CHAPTER I

ARAKELOV THEORY ON SHIMURA VARIETIES.

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Contents

1. Introduction

Arakelov theory is at the crossroad between number theory, algebraic geometry and complex analysis. The starting point of this theory is the analogy between $S = \mathrm{Spec}(\mathscr{O}_K)$, the spectrum of the ring of integer $\mathscr{O}_K$ of a number field K, and a smooth affine curve C defined over an arbitrary field k. A smooth affine curve can always be completed to a smooth projective curve $\overline{C}$ by adding a finite number of points. Similarly, the closed points of S correspond to non-Archimedean places of K and S can be "compactified" by adding the Archimedean places. Of course this two compactifications are very different, the curve $\overline{C}$ has a global structure of a smooth projective curve while $\overline{S}$ is just a patch between Archimedean and non-Archimedean places. Nevertheless there are striking formal analogies between these two pictures. For simplicity of the exposition assume that k is algebraically closed.

The first analogy is the product formula. If $0 \neq f \in k(C)$ is a non zero rational function, then

$$\sum_{p \in \overline{C}(k)} \mathrm{ord}_p(f) = 0,$$

while, if we choose an absolute value in each place of K suitably normalized, then, for $0 \neq f \in K$,

$$\sum_{v \in \mathfrak{M}_K} -\log |f|_v = 0,$$

where $\mathfrak{M}_K$ is the set of all places of K, Archimedean and non-Archimedean. A second example of such analogies is that, using the appropriate notation (see [**Neu99**, Chapter 3]) a variant of the Riemann-Roch theorem for algebraic curves is formally identical to Minkowski theorems on the geometry of numbers. The aim of Arakelov theory is to extend this analogy between algebraic geometry and number theory to higher dimension. We refer the reader to the notes by C. Soulé (Chapter I of this volume) for more details on Arakelov Geometry and we will only recall some points of it.

Consider a projective and flat scheme $\mathscr{X}$ defined over $\operatorname{Spec}(\mathbb{Z})$. For instance $\mathscr{X}$ can be obtained from a projective variety over $\mathbb{Q}$ by clearing denominators of a system of defining equations and taking the irreducible component containing the complex points. In other words, $\mathscr{X}$ is obtained from a variety $X \subset \mathbb{P}^N_{\mathbb{Q}}$ by taking its Zariski closure in $\mathbb{P}^N_{\mathbb{Z}}$. Since $\operatorname{Spec} \mathbb{Z}$ is affine and not complete, we can not expect $\mathscr{X}$ to behave globally as a complete variety. In the same way that we "compactify" $\operatorname{Spec} \mathbb{Z}$ by adding one Archimedian point, we can "compactify" $\mathscr{X}$ by adding one Archimedean fibre $X_{\mathbb{R}} = \mathscr{X} \times \operatorname{Spec}(\mathbb{R})$. The *compound object* $\mathscr{X} \amalg X_{\mathbb{R}}$ should behave formally as a complete variety over a field. This means that many theorems in algebraic geometry should have a number-theoretical analogue in this setting. This analogue will typically involve algebraic geometry in $\mathscr{X}$ and complex analysis in $X_{\mathbb{R}}$. Successful examples of such strategy are the Arithmetic Riemann-Roch Theorem [**GS92**] and the Lefschetz fixed point theorem [**KR01**].

The main objects considered in Arakelov theory are arithmetic cycles $\overline{Z} = (Z, g_Z)$, where Z is an algebraic cycle on $\mathscr{X}$ and g_Z is a *Green current* for Z in $\mathscr{X}(\mathbb{C})$, and hermitian vector bundles $\overline{E} = (E, h)$, where E is a vector bundle on $\mathscr{X}$ and h is a smooth hermitian metric on the complex vector bundle $E_{\mathbb{C}}$ over $\mathscr{X}(\mathbb{C})$.

There is an arithmetic intersection theory of arithmetic cycles [**GS90a**] and a theory of arithmetic characteristic classes [**GS90b**] for hermitian vector bundles. This theory of arithmetic characteristic classes was first developed for smooth hermitian metrics.

Modular and Shimura varieties have many interesting vector bundles that have modular or group theoretical interpretations, like the line bundle of modular forms on a modular curve, or an automorphic vector bundle on a more general Shimura variety. In many instances, these vector bundles come equipped with hermitian metrics that also have modular or group theoretical interpretations. An example of such metrics is the Petersson metric on the line bundle of modular forms.

The geometric and arithmetic invariants of Shimura varieties have a very rich structure. Usually, geometric invariants are related to special values of L-functions, while arithmetic invariants are related to logarithmic derivatives of L functions. Thus, arithmetic invariants give information about the second term in the Taylor expansion of the L function. Kudla's program aims to make precise this idea. See for instance [**KRY06**] and the references therein.

Many Shimura varieties and automorphic vector bundles have models over arithmetic rings (for instance a ring of integers with some primes inverted). Therefore we have at our disposal almost all the ingredients needed to define and compute arithmetic invariants, at least for projective Shimura varieties. But many modular and Shimura varieties are only quasi-projective and not projective, and, in order to be able to define arithmetic invariants for them, it is useful to find suitable compactifications. Under some conditions (see [**Mum77**] for precise details) automorphic vector bundles can be extended to such compactifications, but the *interesting* hermitian metrics do not extend to smooth hermitian metrics on the completed vector bundle, but only to logarithmically singular hermitian bundles.

Arakelov theory developed in [**GS90a**] and [**GS90b**] only deals with smooth hermitian metrics. So in order to apply Arakelov theory to automorphic vector bundles we need to extend it to logarithmically singular hermitian vector bundles. This extension was made in [**BGKK07**] and [**BGKK05**].

The aim of this note is to introduce the basic ingredients of the extension of Arakelov theory to compactifications of quasi-projective Shimura varieties. In section 2 we will recall some basic facts of Hermitian symmetric spaces. In section 3 we will discuss connected Shimura varieties. Section 4 is devoted to equivariant vector bundles and their invariant metrics. In section 5 we will study log-singular metrics and log-log forms and in section 6 we will put everything together to construct an arithmetic intersection theory suitable to study non-compact Shimura varieties.

2. Hermitian symmetric spaces

In this section we will use the modular curve as an illustration of the general theory of hermitian symmetric spaces. We start with the complex upper half plane

$$\mathbb{H} = \{x + iy \in \mathbb{C} \mid y > 0\}$$

and the group $\mathrm{SL}_2(\mathbb{R})$. This group acts on $\mathbb{H}$ by Moebius transforms

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}.$$

The action is transitive and the stabilizer of the point $i \in \mathbb{H}$ is

$$\mathrm{SO}_2(\mathbb{R}) = \left\{ \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \middle| c^2 + s^2 = 1 \right\}.$$

Therefore we can write

$$\mathbb{H} = \mathrm{SL}_2(\mathbb{R})/\,\mathrm{SO}_2(\mathbb{R}). \tag{2.1}$$

There is something odd with this identity: the left hand side is a complex manifold, while the right hand side is a real differentiable manifold. Thus this identity has to be seen as an identity of differentiable manifolds. Then, where does the complex structure come from? The next exercise shows that the complex structure of $\mathbb{H}$ is determined (up to complex conjugation) by the groups $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SO}_2(\mathbb{R})$.

Exercise 1. — Consider the element

$$J = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \in \mathrm{SO}_2(\mathbb{R})$$

1. Show that J has order 8 in $\mathrm{SL}_2(\mathbb{R})$ but as an operator on $\mathbb{H}$ has only order 4.
2. Show that the point $i \in \mathbb{H}$ is a fixed point of J and that the action of J on the tangent space $\mathrm{T}_i\,\mathbb{H}$ is multiplication by i.
3. For $g \in \mathrm{SL}_2(\mathbb{R})$ and $\tau = g \cdot i$, show that gJg^{-1} stabilizes τ and the action of gJg^{-1} on $\mathrm{T}_\tau\,\mathbb{H}$ is multiplication by i.

Conclude that J defines an equivariant integrable almost complex structure (see [**Wel80**, Chapter I, Section 3] for the notion of almost complex structure) on the quotient $\mathrm{SL}_2(\mathbb{R})/\,\mathrm{SO}_2(\mathbb{R})$ such that equation (2.1) becomes an identity of complex manifolds.

