconducting quantum interference devices (SQUIDS) [12]. These possess excellent magnetic sensitivity and have dimensions ranging from microns [13] to tens of nanometers in the case of nano-SQUIDS [14]. In this spirit, atomic-size sensors such as $^{171}\text{Yb}^+$ [15–17] and

$^{40}\text{Ca}^+$ [18] trapped ions, or nitrogen-vacancy centers in diamond [19–21] achieve ultimate size limits for quantum sensors.

Especially interesting is the case of quantum sensors based on $^{171}\text{Yb}^+$ ions that we use as a testbed for our protocol. This ion species encodes the degrees of freedom of the sensor in its $^2S_{1/2}$ spin manifold whose hyperfine levels present a negligible spontaneous emission rate [22]. The latter makes the $^{171}\text{Yb}^+$ ion an ideal atomic-size quantum sensor if properly stabilized against decoherence using dynamical decoupling (DD) methods [23–35]. In particular, owing to its resilience against environmental errors and amplitude fluctuations on the microwave (MW) control, the DD scheme leading to the *dressed state qubit* has been used for quantum information processing [15,17] and quantum sensing [16]. Despite this robustness and in close similarity with other sensing techniques, the dressed state qubit approach is restricted to a narrow range in the amplitudes and frequencies of the target electromagnetic signals. A departure from this regime significantly distorts the sensor response and thus makes impossible a direct linear inference of the external field parameters via, e.g., standard fast Fourier transform (FFT) methods.

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In this article, we present a method that combines DD techniques to stabilize the quantum sensor with Bayesian inference schemes [36,37], which enables the accurate estimation of external field parameters from a complex sensor response. This results in a versatile quantum sensing strategy that permits the reconstruction of electromagnetic signals in a wide parameter range, with a minimal previous knowledge of the signal features, and in realistic scenarios involving noise over the sensor and a low number of measurements. As an example, we consider a $^{171}\text{Yb}^+$ ion and demonstrate that Bayesian inference shows a superior performance over standard analysis techniques, such as FFT and least-squares fits. We stress that our method can be adapted to other atomic-size sensors such as $^{40}\text{Ca}^+$ trapped ions or nitrogen-vacancy centers in diamond.

II. QUANTUM SENSOR

We start describing the main features of our quantum sensor device. The $^2S_{1/2}$ manifold of the $^{171}\text{Yb}^+$ ion comprises four hyperfine levels named $|0\rangle, |\dot{0}\rangle, |1\rangle$, and $|-1\rangle$. In an external static magnetic field B_z , the degeneracy of the $|\dot{0}\rangle, |1\rangle, |-1\rangle$ spin levels is removed, leading to the diagonal Hamiltonian $H_0 = \omega_{\dot{0}}|\dot{0}\rangle\langle\dot{0}| + \sum_{j=-1}^1 \omega_j |j\rangle\langle j|$, with $\omega_{\pm 1} = A/4 \pm (\gamma_e - \gamma_n)B_z/2$, and $\omega_0 = -\omega_{\dot{0}} - A/2 = -3A/4 - [(\gamma_e + \gamma_n)^2/4A]B_z^2$, where $A = (2\pi) \times 12.643$ GHz [22] and γ_e, γ_n are the electronic and nuclear gyromagnetic ratios. We refer the reader to the Supplemental Material (SM) [38], which includes Refs. [39,40], for a detailed derivation of H_0 and its spectrum. Under a set of control MW drivings, the $^{171}\text{Yb}^+$ ion Hamiltonian reads

$$H = H_0 + \mu(t)(|1\rangle\langle 1| - |-1\rangle\langle -1|) + \sum_j \tilde{\Omega}_j [|1\rangle\langle \dot{0}| - |1\rangle\langle 0| + |\dot{0}\rangle\langle -1| + |0\rangle\langle -1| + \text{H.c.}], \quad (1)$$

where $\tilde{\Omega}_j = \Omega_j \cos(\omega_j t + \phi_j)$ denotes the frequency ω_j , phase ϕ_j , and Rabi frequency Ω_j of the j th MW driving, and $\mu(t)$ accounts for fluctuations leading to loss of quantum coherence on the magnetically sensitive levels $|1\rangle$ and $|-1\rangle$ [38].

