Comment on “The Winfree model with non-infinitesimal phase-response curve: Ott–Antonsen theory” [Chaos 30, 073139 (2020)]

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ABSTRACT

In a recent paper [Chaos 30, 073139 (2020)], we analyzed an extension of the Winfree model with nonlinear interactions. The nonlinear coupling function Q was mistakenly identified with the non-infinitesimal phase-response curve (PRC). Here, we assess to what extent Q and the actual PRC differ in practice. By means of numerical simulations, we compute the PRCs corresponding to the Q functions previously considered. The results confirm a qualitative similarity between the PRC and the coupling function Q in all cases.

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In Ref. 1, we studied this generalization of the Winfree model of globally coupled phase oscillators,

$$\dot{\theta}_i = \omega_i + Q(\theta_i, A), \quad (1a)$$

$$A = \frac{\varepsilon}{N} \sum_{j=1}^N P(\theta_j). \quad (1b)$$

Here, A is proportional to the sum over the pulses emitted by the N oscillators of the population. In contrast to the original model,^{2,3} function Q in Eq. (1a) has a nonlinear dependence on the mean field A . The motivation for this is the fact that nonlinearity is an unavoidable consequence of applying phase reduction beyond the first order to oscillator ensembles.⁴ Note that a Taylor expansion of Q to n th order in A yields up to $(n+1)$ -body phase interactions, similar to Ref. 5.

We mistakenly called Q a “non-infinitesimal phase-response curve” in Ref. 1. Properly speaking, function Q is a non-linear “coupling function.”⁴ The aim of this Comment is to clarify to what extent the coupling function Q determines the actual phase-response curve (PRC). The PRC quantifies the phase shift gained by an oscillator in response to an external stimulus.⁶ There is no analytic relation between Q and the PRC beyond the small ε limit; in that case $Q(\theta, A) \simeq \tilde{Q}(\theta)A$, where $\tilde{Q}$ turns out to be the so-called infinitesimal

PRC (iPRC). In consequence, we rely here on numerical simulations to compute the PRC empirically.

The family of functions $Q(\theta, A)$ considered in Ref. 1 was

$$Q(\theta, A) = f_1(A)(1 - \cos \theta) - f_2(A) \sin \theta. \quad (2)$$

Four representative pairs of functions $f_{1,2}(A)$ were studied in detail in Ref. 1, and the corresponding coupling functions $Q(\theta, A)$ were depicted in Fig. 2 of Ref. 1. With the aim of comparing them, we obtain the PRC for each of the four coupling functions Q considered in Ref. 1.

The PRC value depends on the timing as well as on the specific shape of the stimulus, which is not necessarily weak or brief.⁶ Numerically, we obtain the PRC measuring the effect on one oscillator’s phase of a pulse generated by another oscillator. This means that the two oscillators are unidirectionally coupled (i.e., a master-slave configuration). We adopt $\omega = 1$ as the natural frequency for both perturbed and perturbing oscillators, which is the obvious choice as it is the central frequency of the distribution in Ref. 1. Moreover, we follow Ref. 1 and use the same 2π -periodic symmetric unimodal pulse function $P(\theta)$. It vanishes at $\theta = \pm\pi$, and a free parameter $r < 1$ controls the narrowness of P : The height of the pulse is $P(0) = 2/(1-r)$, and $\lim_{r \rightarrow 1} P(\theta) = 2\pi\delta(\theta)$. In this Comment, we consider two different pulse widths: $r = 0.9$ (the value

FIG. 1. PRCs for cases (a)–(d) in Ref. 1. The set of five ε values used is specified in panel (a). The coupling function (2) in each panel is (a) $f_1 = A/(1+A) = f_2/A$; (b) $f_1 = A^2/(1+A) = Af_2$; (c) $f_1 = 0$, $f_2 = A(1-A)/(1+A)$; (d) $f_1 = 0$, $f_2 = A(A-1)/(1+A)$. The pulse acting on the oscillator is $P(t-\pi) = \frac{(1-r)[1+\cos(t-\pi)]}{1-2r\cos(t-\pi)+r^2}$, with $r = 0.9$.

FIG. 2. The same as Fig. 1 with $r = 0.99$.

selected in Ref. 1) and $r = 0.99$ corresponding to an extremely narrow pulse.

The simulation starts at time $t = 0$ with the (slave) oscillator at an initial phase θ_{in} . Then, we let it evolve under the influence of the forcing oscillator. The phase of this one grows linearly such that the input felt by the first oscillator is $A(t) = \varepsilon P(t - \pi)$. Parameter ε determines the strength of the stimulus. The simulation runs from $t = 0$ to $t = 2\pi$, since A exactly vanishes at these times. Note that we do not need to run the simulation further since phase oscillators are governed by first-order differential equations. For a given ε value, we measure the phase shift at $t = 2\pi$ such that $\text{PRC}(\theta_0, \varepsilon) = \theta(t = 2\pi) - (\theta_{\text{in}} + 2\pi)$. The phase θ_0 in the argument of the PRC is the phase value when A attains its maximum, assuming no input exists: $\theta_0 = \theta_{\text{in}} + \pi$. The results are shown in Figs. 1 and 2 for a set of ε values, in each panel for one particular coupling function $Q(\theta, A)$ already adopted in Ref. 1. In all panels, the corresponding iPRC is shown as a reference. Note that the normalization of the y -axis in Figs. 1 and 2 includes a 2π factor—in addition to ε —because this is the integral of the pulse over an interval of length 2π . Figures 1 and 2 are quite similar, though for $r = 0.9$ (Fig. 1) the PRCs remain closer to their iPRCs up to a larger ε value.

The comparison of the PRCs in this Comment with the corresponding Q functions in Fig. 2 of Ref. 1 evidence that $Q(\theta, A)$ is not simply the PRC. Indeed, Q in (2) has only the first harmonic in θ , whereas the non-infinitesimal PRCs in the figures display additional Fourier components. In spite of these dissimilarities, simple visual inspection indicates that the PRC strongly resembles the coupling function Q in all four cases. For example, we observe the same loss of

non-negativeness of the (type-I) iPRC as A increases in panel (a), or the transition from a synchronizing iPRC to a desynchronizing PRC for large enough A in panel (c). Summarizing, our simulations confirm that the main attributes of the coupling function Q are shared by the non-infinitesimal PRC.

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DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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