The only other choice to cook a complex structure on $\mathrm{SL}_2(\mathbb{R})/\,\mathrm{SO}_2(\mathbb{R})$ using elements from $\mathrm{SL}_2(\mathbb{R})$ would have been to use the transpose of

J. This choice would have produced the complex conjugate space of $\mathbb{H}$, that is, the complex lower half plane.

As we explain below, the method of Exercise 1 can be generalized from $\mathrm{SL}_2(\mathbb{R})$ to other groups.

Definition 2.1. — 1. Let D be a complex manifold with almost complex structure J and g a Riemannian structure on D. We say that g is a *hermitian structure* if for every point $p \in D$ and tangent vectors $v, w \in T_pD$, the condition

$$g(Jv, Jw) = g(v, w).$$

holds, A complex manifold with a hermitian structure is called a *hermitian space*. If D is a hermitian space, the group of holomorphic isometries of D will be denoted by $\mathrm{Is}(D)$. The connected component of the identity in this group will be denoted $\mathrm{Is}^+(D)$.

2. Let D be a connected hermitian space. Then D is called a *hermitian symmetric space* if every point $p \in D$ is an isolated fixed point of an involutive holomorphic isometry of D.

Exercise 2. — On the space $\mathbb{H}$ with coordinates x, y we put the Riemannian structure

$$ds^2 = \frac{dx^2 + dy^2}{y^2}. \tag{2.2}$$

1. Show that the Riemannian structure (2.2) is invariant under the action of the group $\mathrm{SL}_2(\mathbb{R})$ and it is a hermitian structure.
2. Show that $\mathbb{H}$ is a hermitian symmetric space. *Hint:* J^2 gives us the involution that fixes i and the other needed involutions are obtained by conjugation with elements of $\mathrm{SL}_2(\mathbb{R})$.

In fact

$$\mathrm{Is}^+(\mathbb{H}) = \mathrm{SL}_2(\mathbb{R})/\{\pm \mathrm{Id}\} = \mathrm{PSL}_2(\mathbb{R}).$$

For proofs of the next results and more information about symmetric spaces and hermitian symmetric spaces the reader is refered to [**AMRT10**, III, §2] and [**Hel78**]. In particular, see [**Hel78**, Chapter VIII] for the definition of irreducible hermitian symmetric spaces and the ones of compact and non-compact type.

Theorem 2.2 ([**AMRT10**, Ch. III §2.1]). — *Let D be a hermitian symmetric space. Then there is a decomposition*

$$D = D_0 \times D_1 \times \cdots \times D_n$$

where

1. *D_0 is the quotient of a complex vector space, with a translation invariant hermitian structure, by a discrete group of translations. This factor is called of* Euclidean type.
2. *Each D_i, $i \neq 0$ is an irreducible hermitian symmetric space that is not of Euclidean type.*

In the previous theorem, the factors D_i, $i \neq 0$, that are compact are called of compact type, while the non-compact ones are called of non-compact type.

Theorem 2.3 ([**Hel78**, Ch. VIII Theorem 6.1])

1. *The irreducible hermitian symmetric spaces of non-compact type are the varieties of the form G/K where G is a connected non-compact simple Lie group with center $\{e\}$ and K is a maximal compact subgroup with non-discrete center.*
2. *The irreducible hermitian symmetric spaces of compact type are the varieties of the form G/K where G is a connected compact simple Lie group with center $\{e\}$ and K is a maximal proper connected subgroup with non-discrete center.*

Let D be a quotient of the form G/K as in the theorem. Let e be the neutral element of G and $o = [e] \in G/K$ its class in D. In both cases of the theorem, to prove that G/K is a hermitian symmetric space we have to construct an almost complex structure and, for each point, an involution as we did for the upper half plane in Exercise 1. To this end one uses the fact that K has non discrete center to find an element J in the center of K whose action in $\mathrm{T}_o D$ has order 4. Then the complex structure on the tangent space at o is induced by J and the involution that fixes o is induced by $s = J^2$. Both operations are translated to the whole space by conjugation. See the proof of [**Hel78**, Ch. VIII Theorem 6.1] for more details.

Example 2.4. — 1. The group $G = \mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm \mathrm{Id}\}$ is a connected non-compact simple Lie group with center $\{e\}$. Moreover, $K = \mathrm{SO}_2(\mathbb{R})/\{\pm \mathrm{Id}\}$ is a maximal connected compact subgroup isomorphic to $\mathrm{U}(1)$. It is abelian and non-discrete. Thus its center is non-discrete. The quotient G/K is the hermitian symmetric space of non-compact type $\mathbb{H}$.

2. The group $G = \mathrm{SU}(2)/\{\pm \mathrm{Id}\}$ is a connected compact simple Lie group with center $\{e\}$ and K as before is a maximal connected

proper subgroup isomorphic to U(1). Again it is abelian and non-discrete. Thus its center is non-discrete. The quotient G/K is a hermitian symmetric space of compact type isomorphic to $\mathbb{P}^1(\mathbb{C})$.

The spaces $\mathbb{H}$ and $\mathbb{P}^1(\mathbb{C})$ of Example 2.4 are very close to each other. In fact they are in duality. This is a general procedure that we describe next.

Let D be an irreducible hermitian symmetric space of non-compact type and $o \in D$. Put

$$\begin{aligned} &G = \mathrm{Is}^+(D), \text{ a simple connected Lie group,}\\ &K = \mathrm{Stab}(o), \text{ the maximal compact subgroup,}\\ &s_o \in K \text{ the involution fixing } o. \end{aligned}$$

Then s_o induces an inner automorphism σ of G. The subgroup G^σ of elements fixed by σ agrees with K. Set

$$\begin{aligned} &\mathfrak{g} = \mathrm{Lie}(G)\\ &\mathfrak{k} = \mathrm{Lie}(K) = \text{ subspace of } \mathfrak{g} \text{ where } \sigma \text{ acts as } +1,\\ &\mathfrak{p} = \text{ subspace where } \sigma \text{ acts as } -1, \end{aligned}$$

The group G is the identity component of the set of real points of an algebraic group $\mathscr{G}$. Indeed, consider the adjoint action

$$G \longrightarrow \mathrm{GL}(\mathfrak{g})$$

and let $\mathscr{G}$ be the Zariski closure of the image of G. It is a closed subgroup of $\mathrm{GL}(\mathfrak{g})$, hence algebraic and G is the identity component of the group of real points $\mathscr{G}(\mathbb{R})^+$. Write $G_{\mathbb{C}} = \mathscr{G}(\mathbb{C})$.

Inside $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C} = \mathrm{Lie}(\mathscr{G}(\mathbb{C}))$ we write

$$\begin{aligned} \mathfrak{k}_c &= \mathfrak{k},\\ \mathfrak{p}_c &= i\mathfrak{p},\\ \mathfrak{g}_c &= \mathfrak{k}_c \oplus \mathfrak{p}_c. \end{aligned}$$

Then $\mathfrak{g}_c$ is a real form of $\mathfrak{g}_{\mathbb{C}}$ that determines a compact group G_c. The compact dual of D is the symmetric space of compact type

$$\check{D} = G_c/K.$$

This construction can be reversed and the irreducible non Euclidean hermitian symmetric spaces come in pairs, one compact and one non compact (see [**AMRT10**, Ch. III §2.1]).

Definition 2.5. — Let D be an irreducible hermitian symmetric space. If D is of non-compact type, its *compact dual* is the previously constructed hermitian symmetric space, while if D is of compact type, its *compact dual* is D itself. If D is a hermitian symmetric space with no Euclidean factors, its *compact dual* is the product of the compact duals of all its factors.

Exercise 3. — Consider the case $D = \mathbb{H}$.

1. Show that, in this case $\mathfrak{g}$ is the Lie algebra of 2 by 2 real matrices of trace zero, $\mathfrak{k}$ the subalgebra of skew-symmetric matrices and $\mathfrak{p}$ the subspace of symmetric trace zero matrices.
2. Show that $\mathfrak{g}_c$ is the Lie algebra of 2 by 2 skew-hermitian complex matrices of zero trace. Thus $G_c = \mathrm{SU}(2)$ and the compact dual of $\mathbb{H}$ is isomorphic to $\mathbb{P}^1(\mathbb{C})$.