A. Refined atomic-size sensor

To stabilize the quantum sensor, one has to remove the impact of magnetic field fluctuations from the dynamics, i.e., the term $\mu(t)(|1\rangle\langle 1| - |-1\rangle\langle -1|)$ in Eq. (1). To this end, we tune one of the MW controls in resonance with the $|0\rangle \leftrightarrow |1\rangle$ hyperfine transition, while the other MW control resonates with $|0\rangle \leftrightarrow |-1\rangle$. Now, a target electromagnetic field (or signal) can be detected by using either the

transition $|\dot{0}\rangle \leftrightarrow |1\rangle$ or $|\dot{0}\rangle \leftrightarrow |-1\rangle$. Note that a target signal induces the term $\Omega_{\text{tg}} \cos(\omega_{\text{tg}} t + \phi_{\text{tg}})[|1\rangle\langle \dot{0}| - |1\rangle\langle 0| + |\dot{0}\rangle\langle -1| + |0\rangle\langle -1| + \text{H.c.}]$ in Eq. (1).

The standard procedure to estimate Ω_{tg} is illustrated in Ref. [16]. This assumes the target field to be on resonance with the $|\dot{0}\rangle \leftrightarrow |1\rangle$ transition (that is, $\omega_{\text{tg}} = \omega_1 - \omega_{\dot{0}}$), leading to [38]

$$H = -\frac{\mu(t)}{\sqrt{2}}(|D\rangle\langle u| + |D\rangle\langle d| + \text{H.c.}) + \frac{\Omega}{\sqrt{2}}(|u\rangle\langle u| - |d\rangle\langle d|) + \frac{\Omega_{\text{tg}}}{4}(|u\rangle\langle \dot{0}| + |d\rangle\langle \dot{0}| - \sqrt{2}|D\rangle\langle \dot{0}| + \text{H.c.}).$$

The new basis $\{|u\rangle, |d\rangle, |D\rangle, |\dot{0}\rangle\}$ is $|u\rangle = (1/\sqrt{2})(|B\rangle + |0\rangle)$, $|d\rangle = (1/\sqrt{2})(|B\rangle - |0\rangle)$, $|D\rangle = (1/\sqrt{2})(|-1\rangle - |1\rangle)$, $|\dot{0}\rangle = |\dot{0}\rangle$, with $|B\rangle = (1/\sqrt{2})(|1\rangle + |-1\rangle)$ [15–17]. The noisy term $-(\mu(t)/\sqrt{2})(|D\rangle\langle u| + |D\rangle\langle d| + \text{H.c.})$ can be removed since, in the rotating frame defined by the operator $(\Omega/\sqrt{2})(|u\rangle\langle u| - |d\rangle\langle d|)$, it rotates at a speed $\propto \Omega$ that allows one to apply the rotating wave approximation (RWA). Analogously, the terms $-(\mu(t)/\sqrt{2})(|D\rangle\langle u| + |D\rangle\langle d| + \text{H.c.})$ and $(\Omega_{\text{tg}}/4)(|u\rangle\langle \dot{0}| + |d\rangle\langle \dot{0}| + \text{H.c.})$ average out by invoking the RWA if $\Omega_{\text{tg}} \ll \Omega$. One thus finds that

$$H = -\frac{\Omega_{\text{tg}}}{2\sqrt{2}}(|D\rangle\langle \dot{0}| + |\dot{0}\rangle\langle D|). \quad (2)$$

Equation (2) induces Rabi oscillations between $|D\rangle$ and $|\dot{0}\rangle$ at a rate $\propto \Omega_{\text{tg}}$. This allows one to find the amplitude of the electromagnetic signal Ω_{tg} by monitoring, e.g., the population $P_D(t)$ of state $|D\rangle$ at a time t . In particular, from Eq. (2) and for $P_D(0) = 1$, one finds that

$$P_D(t) = \cos^2(\pi t/t_R) \quad (3)$$

with $t_R = 2\pi\sqrt{2}/\Omega_{\text{tg}}$. An example of this purely oscillatory response of the sensor is given in Fig. 1(a). However, a departure from the regime leading to Eq. (2) induces significant deviations with respect to the periodic behavior predicted by Eq. (3). An example of such deviations is given in Fig. 1(b). As we will see later, this challenges the estimation of Ω_{tg} .

A rigorous treatment of the Hamiltonian in Eq. (1) leads to a more involved expression. The resulting Hamiltonian is denoted by H_r and reproduced in the Appendix for completeness, Eq. (A1), while we refer the reader to the SM [38] for further details of the derivation of Eq. (A1). The Hamiltonian H_r is our refined model that describes the quantum sensor dynamics in a wide parameter regime. In particular, H_r exhibits a nontrivial dependence on Ω_{tg} , as

FIG. 1. Dynamical behavior of the population $P_D(t)$ with $P_D(0) = 1$, $\Omega = 2\pi \times 37.27$ kHz as in Ref. [16], and when interacting with a rf signal with (a) $\Omega_{\text{tg}} = 2\pi \times 1$ kHz, $\omega_{\text{tg}} = 2\pi \times 14$ MHz ($B_z = 1$ mT), and $\xi = 0$, or (b) $\Omega_{\text{tg}} = 2\pi \times 8$ kHz, $\omega_{\text{tg}} = 2\pi \times 5.6$ MHz ($B_z = 0.4$ mT) with $\xi = 2\pi \times 0.25$ kHz. The solid (black) line corresponds to the realistic signal, obtained using H_r from Eq. (A1), while the dashed (red) line is obtained upon various approximations as shown in Eq. (3) with $t_R = 2\pi\sqrt{2}/\Omega_{\text{tg}}$. Note the significant deviations with respect to the Rabi oscillations in (b).

well as on the detuning ξ of the signal with respect to the resonant condition, i.e., $\omega_{\text{tg}} = \omega_1 - \omega_0 + \xi$. Contrary to Eq. (2), H_r does not allow us to find analytical expressions for the dynamics of observables such as $P_D(t)$ [cf. Eq. (3)]. However, as we demonstrate later, a specific use of Bayesian methods permits an accurate estimation of target signals in the wide parameter regime described by H_r that surpasses the performance of standard techniques such as FFT or least-squares methods.

Regarding the noise sources included in H_r , we have verified that their effect on the sensor dynamics during the time scales considered in this work is negligible. In this respect, one should note that the scheme in H_r includes two MW drivings that eliminate the noise effects induced by, firstly, $\mu(t)$ and, secondly, by Rabi frequency fluctuations. A specific assessment on this—including noise sources taken from Ref. [16]—can be found in the SM [38] that includes Refs. [41–45].

In order to simulate an experimental acquisition of data, we proceed as follows. The data, denoted by $\mathbf{D}$, are generated by computing the evolution of the quantum sensor state with Hamiltonian H_r at different times t_k with $k = 1, \dots, N_p$. The set $\mathbf{D}$ contains the string of N_m binary outcomes $x_{n;k} \in \{0, 1\}$ for each time instant t_k with $n = 1, \dots, N_m$, that is, the $x_{n;k}$ are random variables drawn from a Bernoulli distribution $B(P_k)$ where the success probability P_k is obtained from the dynamics of Hamiltonian H_r . We denote by $X_k = \sum_{n=1}^{N_m} x_{n;k}$ the number of successes recorded at time t_k , so that $P_k^s = X_k/N_m$ is the estimation of P_k from $\mathbf{D}$. In particular, we initialize the system in the state $|D\rangle$ at time $t = 0$, and compute the probability P_k of finding it in $|D\rangle$ at time t_k , from where the values $x_{n;k}$ are obtained.