When working with a general hermitian symmetric space G/K it may be difficult to visualize the complex structure, unless we find a nice representation of it as in the case of the upper half plane. The big advantage of the compact dual is that its complex structure is easier to visualize. Moreover, a non-compact hermitian symmetric space can always be embedded in its compact dual making also apparent its complex structure.

Let $D = G/K$ be a connected hermitian symmetric space without Euclidean factor, and $o \in D$ the image of $e \in G$. Assume that $G = \mathrm{Is}^+(D)$. Then there is a map $u_o \colon \mathrm{U}(1) \to G$ such that $u_o(z)$ acts as multiplication by z in $\mathrm{T}_o D$. Then $J = u_o(i)$ defines the complex structure of $\mathrm{T}_o D$ and $s_o = u_o(-1)$ is the involution that fixes o. The subgroup K is the centralizer of $u_o(\mathrm{U}(1))$ and, if D is irreducible, $u_o(\mathrm{U}(1))$ is the center of K. The subspace $\mathfrak{p} \subset \mathfrak{g}$ can be identified with $\mathrm{T}_o D$ and the action of K on $\mathrm{T}_o D$ agrees with the adjoint action of K on $\mathfrak{p}$. Let

$$\mathfrak{p}_{\mathbb{C}} \coloneqq \mathfrak{p} \otimes \mathbb{C} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$$

be the decomposition of $\mathfrak{p}_{\mathbb{C}}$ into $\pm i$ eigenspaces with respect to J (recall that $\mathfrak{p}$ is the -1-eigenspace of $s_o = J^2$). Denote by $P_\pm$ the subgroup of $G_{\mathbb{C}}$ generated by $\exp(\mathfrak{p}_\pm)$. Then $K_{\mathbb{C}}$ normalizes $P_\pm$ and $K_{\mathbb{C}}\Delta P_-$is a parabolic subgroup of $G_{\mathbb{C}}$ with unipotent radical P_-. Hence $G_{\mathbb{C}}/K_{\mathbb{C}}\Delta P_-$ is a projective algebraic variety that we temporarily denote by X. The following theorem exhibits X as a complex manifold. For a proof of see [**Hel78**, Ch.8 §7], See also [**AMRT10**, Ch. III Theorem 2.1] for an outline.

Theorem 2.6 **(Borel and Harish-Chandra embedding theorem)**

1. *The map $P_+ \times K_{\mathbb{C}} \times P_- \to G_{\mathbb{C}}$ given by multiplication is injective and its image contains G. Moreover $(K_{\mathbb{C}} \cdot P_-) \cap G = K$.*
2. *We obtain maps*

$$\begin{array}{ccccc} G/K & \longrightarrow & P_+ \times K_{\mathbb{C}} \times P_-/K_{\mathbb{C}} \times P_- & \longrightarrow & G_{\mathbb{C}}/K_{\mathbb{C}} \cdot P_- \\ \simeq\uparrow & & \simeq\uparrow & & \simeq\uparrow \\ D & & P_+ & & X \\ & & \simeq\uparrow \exp & & \\ & & \mathfrak{p}_+ & & \end{array}$$

   They are all open holomorphic immersions. The image of D in $\mathfrak{p}_+$ is a bounded domain and the image of $\mathfrak{p}_+$ in X is a dense open Zariski subset.
3. *The compact form G_c of G acts transitively on X. Moreover $(K_{\mathbb{C}} \cdot P_-) \cap G_c = K$. Therefore $X = G_c/K = \check{D}$ is the compact dual of D.*

Exercise 4. — We go back to the example $D = \mathbb{H}$, $G = \mathrm{PSL}_2(\mathbb{R})$, $K = \mathrm{SO}_2(\mathbb{R})/\pm\mathrm{Id}$. It will be easier to work with the double coverings $\widetilde{G} = \mathrm{SL}_2(\mathbb{R})$ and $\widetilde{K} = \mathrm{SO}_2(\mathbb{R})$. The groups P_+ and P_- can be defined as subgroups of $\widetilde{G}_{\mathbb{C}}$, but the map $u_o\colon U(1) \to G$ does not lift to a map to $\widetilde{G}$.

1. Show that $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are the one dimensional subspaces of $\mathfrak{g}_{\mathbb{C}}$ generated respectively by the matrices

$$\frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \text{ and } \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$$

   Therefore the subgroups $P_\pm$ are given by

$$P_\pm = \left\{ P_\pm(z) := \begin{pmatrix} 1+z/2 & \pm iz/2 \\ \pm iz/2 & 1-z/2 \end{pmatrix} \middle| z \in \mathbb{C} \right\}.$$

2. Consider the action of P_+ on $\mathbb{P}^1(\mathbb{C})$ by Moebius transformations and show that

$$P_+(z) \cdot i = \frac{i(1+z)}{1-z}.$$

   Conclude that the open immersion $\mathbb{H} \to \mathfrak{p}_+$ is given by

$$\tau \mapsto \frac{\tau - i}{\tau + i}$$

and the image of the upper half plane is the interior of the unit disk.

3. The adjoint action of $\mathrm{SO}_2(\mathbb{R})$ on $\mathfrak{p}_+$ is given by
$$\mathrm{Ad}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}(M) = \frac{1}{(a+ib)^2}M.$$
4. Let $t \in \mathbb{C}^\times$, the matrix
$$K(t) := \frac{1}{2t}\begin{pmatrix} t^2+1 & i(t^2-1) \\ i(1-t^2) & t^2+1 \end{pmatrix}$$
belongs to $\mathrm{SO}_2(\mathbb{C})$. The map $t \mapsto K(t)$ is a group isomorphism. The adjoint action of $K(t)$ on $\mathfrak{p}_+$ is given by multiplication by t^{-2}.
5. Let
$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \widetilde{G},$$
and denote by s_1 the holomorphic map given by the composition
$$\mathbb{H} \to \mathfrak{p}_+ \xrightarrow{\exp} P_+ \to G_{\mathbb{C}}.$$
Then
$$\gamma \cdot s_1(\tau) = s_1(\gamma \cdot \tau) \cdot K(j_1(\gamma, \tau)) \cdot p,$$
where $p \in P_-$ and
$$j_1(\gamma, \tau) = \frac{(a\tau + b) + i(c\tau + d)}{\tau + i}.$$
6. Show that j_1 satisfies the cocycle condition for the action of $\widetilde{G}$ on $\mathbb{H}$. That is
$$j_1(\gamma \cdot \gamma', \tau) = j_1(\gamma, \gamma' \cdot \tau) j_1(\gamma', \tau).$$
7. Let $f\colon \mathbb{H} \to \mathbb{C}^\times$ be the function $f(\tau) = (\tau + i)$. Show that the cocycle
$$j(\gamma, \tau) = j_1(\gamma, \tau) f(\tau) f(\gamma \cdot \tau)^{-1}$$
is given by
$$j(\gamma, \tau) = (c\tau + d).$$
Conclude that the holomorphic map $s\colon \mathbb{H} \to \widetilde{G}_{\mathbb{C}}$ given by
$$s(\tau) = s_1(\tau) K(\tau + i) \tag{2.3}$$
satisfies
$$\gamma s(\tau) = s(\tau) \cdot K(c\tau + d) \cdot p'$$
for some element $p' \in P_-$.

3. Connected Shimura varieties

It is time to consider the quotient of a hermitian symmetric space by some interesting discrete subgroups.

Definition 3.1. — 1. Let G be a group. Two subgroups Γ_1 and Γ_2 are said to be *commensurable* if $\Gamma_1 \cap \Gamma_2$ has finite index in both Γ_1 and Γ_2.
2. Let G be an algebraic group defined over $\mathbb{Q}$ that admits a closed embedding $G \hookrightarrow \mathrm{GL}_n$ also defined over $\mathbb{Q}$. An *arithmetic subgroup* of $G(\mathbb{R})$ is any subgroup of $G(\mathbb{Q})$ that is commensurable with $G(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z})$.
3. An arithmetic subgroup $\Gamma \subset G(\mathbb{Q}) \subset \mathrm{GL}_n(\mathbb{C})$ is called *neat* if for every $x \in \Gamma$, the subgroup of $\mathbb{C}^\times$ generated by the eigenvalues of x is torsion free.