III. BAYESIAN INFERENCE AND MAGNETOMETRY

In the following, we provide the basics of Bayesian inference as relevant to our method (see, for example, Refs. [36,37] for further details). Let us denote by $\Theta = \{\theta_1, \dots, \theta_M\}$ the set of M unknown parameters that we aim to determine using our quantum sensor from the measured data $\mathbf{D}$. From Bayes' theorem, the probability $p(\Theta|\mathbf{D}) \propto p(\mathbf{D}|\Theta)p(\Theta)$ (referred to as the *posterior* probability) contains the information we can extract from the data given the prior knowledge $p(\Theta)$, and the likelihood $p(\mathbf{D}|\Theta)$. The observations X_k that form the data $\mathbf{D}$ obey a Bernoulli distribution, i.e., $p(\mathbf{D}|\Theta) = \prod_{k=1}^{N_p} f[X_k, N_m, \tilde{P}_k(t_k; \Theta)]$, where $f(x, n, p) = n!/[x!(n-x)!]p^x(1-p)^{n-x}$ accounts for the probability of having recorded exactly x successes from n trials drawn from $B(p)$, while $\tilde{P}_k(t_k; \Theta)$ denotes the expected probability computed using the Hamiltonian H_r , given in Eq. (A1), at time t_k and with parameters Θ . For illustration purposes, we show the data $\mathbf{D}$ as $P_k^s(t_k)$ together with the shot-noise uncertainties σ_k [38]. It is worth remarking that, while magnetic field and intensity fluctuations have been taken into account to generate the data $\mathbf{D}$, their effect is negligible in the considered parameter regime [38]. For the Bayesian inference, the populations $\tilde{P}_k(t_k; \Theta)$ are computed without including these noise sources. Having the posterior distribution, one can obtain the estimated mean and variance value of the unknown parameter θ_j via the marginal distribution $p(\theta_j|\mathbf{D})$, as $\theta_j^{\text{est}} = \int d\theta_j \theta_j p(\theta_j|\mathbf{D})$ and $(\delta\theta_j^{\text{est}})^2 = \int d\theta_j (\theta_j - \theta_j^{\text{est}})^2 p(\theta_j|\mathbf{D})$, respectively, where the marginal reads as $p(\theta_j|\mathbf{D}) = \int \prod_{i \neq j} d\theta_i p(\Theta|\mathbf{D})$.

We exemplify the superior performance of our method over standard analysis techniques with two illustrative cases. For a simplified situation (case I) in which Eq. (3) applies leading to a periodic response, we demonstrate that Bayesian inference can handle situations with even single-shot measurements providing good estimates. When dealing with more complex signals (case II), we show that Bayesian inference from a few measurements is able to provide reliable estimates where standard analysis techniques are not applicable in general.

A. Case I

In this first scenario, $\Theta = \{\Omega_{\text{tg}}\}$ is the only unknown parameter. Assuming that the RWA can be safely applied and that the target signal is resonant, i.e., $\xi = 0$, the sensor is well approximated by Eq. (2) [cf. Fig. 1(a)] [16]. This allows us to compute the posterior $p(\Omega_{\text{tg}}|\mathbf{D})$ by scanning distinct Ω_{tg} values, from which $\Omega_{\text{tg}}^{\text{est}}$ and $\delta\Omega_{\text{tg}}^{\text{est}}$ can be inferred directly. Here we test our method in the worst-case scenario, that is, when no preknowledge about the unknown parameter is available. For that, we consider an uninformative prior, i.e., a flat probability distribution,

FIG. 2. (a) Simulated $N_p = 18$ observed populations $P_k^s(t_k) = X_k/N_m$ with $N_m = 4$ measurements per point, (points) with error bars indicating a standard deviation due to shot noise [38], together with the reconstructed signal (line) using Bayesian inference for a single unknown parameter Ω_{tg} . (b) Posterior probability distributions for the data in (a) (solid red line) and for the same parameters but $N_m = 1$ (dotted green) and $N_m = 20$ (dashed blue). See the main text for the values of $\Omega_{\text{tg}}^{\text{est}}$, and Fig. 1(a) for the remaining parameters.

and an observed signal measured at equally spaced time instances t_k separated by Δt , such that $p(\Omega_{\text{tg}}) \propto 1$ for $0 \leq \Omega_{\text{tg}} \leq \Omega_{\text{tg}}^{\text{max}}$, where $\Omega_{\text{tg}}^{\text{max}} = 2\pi/\sqrt{2}\Delta t$. This method can be trivially extended to handle undersampled or unevenly sampled data [38].