Remark 3.2. — 1. The notion of arithmetic subgroup depends on the rational structure of G, but once this is fixed, it is independent of the choice of closed embedding $G \hookrightarrow \mathrm{GL}_n$.
2. Any neat arithmetic subgroup is torsion free. Every arithmetic subgroup has a neat subgroup of finite index.

We are interested in locally symmetric spaces of the form $\Gamma\backslash D$ for $D = G(\mathbb{R})/K$ a hermitian symmetric space, with G an algebraic group over $\mathbb{Q}$ and Γ an arithmetic subgroup of $G(\mathbb{Q})$.

It turns out that, if the image of Γ in $\mathrm{Is}^+(D)$ is torsion free, then it acts freely on D and therefore the quotient $\Gamma\backslash D$ is a smooth complex manifold. Moreover, the fact that Γ is arithmetic implies that $\Gamma\backslash D$ has finite volume. Even more, we not only obtain a complex manifold, but a smooth algebraic variety over $\mathbb{C}$.

Theorem 3.3 **(Baily-Borel [BB66]).** — *Let $D(\Gamma) = \Gamma\backslash D$ be the quotient of a hermitian symmetric space by a torsion free arithmetic subgroup Γ of $\mathrm{Is}^+(D)$. Then $D(\Gamma)$ has a canonical realization as a Zariski-open subset of a projective algebraic variety $D(\Gamma)^*$. In particular, it has a canonical structure of a quasi-projective algebraic variety over $\mathbb{C}$.*

Remark 3.4. — In fact, $D(\Gamma)$ has a structure of algebraic variety defined over a number field.

We can now give a definition of Shimura variety. The precise definition of Shimura varieties is rather involved as it describes a family of varieties of a very precise form and requires the language of adéles.

We refer the reader to the expository paper by Milne [**Mil04**] for the precise definition and a different presentation of Shimura varieties. In these notes we just will present the definition of *connected Shimura varieties* using congruence subgroups.

Definition 3.5. — Let G be an algebraic reductive group defined over $\mathbb{Q}$. Choose an embedding $G \hookrightarrow \mathrm{GL}_n$ and, for each integer $N \geq 1$, define

$$\Gamma(N) = G(\mathbb{Q}) \cap \{g \in \mathrm{GL}_n(\mathbb{Z}) \mid g \equiv \mathrm{Id} \mod N\}.$$

A *congruence subgroup* of G is a subgroup that contains some $\Gamma(N)$ with finite index.

Every congruence subgroup is an arithmetic subgroup but the converse is not true in general. For instance SL_2 has infinitely many arithmetic subgroups that are not congruence subgroups, see for instance [**LS03**, Chs. 6,7].

Definition 3.6. — A *connected Shimura datum* is the data of

1. a semisimple algebraic group G defined over $\mathbb{Q}$ of non-compact type,
2. a connected hermitian symmetric space D, and
3. an action of $G^{\mathrm{ad}}(\mathbb{R})^+$ on D defined by a surjective homomorphism $G^{\mathrm{ad}}(\mathbb{R})^+ \to \mathrm{Is}^+(D)$.

Example 3.7. — The data of $G = \mathrm{SL}_2$, $D = \mathbb{H}$ and the action of G on $\mathbb{H}$ by Moebius transforms is a connected Shimura datum.

Definition 3.8. — The *connected Shimura variety* $\mathrm{Sh}^\circ(G, D)$ is the inverse system of locally symmetric varieties $(\Gamma\backslash D)_\Gamma$, where Γ runs over the set of torsion-free arithmetic subgroups of $G^{\mathrm{ad}}(\mathbb{R})$ whose preimage in $G(\mathbb{R})^+$ is a congruence subgroup.

This definition has some disadvantages. First, for some number theoretic applications it is better to start with a reductive group and not a semisimple group. Second, each space on the tower $(\Gamma\backslash D)_\Gamma$ has a model over a number field, but the number field may change with the subgroup. Therefore, the inductive limit may only be defined over an infinite extension of $\mathbb{Q}$. For instance the modular curve $\Gamma(N)\backslash\mathbb{H}$ has a canonical model over $\mathbb{Q}[\xi_N]$, where ξ_N is a primitive N-th root of 1. Therefore, the connected Shimura variety $\mathrm{Sh}^\circ(\mathrm{SL}_2, \mathbb{H})$ is defined over the whole cyclotomic extension of $\mathbb{Q}$.

The definition of Shimura varieties (as opposed to connected Shimura varieties) solves these disadvantages. In particular Shimura varieties are defined over number fields. See [**Mil04**, §5] for more details.

In many cases, the algebraic variety $X = D(\Gamma) = \Gamma\backslash D$ is non-compact and it is useful to compactify it. Part of the content of Theorem 3.3 is that there exists a canonical compactification $D(\Gamma)^*$ that is a projective variety. This compactification is canonical but is highly singular. Of course we can appeal to the general theorem of resolution of singularities to find a non-singular compactification. But then we lose control on the compactification. The theory of toroidal compactifications [**AMRT10**] gives us a controlled family of smooth compactifications.

Theorem 3.9. — *Let $D(\Gamma) = \Gamma\backslash D$ be the quotient of a hermitian symmetric space by an arithmetic subgroup Γ of $\mathrm{Is}^+(D)$. Then $D(\Gamma)$ admits a family of compactifications that can be described combinatorially in terms of the groups G, K and Γ and are called toroidal compactifications. If Γ is neat, there are smooth projective toroidal compactifications $\overline{X}$ of X with $\overline{X} \setminus X$ a simple normal crossings divisor.*

Example 3.10. — We recall how the general theory of compactifications of locally symmetric spaces applies to the modular curve. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a subgroup of finite index such that $-\mathrm{Id} \not\in \Gamma$ and write $Y(\Gamma) = \Gamma\backslash\mathbb{H}$. The group Γ acts on $\mathbb{P}^1(\mathbb{Q})$. The curve $Y(\Gamma)$ can be compactified by adding one point for each equivalence class for the action of Γ on $\mathbb{P}^1(\mathbb{Q})$. These points are called cusps. The compactified curve is denoted $X(\Gamma)$.

$$X(\Gamma) = Y(\Gamma) \cup \{\text{cusps}\}$$

In order to understand the algebraic structure of $X(\Gamma)$, we give a local coordinate on $X(\Gamma)$ around each cusp. Any cusp can be sent to the point ∞ by an element $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Replacing Γ by $\gamma\Gamma\gamma^{-1}$, it is enough to understand what happens at the cusp $\infty = (1:0) \in \mathbb{P}^1(\mathbb{Q})$.

The stabilizer of the point ∞ in Γ is an infinite cyclic group

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & nw \\ 0 & 1 \end{pmatrix} \middle| n \in \mathbb{Z} \right\}$$

The integer $w > 0$ is called the width of the cusp ∞. Let Δ be the interior of the unit disk and Δ^* the interior of the unit disk with the point 0 removed. The map

$$\begin{array}{ccc} \mathbb{H} & \longrightarrow & \Delta^* \\ \tau & \longmapsto & \exp(2\pi i\tau/w) \end{array}$$

induces an isomorphism $\Gamma_\infty\backslash\mathbb{H} \to \Delta^*$. The map

$$\Gamma_\infty\backslash\mathbb{H} \to \Gamma\backslash\mathbb{H}$$

sends a neighborhood of $0 \in \Delta$ to a neighborhood of the cusp ∞ in $X(\Gamma)$. Thus we can use $q = \exp(2\pi i\tau/w)$ as a local coordinate around the cusp ∞.

4. Equivariant vector bundles and invariant metrics

We next move our attention to the construction of holomorphic equivariant vector bundles on a hermitian symmetric space. Being equivariant they will descend to vector bundles on any quotient by an arithmetic subgroup.

Let $D = G/K$ be a hermitian symmetric space, V a complex vector space of finite dimension and $\rho\colon K \to \mathrm{GL}(V)$ a representation of K. The group K acts on G on the right and the map $G \to D$ is a principal K-bundle. Through the representation ρ, K also acts on V. This time on the left. Thus we can form the space

$$\mathscr{V} = G \underset{K}{\times} V \coloneqq G \times V/\sim$$

where $\sim$ is the equivalence relation

$$(g \cdot k, v) \sim (g, \rho(k)(v)), \text{ for all } g \in G, v \in V, k \in K.$$

The map $\mathscr{V} \to D$ given by $[(g, v)] \mapsto g \cdot o$ is well defined and $\mathscr{V}$ is a differentiable vector bundle over D with fiber V. Moreover $\mathscr{V}$ is a G-equivariant vector bundle with the G-action given by

$$g \cdot [(g', v)] = [(gg', v)].$$

But, is it possible to give to $\mathscr{V}$ the structure of an equivariant *holomorphic* vector bundle? The answer is yes, but not in a unique way.

To this end, we can complexify ρ to a representation of $\rho\colon K_{\mathbb{C}} \to \mathrm{GL}(V)$ and we can extend it to a representation of $K_{\mathbb{C}} \cdot P_-$. One possible way to do this is to declare that the action of P_- is trivial.

We now repeat the previous process to obtain a holomorphic vector bundle

$$\mathscr{V}_h \coloneqq G_{\mathbb{C}} \underset{K_{\mathbb{C}}\cdot P_-}{\times} V$$

on the compact dual $\check{D}$ of D. The restriction of $\mathscr{V}_h$ to D is a holomorphic vector bundle that has the same underlying differentiable structure as $\mathscr{V}$. We will identify both vector bundles and think of $\mathscr{V}$ as an equivariant holomorphic vector bundle.

Remark 4.1. — The condition that the action of P_- is trivial gives a particular choice of holomorphic structure. The holomorphic vector bundles produced using this condition are called *fully decomposable vector bundles*. One has to be careful that some interesting equivariant holomorphic vector bundles on D are not fully decomposable.

Denote by $\pi\colon G_{\mathbb{C}} \to \check{D}$ the projection deduced from Theorem 2.6 and let $s\colon D \to G_{\mathbb{C}}$ be a holomorphic map such that the composition $\pi \circ s\colon D \to \check{D}$ agrees with the embedding of D in $\check{D}$. For instance we can take s as the composition $D \hookrightarrow \mathfrak{p}_+ \xrightarrow{\exp} P_+ \hookrightarrow G_{\mathbb{C}}$.

The holomorphic vector bundle $\pi^*\mathscr{V}_h$ over $G_{\mathbb{C}}$ has a canonical trivialization $\pi^*\mathscr{V}_h = G_{\mathbb{C}} \times V$. Thus $\mathscr{V} \simeq s^*G_{\mathbb{C}} \times V \simeq D \times V$ is a trivialized vector bundle. Although the trivialization depends on the map s.

Let $g \in G$. Since $\pi(g \cdot s(x)) = \pi(s(g \cdot x))$, we deduce that

$$gs(x) = s(gx)j(g,x)p_-(g,x)$$

for well defined elements $j(g,x) \in K_{\mathbb{C}}$ and $p_-(g,x) \in P_-$. The map j is a cocycle in the sense that

$$j(gg',x) = j(g, g' \cdot x)j(g',x).$$

This cocycle depends on the choice of s. The transformation rule for p_- involves conjugating by k, but we will not need it.

Exercise 5. — Prove that, in the trivialization determined by s, the action of g on $\mathscr{V}$ is given by

$$g \cdot (x,v) = (g \cdot x, \rho(j(g,x))(v)$$

Example 4.2. — We go back to the example $D = \mathbb{H}$. Both groups K and $\widetilde{K}$ are isomorphic to $\mathrm{U}(1)$. The irreducible representations of $U(1)$ are one dimensional and classified by its weight. The representation of weight $k \in \mathbb{Z}$ is given, for $z \in \mathrm{U}(1) \subset \mathbb{C}^\times$ by

$$\rho_k(z)(v) = z^k v.$$

Since the map $\widetilde{K} \to K$ is a covering of order 2, a representation of K of weight n determines a representation of $\widetilde{K}$ of weight $2n$.

Let k be an even integer. The one-dimensional representation of K of weight $k/2$ induces the representation of $\widetilde{K} = \mathrm{SO}_2(\mathbb{R})$ of weight k. Let $\mathscr{L}_k$ be the corresponding G-equivariant line bundle on $\mathbb{H}$. We use the map (2.3) to trivialize $\mathscr{L}_k$. By Exercise 4 (7) we deduce that the action of G on $\mathscr{L}_k$ is given, in this trivialization, by

$$g \cdot (\tau, v) = (g \cdot \tau, (c\tau + d)^k v).$$

Exercise 6. — In this exercise we want to recover the classical definition of modular forms. Let k be an even integer and let $\rho_k\colon \mathrm{SO}_2(\mathbb{R}) \to \mathrm{GL}_1(\mathbb{C})$ be the representation of weight k. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a neat subgroup of finite index, and write $Y(\Gamma) = \Gamma\backslash\mathbb{H}$ for the corresponding open modular curve. Since Γ is neat, $Y(\Gamma)$ is a smooth quasi-projective curve. Let $\mathscr{L}_k$ be the equivariant line bundle on $\mathbb{H}$ determined by ρ_k and let $\mathscr{M}_k(\Gamma)$ the induced line bundle in $Y(\Gamma)$. Prove that a section s of $\mathscr{L}_k$ descends to a section of $\mathscr{M}_k(\Gamma)$ if and only if it is invariant under Γ. That is, for any $\gamma \in \Gamma$,

$$\gamma s(\tau) = s(\gamma \cdot \tau).$$

Using the trivialization of Exercise 5 we can identify sections of $\mathscr{L}_k$ with holomorphic functions on $\mathbb{H}$. Prove that a holomorphic function f defines a section of $\mathscr{M}_k(\Gamma)$ if and only if, for any $\gamma \in \Gamma$,

$$f(\gamma \cdot \tau) = (c\tau + d)^k f(\tau). \tag{4.1}$$

Remark 4.3. — Of course, the condition (4.1) makes sense for non-necessarily neat arithmetic subgroups and for odd weight k. But not always such functions can be interpreted as sections of a line bundle on $Y(\Gamma)$.

The next task is to put metrics on the vector bundles we have constructed. Since the group K is compact, given any complex representation $\rho\colon K \to \mathrm{GL}(V)$ there is a (non-unique) hermitian metric on V that is invariant under the action of K. Let $\langle\cdot,\cdot\rangle$ be one such K-invariant metric on V. Then there is a unique way to define a hermitian metric on the vector bundle $\mathscr{V} \coloneqq G \underset{K}{\times} V \to D$ such that:

1. the restriction to the fiber $\mathscr{V}_o = V$ over the point o agrees with the given metric;
2. it is invariant under the action of G. That is, if $g \in G$, $p \in D$, $\mathscr{V}_p$ the fiber of $\mathscr{V}$ over p and $v, w \in \mathscr{V}_p$, then

$$\langle u, v\rangle_p = \langle g\cdot u, g\cdot v\rangle_{g\cdot p}.$$

In fact, given $p \in D$ and $v, w \in \mathscr{V}_p$, we can choose a $g \in G$ with $g \cdot p = o$ and we necessarily have

$$\langle u, v\rangle_p = \langle g\cdot u, g\cdot v\rangle_o. \tag{4.2}$$

So such invariant metric is unique once we have fixed the metric on V. If g' is another element such that $g' \cdot p = o$, then $g' = kg$ with $k \in K$, and

$$\langle g'\cdot u, g'\cdot v\rangle_o = \langle kg\cdot u, kg\cdot v\rangle_o = \langle g\cdot u, g\cdot v\rangle_o.$$

Thus the product (4.2) does not depend on the choice of g, and the invariant hermitian metric is well defined.

Exercise 7. — 1. Let $\tau \in \mathbb{H}$ and $g \in \mathrm{SL}_2(\mathbb{R})$. Show that

$$\Im(g \cdot \tau) = \frac{\Im\tau}{|c\tau + d|^2},$$

where $\Im\tau$ denotes the imaginary part of τ.