We simulate an experimental interrogation of the quantum sensor, recording N_m measurements per each of the N_p different time instances. Since $\Delta t \approx 1/6$ ms, it follows that $\Omega_{\text{tg}}^{\text{max}} \approx 2\pi \times 4.2$ kHz. An example is plotted in Fig. 2(a), together with the estimated signal, while the posterior distributions for different observations are illustrated in Fig. 2(b). We obtain very precise estimators even with large shot noise, such as the extreme case of single shots (i.e., $N_m = 1$). In particular, using the same parameters as those in Fig. 1(a) ($\Omega_{\text{tg}} = 2\pi \times 1$ kHz), we find that $\Omega_{\text{tg}}^{\text{est}} = 2\pi \times 1.011(42)$, $0.988(14)$, and $1.0048(76)$ kHz for three distinct realizations with $N_m = 1$, 4, and 20 measurements, respectively, where the uncertainty is given by $\delta\Omega_{\text{tg}}^{\text{est}}$. See the SM [38] for further details on the precision of the inferred amplitude $\Omega_{\text{tg}}^{\text{est}}$ and the string of outcomes for these realizations. In this simple case and for a moderate or large number of measurements, a least-squares fit provides, in average, slightly less accurate results, e.g., $\Omega_{\text{tg}}^{\text{est}} = 2\pi \times 0.947(20)$ kHz for $N_m = 4$ (see the SM [38] for further realizations and details). As the prior probability distribution is flat, the Bayesian estimators simply correspond to maximum likelihood estimators, which are known to outperform least-square fits [46]. In addition, note that an analysis using standard FFT methods leads to worse estimators. In particular, for the case in Fig. 2, one obtains $\Omega_{\text{tg}}^{\text{est}} = 2\pi \times 0.94(12)$ kHz [38], which further demonstrates the suitability of Bayesian inference techniques.

B. Case II

A more realistic situation needs to account for potential nonresonant radiation as well as off-resonant transitions within the quantum sensor. Thus, $\Theta = \{\Omega_{\text{tg}}, \xi\}$, where ξ denotes a detuning with respect to the resonant condition, and Eq. (A1) is required [cf. Fig. 1(b)]. In addition, Markov chain Monte Carlo (MCMC) methods will be employed to efficiently sample the posterior $p(\Theta|\mathbf{D})$ [36,47]. For that, we consider independent priors, namely, $p(\Omega_{\text{tg}}, \xi) = p(\Omega_{\text{tg}})p(\xi)$, taking again $p(\Omega_{\text{tg}})$ completely uninformative in the region $0 \leq \Omega_{\text{tg}} \leq 2\pi \times 50$ kHz, while $p(\xi) = \mathcal{N}(0, \sigma_\xi^2)$ with $\sigma_\xi = 2\pi \times 0.25$ kHz, as we expect close to resonant rf fields. By randomly choosing an initial point Θ_0 from the prior, we rely on a standard Metropolis algorithm to sample the posterior [47]. After j steps, the proposed point Θ_{j+1} obtained from $\mathcal{N}(\Theta_j, \tilde{\sigma}_p^2)$, where $\tilde{\sigma}_p^2 = \{\tilde{\sigma}_\Omega^2, \tilde{\sigma}_\xi^2\}$ refers to the variance in the proposal distributions, is accepted with probability $\alpha = \min[1, p(\Theta_{j+1}|\mathbf{D})/p(\Theta_j|\mathbf{D})]$. After a sufficient number of steps, $N_{\text{MC}} \gg 1$, the recorded Θ values provide an accurate sampling of $p(\Theta|\mathbf{D})$ and the marginals can be easily computed. Convergence of the MCMC can be checked by the mixing of different Markov chains [38,47]. Although we illustrate the working method for this case of study with a single example, we stress that the following procedure is general and can be applied to different situations.