2. Let $\rho_k\colon \mathrm{SO}_2(\mathbb{R})$ be the representation of $\mathrm{SO}_2(\mathbb{R})$ of weight k and let $\mathscr{L}_k$ the associated equivariant line bundle over $\mathbb{H}$ and let $C > 0$ be a real positive constant. On the trivialization of Example 4.2, show that the norm

$$\|(\tau, v)\|^2 = |v|^2 (4\pi\Im\tau)^k \tag{4.3}$$

defines an invariant metric on $\mathscr{L}_k$. Show that every invariant metric on $\mathscr{L}_k$ is a constant multiple of this one. The metric on $\mathscr{L}_k$ given by (4.3) is called the *Petersson metric.*

5. Log-singular metrics and log-log forms

Let $D = G/K$ be a hermitian symmetric space, where G is the connected component of the identity of the set of real points of a semisimple algebraic group defined over $\mathbb{Q}$, and let $\Gamma \subset G$ be an arithmetic subgroup of G. Put $X = \Gamma\backslash D$ for the quotient of the symmetric space by the arithmetic group.

Let $\rho\colon K \to \mathrm{GL}(V)$ be a complex representation of K. Since the associated fully decomposable vector bundle $\mathscr{V}$ over D is G-equivariant, it is also Γ equivariant. Thus it descends to a vector bundle E on X. If V has a K-invariant metric, the vector bundle $\mathscr{V}$ has an induced invariant metric, that descends to a metric on E. Let $\overline{X}$ be a toroidal compactification of X as in Theorem 3.9. Once we have compactified X, it is a natural question to extend to $\overline{X}$ the hermitian vector bundle we have constructed. Let us first examine the case of the modular curve.

Example 5.1. — Let s be a section of the line bundle $\mathscr{M}_k(\Gamma)$ on $Y(\Gamma)$ and let f be the corresponding holomorphic function given in Exercise 6. By the transformation rule (4.1), we see that f is invariant under the action of Γ_∞. By Fourier analysis we have an expansion

$$f = \sum_{n\in\mathbb{Z}} a_n q^n \tag{5.1}$$

with $q = \exp(2\pi i\tau/w)$ as in Example 3.10. The section s is said to be meromorphic at the cusp ∞ if the expansion (5.1) has only a finite number of negative terms and is said to be holomorphic if this expansion has only non-negative terms. In fact we define

$$\operatorname{ord}_\infty(s) = \min\{n \mid a_n \neq 0\}.$$

This procedure specifies an extension of $\mathscr{M}_k(\Gamma)$ to $X(\Gamma)$.

Exercise 8. — Let s be the section of $\mathscr{M}_k(\Gamma)$ of Example 5.1 and f the corresponding function. Let $\|\cdot\|$ denote the Petersson metric (4.3).

1. Show that, in a neighborhood of the cusp ∞
$$\|s\|^2 = |f|^2(-w\log|q|^2)^k = \varphi(q)|q|^{2\operatorname{ord}_\infty s}(\log|q|^2)^k,$$
where φ is a continuous function with $\varphi(0) \neq 0$ and w is the width of the cusp.
2. The first Chern form of the line bundle $\mathscr{M}_k$ with the metric $\|\cdot\|$ is given by
$$c_1(\mathscr{M}_k, \|\cdot\|) = \frac{i}{2\pi}\partial\bar\partial(-\log\|s\|^2).$$
Show that, locally around the cusp, this first Chern form can be written as

$$c_1(\mathscr{M}_k, \|\cdot\|) = \frac{ik}{2\pi}\frac{dq\wedge d\bar q}{q\bar q(\log q\bar q)^2}. \tag{5.2}$$

The $(1,1)$-form given in equation (5.2) determines a metric on $Y(\Gamma)$ called the *hyperbolic metric* or also the *Poincaré metric*.

As Exercise 8 shows, we can not extend the line bundle $\mathscr{M}_k(\Gamma)$ in such a way that the Petersson metric extends to a smooth (or even continuous) metric on the whole $X(\Gamma)$. The problematic term being $(\log|q|^2)^k$. The best we can hope for is a log-singular metric.

Log-log growth forms is a class of singular differential forms. Among them one finds functions like $\log(-\log|q|)$ and the Poincaré metric (5.2). Log-log growth forms form an algebra and the complex of log-log growth forms has nice cohomological properties. To study the properties of log-log growth forms, until the end of the section, we forget about symmetric spaces and work with arbitrary complex manifolds and normal crossings divisors. Thus we change the notation accordingly.

Let now X be a complex manifold, we will denote by $\mathscr{E}^*_X$ the sheaf of smooth complex valued differential forms and by $\mathscr{E}^*_{X,\mathbb{R}}$ the subsheaf of real valued forms. We will use roman typography to denote the space of global sections. Namely

$$E_X^* = \Gamma(X, \mathscr{E}_X^*).$$

Recall that E_X^* has a bigrading,

$$E_X^n = \bigoplus_{p+q=n} E_X^{p,q},$$

where $\eta \in E_X^{p,q}$ if it can be locally written as

$$\eta = \sum_{I,J} f_{I,J}(z_1, \dots, z_d) dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q},$$

where the sum runs over subsets $I = \{i_1, \dots, i_p\}$ and $J = \{j_1, \dots, j_q\}$ of $\{1, \dots, d\}$, and $f_{I,J}$ are $\mathscr{C}^\infty$-functions.

There is also a Hodge filtration F given by

$$F^p E_X^n = \bigoplus_{p' \geq p} E_X^{p', n-p'},$$

and a decomposition $d = \partial + \bar{\partial}$, where

$$\partial \colon E_X^{p,q} \to E_X^{p+1,q} \quad \text{and} \quad \bar{\partial} \colon E_X^{p,q} \to E_X^{p,q+1}.$$

Let $D \subset X$ a normal crossings divisor. This means that locally, D is given by the equation

$$z_1 \cdots z_k = 0. \tag{5.3}$$

That is, the divisor D is locally like a collection of coordinate hyperplanes. We will say that an open coordinate neighbourhood $U \subset X$ with coordinates $(z_1, \dots, z_k)$ is a *small coordinate neighbourhood adapted* to D if D has an equation of the form (5.3) for some $k \leq n$ and each point $p \in U$ has coordinates $(z_1, \dots, z_i)$ satisfying $|z_i| \leq 1/(2e)$.

In the next definition we will use multi-index notation. That is, for any multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}_{\geq 0}^d$, we write

$$|\alpha| = \sum_{i=1}^d \alpha_i, \qquad z^\alpha = \prod_{i=1}^d z_i^{\alpha_i}, \qquad \bar{z}^\alpha = \prod_{i=1}^d \bar{z}_i^{\alpha_i},$$

$$r^\alpha = \prod_{i=1}^d r_i^{\alpha_i}, \qquad (\log(1/r))^\alpha = \prod_{i=1}^d (\log(1/r_i))^{\alpha_i},$$

$$\frac{\partial^{|\alpha|}}{\partial z^\alpha} f = \frac{\partial^{|\alpha|}}{\prod_{i=1}^d \partial z_i^{\alpha_i}} f, \qquad \frac{\partial^{|\alpha|}}{\partial \bar{z}^\alpha} f = \frac{\partial^{|\alpha|}}{\prod_{i=1}^d \partial \bar{z}_i^{\alpha_i}} f.$$

If α and β are multi-indices, we denote by $\alpha+\beta$ the multi-index with components $\alpha_i+\beta_i$. If α is a multi-index and $k\geq 1$ is an integer, we will denote by $\alpha^{\leq k}$ the multi-index

$$\alpha_i^{\leq k}=\begin{cases}\alpha_i, & i\leq k,\\ 0, & i>k.\end{cases}$$

The following definition is taken from [**BGKK05**, Definition 2.17].

Definition 5.2. — Let X be a complex manifold and $D\subset X$ a normal crossings divisor. Let U be a small open coordinate neighborhood adapted to D. For every integer $K\geq 0$, we say that a smooth complex function f on $V\setminus D$ has *log-log growth along D of order K*, if there exists an integer N_K such that, for every pair of multi-indices $\alpha,\beta\in\mathbb{Z}^d_{\geq 0}$ with $|\alpha+\beta|\leq K$, it holds the inequality

$$\left|\frac{\partial^{|\alpha|}}{\partial z^{\alpha}}\frac{\partial^{|\beta|}}{\partial \bar{z}^{\beta}}f(z_1,\dots,z_d)\right|\prec\frac{\left|\prod_{i=1}^{k}\log(\log(1/r_i))\right|^{N_K}}{|z^{\alpha^{\leq k}}\bar{z}^{\beta^{\leq k}}|}. \tag{5.4}$$

We say that f has *log-log growth along D of infinite order*, if it has log-log growth along D of order K for all $K\geq 0$. The *sheaf of differential forms on X with log-log growth along D of infinite order* is the subalgebra of $\iota_*\mathscr{E}^*_U$ generated, in each small coordinate neighborhood U adapted to D, by the functions with log-log growth along D and the differentials

$$\begin{aligned}&\frac{dz_i}{z_i\log(1/r_i)},\ \frac{d\bar{z}_i}{\bar{z}_i\log(1/r_i)}, && \text{for } i=1,\dots,k,\\ &dz_i,\ d\bar{z}_i, && \text{for } i=k+1,\dots,d.\end{aligned}$$

A differential form with log-log growth along D of infinite order will be called a *log-log growth form.*

A differential form ω on $X\setminus D$ is called *log-log* if ω, $\partial\omega$, $\bar{\partial}\omega$ and $\partial\bar{\partial}\omega$ are log-log growth forms. The sheaf of log-log forms is denoted $\mathscr{E}^*\langle\langle D\rangle\rangle$ and the space of goblal sections as $E^*\langle\langle D\rangle\rangle$.

Log-log forms have very nice properties. In fact, they are almost as good as smooth forms for many purposes. First $\mathscr{E}^*\langle\langle D\rangle\rangle$ is an algebra and has a bigrading

$$\mathscr{E}^*\langle\langle D\rangle\rangle=\bigoplus_{p+q=n}\mathscr{E}^{p,q}\langle\langle D\rangle\rangle$$

and an associated Hodge filtration

$$F^p\mathscr{E}^*\langle\langle D\rangle\rangle = \bigoplus_{p'\geq p}\mathscr{E}^{p',n-p'}\langle\langle D\rangle\rangle.$$

We will denote by $\mathscr{D}^*_X$ the sheaf of currents on X. It has also a bigrading and a Hodge filtration. Currents are a fundamental tool in Arakelov theory. For more details about currents, the reader is referred to de Rham book [**dR73**], also the fourth chapter of [**GH94**] or, for a quick introduction the first section of [**GS90a**].

Recall that a morphisms of complexes (in any abelian category) $f\colon A^*\to B^*$ is called a quasi-isomorphism if it induces an isomorphism in cohomology objects. A morphism of filtered complexes

$$f\colon (A^*,F)\to(B^*,F),$$

is called a filtered quasi-isomorphism if all the induced morphisms

$$F^pf\colon F^pA^*\to F^pB^*$$

are quasi-isomorphisms.

The next result summarizes the main properties of the complex of log-log forms.

Theorem 5.3. — 1. *Every log-log form is locally integrable. Moreover, they have no residue, this means that the map $\mathscr{E}^*_X\langle\langle D\rangle\rangle\to\mathscr{D}^*_X$ is a morphism of complexes.*

2. *The maps*

$$(\mathscr{E}^*_X,F)\longrightarrow(\mathscr{E}_{X^*}\langle\langle D\rangle\rangle,F)\longrightarrow(\mathscr{D}^*_X,F)$$

are filtered quasi-isomorphisms. Therefore, if X is a smooth projective variety, we can use the complex $E^*_X\langle\langle D\rangle\rangle$ to compute the complex cohomology of X with its real structure and its Hodge filtration.

Proof. — The first statement is proved in [**Mum77**] for good forms, that are very close to log-log forms. The proof in *loc. cit* can easily be extended to log-log forms.

The second statement is [**BGKK05**, Theorem 2.23]. □

Recall that, to any holomorphic hermitian vector bundle we can associate a collection of characteristic forms that represent the characteristic classes of the bundle. Let E be a holomorphic vector bundle on a complex manifold X. Choose a local holomorphic frame for E. The hermitian metric is represented in this frame by a matrix of smooth functions h. The curvature matrix of the metric is the matrix of $(1,1)$-forms

$$K = \bar{\partial}(\partial h \cdot h^{-1}).$$

The coefficients of the characteristic polynomial of K do not depend on the frame and define global differential forms that represent the characteristic classes of E. For more details, see for instance [**GH94**, Chapter 3, §3] or [**Wel80**, III §3].

Definition 5.4. — Let X be a complex manifold, D a normal crossings divisor and E a holomorphic vector bundle on X with a smooth hermitian metric on $X \setminus D$. The metric is said to be *log-singular* along D if, for each small coordinate chart of X adapted to D, and holomorphic frame for E, the following estimates hold.

1. The functions $h_{i,j}$ and $(\det h)^{-1}$ grow at most logarithmically along D.
2. The 1-forms $(\partial h \cdot h^{-1})_{i,j}$ are log-log forms along D.

The interest for us of log-singular metrics and log-log forms comes from the following two theorems by Mumford [**Mum77**, Theorem 1.4] and [**Mum77**, Theorem 3.1]. See also [**BGKK05**].

Theorem 5.5. — *Let X be a complex manifold and D a normal crossing divisor. Let E be a vector bundle on X with a smooth metric $\|\cdot\|$ on $X \setminus D$ that is log singular along D. Then the characteristic forms of $(E, \|\cdot\|)$ are log-log forms. In particular they are locally integrable and the associated currents represent the Chern classes of E.*

Theorem 5.6. — *Let $D = G/K$ be a hermitian symmetric space, with G a semisimple algebraic group defined over $\mathbb{Q}$, Γ a neat arithmetic subgroup of G, V a complex vector space with a hermitian metric and ρ a unitary representation of K on V. To these data we have associated a smooth quasi-projective complex variety $X = \Gamma\backslash G/K$ and a hermitian vector bundle E on X with hermitian metric h. Let $\overline{X}$ be a toroidal compactification of X with $D = \overline{X} \setminus X$ a simple normal crossing divisor. Then E can be extended uniquely to a vector bundle $\overline{E}$ over $\overline{X}$ such that h is log singular along D.*

6. Arakelov geometry with log-log forms

Following the ideas of the previous section, in order to extend Arakelov theory to Shimura varieties, it is enough to replace smooth forms by log-log forms. The theory is almost identical to the classical theory by Gillet and Soulé [**GS90a**]. For more details the reader can follow Chapter I by Soulé of this volume. We summarize the results in the case of varieties over Spec($\mathbb{Z}$). All the changes occur at the generic fibre. Thus there is no difficulty to extend everything to arbitrary arithmetic rings. A variant of the theory presented here has been developed in [**BGKK05**] and [**BGKK07**].

Let $\mathscr{X}$ be a regular flat projective variety over $\operatorname{Spec}\mathbb{Z}$. Let $X_{\mathbb{C}}$ be the associated smooth projective complex variety and let $D \subset X_{\mathbb{C}}$ be a normal crossing divisor defined over $\mathbb{R}$. There is an antilinear involution $F_\infty \colon X_{\mathbb{C}} \to X_{\mathbb{C}}$ corresponding to complex conjugation of the coordinates. Since the divisor D is defined over $\mathbb{R}$ it is invariant under this involution.

Definition 6.1. — Let Z be a codimension p cycle on $\mathscr{X}$. A *log-log Green current* for Z is a current $g_Z \in D_X^{p-1,p-1}$ satisfying a symmetry condition with respect complex conjugation

$$F_\infty^* g_Z = (-1)^{p-1} g_Z$$

and the equation

$$\omega_Z \coloneqq dd^c g_Z + \delta_Z \in E_X^{p,p}\langle\langle D\rangle\rangle.$$

A *codimension p log-log arithmetic cycle* is a pair (Z, g_Z) with Z a codimension p cycle and g_Z a log-log Green current for Z. We denote by $\widehat{Z}^p(\mathscr{X}, \langle\langle D\rangle\rangle)$ the group of codimension p log-log arithmetic cycles.