In Fig. 3 we have considered a set of data $\mathbf{D}$ obtained for a rf signal with $\omega_{\text{tg}} \approx 2\pi \times 7$ MHz ($B_z = 0.5$ mT), a detuning of $\xi = 2\pi \times 0.1$ kHz and amplitude $\Omega_{\text{tg}} = 2\pi \times 12$ kHz, and $\Omega = 2\pi \times 37.27$ kHz that protects the sensor against magnetic field fluctuations, while the data have been generated with $N_p = N_m = 20$ [cf. Fig. 3(a)]. For the MCMC, we observe that $\tilde{\sigma}_\Omega = 10\tilde{\sigma}_\xi = 2\pi \times 0.1$ kHz yields a good mixing [cf. Fig. 3(b) and the SM [38]], so that the effective size of the MCMC (number of accepted points) amounts approximately to $N_{\text{MC}}/2$ [cf. Fig. 3(b)]. We remove the first 200 steps to avoid the burn-in regime [36,47]. In Figs. 3(c) and 3(d) we show the marginals $p(\Omega_{\text{tg}}|\mathbf{D})$ and $p(\xi|\mathbf{D})$, respectively, obtained upon $N_{\text{MC}} = 10^4$ steps for five independent Markov chains, which lead to $\Omega_{\text{tg}}^{\text{est}} = 2\pi \times 11.90(17)$ kHz and $\xi^{\text{est}} = 2\pi \times 0.169(39)$ kHz, very close to the ideal values.

The complex and nonharmonic response of the sensor challenges the determination of the unknown parameters for single-shot acquisitions [38]. However, for a reduced number of measurements per point, e.g., $N_m = 4$, we still find good estimates for the amplitude $\Omega_{\text{tg}}^{\text{est}} = 2\pi \times 12.91(44)$ kHz, although the data may be better explained under distinct detunings, $\xi^{\text{est}} = 2\pi \times -0.112(90)$ kHz. In a similar manner, by reducing the shot noise, more accurate estimates can be obtained, e.g., $\Omega_{\text{tg}}^{\text{est}} = 2\pi \times 12.05(12)$ kHz and $\xi^{\text{est}} = 2\pi \times 0.111(27)$ kHz for $N_m = 40$ measurements per point [cf. Figs. 3(c) and 3(d)]. We provide

FIG. 3. (a) Simulated observations $P_D(t_k)$ with $N_p = N_m = 20$ (points) together with the reconstructed signal (line) using Bayesian inference and MCMC for Ω_{tg} and ξ unknown. The data $\mathbf{D}$ are generated setting $B_z = 0.5$ mT, with a signal of frequency $\omega_{\text{tg}} \approx 2\pi \times 7$ MHz, with detuning $\xi = 2\pi \times 0.1$ kHz and Rabi frequency $\Omega_{\text{tg}} = 2\pi \times 12$ kHz. (b) Scatter plot of the recorded values Θ_j during the evolution of the MCMC for two independent chains (red and green dots) with $N_{\text{MC}} = 10^4$ but removing the burn-in regime. The ideal value is indicated with a black square. Panels (c) and (d) show the histograms for the marginals $p(\Omega_{\text{tg}}|\mathbf{D})$ and $p(\xi|\mathbf{D})$, respectively, obtained from the MCMC illustrated in (b). The solid red (dashed blue) line corresponds to a Gaussian distribution with equal first and second moments as the marginals for $N_m = 20$ ($N_m = 40$), while the dotted black line indicates the ideal value. The estimated values for the case plotted in (a) are $\Omega_{\text{tg}}^{\text{est}} = 2\pi \times 11.90(17)$ kHz and $\xi^{\text{est}} = 2\pi \times 0.169(39)$ kHz.

the string of outcomes $\mathbf{D}$ for each of the realizations and more examples in the SM [38]. Finally, it is worth mentioning that neither least-squares nor FFT techniques are useful in this case due to the complex signal structure. As illustrated in the SM [38], a nonlinear least-squares fit to the dynamics dictated by H_r is unable to find suitable parameters unless initialized close to the ideal values and unsuitable to tackle more complex cases such as in bimodal posterior distributions [38], while at the same time FFT methods exhibit an intricate frequency spectrum of the data $\mathbf{D}$, hindering the identification of the unknown parameters.