The group of rational cycles does not change with respect to the classical Gillet and Soulé theory:

$$\widehat{\operatorname{Rat}}^p(\mathscr{X}, \langle\langle D\rangle\rangle) = \widehat{\operatorname{Rat}}^p(\mathscr{X}) = \operatorname{Span}\{(\operatorname{div}(f), [-\log|f|]) + (0, \partial u + \bar{\partial} v)\}$$

Definition 6.2. — The *log-log arithmetic Chow groups* are defined as

$$\widehat{\operatorname{CH}}^p(\mathscr{X}, \langle\langle D\rangle\rangle) = \widehat{Z}^p(\mathscr{X}, \langle\langle D\rangle\rangle) / \widehat{\operatorname{Rat}}^p(\mathscr{X}).$$

A consequence of Theorem 5.3 is that any log-log Green current for a cycle Z, that meets D properly, can be represented by a differential forms with logarithmic singularities along Z and that is log-log along D. This allow one to prove that most of the properties of the arithmetic Chow rings carry over to the log-log arithmetic Chow groups.

For instance, one can define the intersection product as in the classical case, but there is the caveat that we have to move our cycles to meet properly D as well.

Let Z and W be cycles of codimension p and q respectively in $\mathscr{X}$. Assume that $Z_{\mathbb{C}}$ and $W_{\mathbb{C}}$ intersect properly and that both intersect properly with D. Let g_Z and g_W be log-log Green currents for Z and W respectively and represent them with differential forms with log and log-log singularities. Then the $*$-product

$$g_Z * g_W = g_Z \wedge \delta_W + \omega_Z \wedge g_W$$

is well defined. As in the classical case, we face the technical problem that Z and W intersect properly on the generic fibre but they do not need to intersect properly globally. In fact the moving lemma is only known for varieties over a field. To remedy this problem following **[GS90a]**, one introduce the Chow groups of $\mathscr{X}$ of cycles that do not meet $\mathscr{X}_{\mathbb{Q}}$. Denote these groups as $\mathrm{CH}^{p+q}(\mathscr{X})_{\mathrm{fin},\mathbb{Q}}$. Then there is a well defined intersection product

$$(Z, g_Z) \cdot (W, g_W) \in \mathrm{CH}^{p+q}(\mathscr{X})_{\mathrm{fin},\mathbb{Q}} \oplus \widehat{Z}^{p+q}(\mathscr{X}_{\mathbb{Q}}, \langle\langle D \rangle\rangle).$$

By the moving lemma, this induces an algebra structure on

$$\widehat{\mathrm{CH}}^*(\mathscr{X}, \langle\langle D \rangle\rangle)_{\mathbb{Q}} = \bigoplus_p \widehat{\mathrm{CH}}^p(\mathscr{X}, \langle\langle D \rangle\rangle) \otimes \mathbb{Q}.$$

The inverse image also needs some compatibility with the divisor D. Let $f\colon \mathscr{Y} \to \mathscr{X}$ be a morphism of varieties over $\mathbb{Z}$. Write $Y = \mathscr{Y}_{\mathbb{C}}$ and $X = \mathscr{X}_{\mathbb{C}}$ for the associated complex varieties, and let $f_{\mathbb{C}}\colon Y \to X$ be the induced morphism of complex varieties. Clearly, if the image by $f_{\mathbb{C}}$ of any component of Y is contained in $D \subset X$, then there is no hope to define the inverse image. If we add the hypothesis that $E = f_{\mathbb{C}}^{-1}(D)$ is a normal crossing divisor of Y, then there is a well defined inverse image map

$$f^*\colon \widehat{\mathrm{CH}}^*(\mathscr{X}, \langle\langle D \rangle\rangle) \longrightarrow \widehat{\mathrm{CH}}^*(\mathscr{Y}, \langle\langle E \rangle\rangle).$$

The direct image can be more complicated and we discuss only the case of morphisms to points. Assume that $\mathscr{X}$ is equidimensional of dimension $d+1$ and let $\pi\colon \mathscr{X} \to \operatorname{Spec}\mathbb{Z}$ be the structural map. Then there are well defined direct images

$$\pi_*\colon \widehat{\mathrm{CH}}^p(\mathscr{X}, \langle\langle D \rangle\rangle) \longrightarrow \widehat{\mathrm{CH}}^{p-d}(\operatorname{Spec}\mathbb{Z}).$$

Since

$$\widehat{\mathrm{CH}}^p(\operatorname{Spec}\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } p = 0, \\ \mathbb{R}, & \text{if } p = 1, \\ 0, & \text{otherwise.} \end{cases}$$

only the groups $\widehat{\mathrm{CH}}^{d+1}(\mathscr{X}, \langle\langle D \rangle\rangle)$ and $\widehat{\mathrm{CH}}^{d}(\mathscr{X}, \langle\langle D \rangle\rangle)$ have a non-zero direct image to $\operatorname{Spec}\mathbb{Z}$.

In the paper [**BGKK05**] it is developed a theory of arithmetic characteristic classes for log-singular hermitian vector bundles extending the theory of [**GS90b**]. The essential ingredients for this extension are Mumford's theorems 5.6 and 5.5.

Theorem 6.3 **([BGKK05]).** — *Let $\mathscr{X}$ be a regular projective arithmetic variety over* $\operatorname{Spec}\mathbb{Z}$ *such that $X = \mathscr{X}_{\mathbb{C}}$ is a finite union of smooth toroidal compactifications of locally symmetric hermitian spaces. Let D be the boundary divisor that we assume to be a normal crossing divisor. Let $\overline{E}$ be a vector bundle on $\mathscr{X}$ with a singular hermitian metric such that, on each component of X, it is the fully decomposable holomorphic vector bundle associated to a unitary representation of the compact subgroup as described in §3. Then the theory of arithmetic characteristic classes can be extended to define log-log arithmetic characteristic classes*

$$\widehat{\mathrm{c}}_i(\overline{E}) \in \widehat{\mathrm{CH}}^i(\mathscr{X}, \langle\langle D\rangle\rangle).$$

Corollary 6.4. — *Let $\mathscr{X}$ be an integral model of a Shimura variety of dimension d, Y a codimension p cycle on $\mathscr{X}$ and $\overline{L}_0, \dots, \overline{L}_{d-p}$ be automorphic line bundles with their Petersson metric. Then the height*

$$h_{\overline{L}_0,\dots,\overline{L}_{d-p}}(Y)$$

is defined.

Example 6.5. — This example is taken from [**Küh01**]. Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. This is an arithmetic group, but it is not neat and the quotient $Y(1) = \Gamma\backslash\mathbb{H}$ has elliptic fixed points. We can ignore the orbitfold structure coming from the elliptic fixed points and pretend that $Y(1) \simeq \mathbb{A}^1$. Or, more precisely, we can choose a neat subgroup $\Gamma' \subset \Gamma$ and work on a covering of $Y(1)$ but it is easier to work directly with $Y(1)$. We compactify $Y(1)$ by adding one cusp to obtain a complete curve $X(1)$. Let k be a positive integer divisible by 12. Then $\mathscr{M}_k(1)$ is a line bundle on $X(1)$ with a log singular hermitian metric on the cusp: the so called the Petersson metric. The fact that we need to go to weight 12 is related with the orbitfold structure of $Y(1)$. We can choose a model $\mathscr{X}(1) \simeq \mathbb{P}^1_{\operatorname{Spec}\mathbb{Z}}$ of $X(1)$ and $\mathscr{M}_k(1)$ can be extended to a line bundle on $\mathscr{X}(1)$. See [**Küh01**] for details. We denote by $\overline{\mathscr{M}_k(1)}$ the line bundle on the model with the log singular hermitian metric. Then

$$\widehat{\mathrm{c}}_1(\overline{\mathscr{M}_k(1)}) \cdot \widehat{\mathrm{c}}_1(\overline{\mathscr{M}_k(1)}) = k^2 \left(\frac{1}{2}\zeta_{\mathbb{Q}}(-1) + \zeta'_{\mathbb{Q}}(-1)\right)$$

where $\zeta_{\mathbb{Q}}$ is Riemann zeta function. As we see in this example, not only the value of the Riemann zeta function appears, but also the value of its derivative.

For more examples of arithmetic intersection numbers on Shimura varieties and its relation with Kudla's program, see [**KRY06**], [**BBGK07**], [**BHY15**], [**BO10**], [**BY09**], [**How15**].

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