IV. CONCLUSIONS

We have presented a protocol relying on Bayesian methods that significantly enhances the performance of quantum sensors in realistic scenarios. In particular, we have demonstrated that a quantum sensor can be used even when the character of target signals, as well as the presence of noise and a reduced number of measurements, spoil its ideal functioning, leading to strong deviations of the sensor from a simple harmonic response. We illustrate this scheme using a $^{171}\text{Yb}^+$ trapped ion, and relying on standard MCMC methods if so required by the parameter regime. Our results showcase the suitability of Bayesian inference with respect to standard analysis techniques for parameter estimation. Our method therefore paves the way to use quantum sensors under realistic conditions, significantly extending their working region and reducing the detection times, thus enhancing their adaptability to different scenarios.

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APPENDIX: REFINED MODEL FOR THE ATOMIC-SIZE SENSOR

A more rigorous treatment of the Hamiltonian given in Eq. (1) can be written in the basis $\{|u\rangle, |d\rangle, |D\rangle, |\dot{0}\rangle\}$ as

$$H_r = \frac{-\mu(t)}{\sqrt{2}}(|D\rangle\langle u| + |D\rangle\langle d| + \text{H.c.}) + \frac{\Omega}{\sqrt{2}}(|u\rangle\langle u| - |d\rangle\langle d|) - \left[\frac{\Omega}{2\sqrt{2}}(|u\rangle\langle u| - |d\rangle\langle d|) + \frac{\Omega}{4}(|u\rangle\langle D| + |D\rangle\langle d|) - \frac{\Omega}{4}(|D\rangle\langle u| + |d\rangle\langle D|) \right] e^{i\gamma_e B_z t} + \left[\frac{\Omega_{\text{tg}}}{4}(|u\rangle\langle \dot{0}| + |d\rangle\langle \dot{0}|) - \frac{\Omega_{\text{tg}}}{2\sqrt{2}}|D\rangle\langle \dot{0}| \right] e^{-i\xi t} + \text{H.c.}$$

$$\begin{aligned}
& + \frac{\Omega_{\text{tg}}}{2} \left(\frac{1}{2} |u\rangle \langle \dot{0}| + \frac{1}{2} |d\rangle \langle \dot{0}| - \frac{1}{\sqrt{2}} |D\rangle \langle \dot{0}| \right) e^{2i(\gamma_e B_z/2 - \gamma_e^2 B_z^2/4A)t} e^{i\xi t} + \frac{\Omega_{\text{tg}}}{2} \left(\frac{1}{2} |\dot{0}\rangle \langle u| + \frac{1}{2} |\dot{0}\rangle \langle d| + \frac{1}{\sqrt{2}} |\dot{0}\rangle \langle D| \right) e^{i\gamma_e B_z t} e^{i\xi t} + \text{H.c.} \\
& + \frac{\Omega_{\text{tg}}}{2} \left(\frac{1}{2} |\dot{0}\rangle \langle u| + \frac{1}{2} |\dot{0}\rangle \langle d| + \frac{1}{\sqrt{2}} |\dot{0}\rangle \langle D| \right) e^{i(\gamma_e^2/2A)B_z^2 t} e^{-i\xi t} + \text{H.c.}, \tag{A1}
\end{aligned}$$

where the first term accounts for magnetic field fluctuations in the states $|D\rangle$ and $|u\rangle$. The rest of the terms appear due to both a nonresonant target signal $\omega_{\text{tg}} = \omega_1 - \omega_0 + \xi$ with a detuning ξ , as well as a large Rabi frequency Ω_{tg} compared to the frequency ω_{tg} of the rf signal. The previous Hamiltonian includes two MW controls with amplitudes Ω . See the SM [38] for details of the derivation.

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