Towards a standard completeness for a probabilistic logic on infinite-valued events

Tommaso Flaminio

Artificial Intelligence Research Institute (IIIA - CSIC)
Campus de la Univ. Autònoma de Barcelona s/n, 08193 Bellaterra, Spain.
`tommaso@iiaa.csic.es`

Abstract. MV-algebras with internal states, or SMV-algebras for short, are the equivalent algebraic semantics of the logic $\text{SFP}(\mathbf{L}, \mathbf{L})$ which allows to represent and reason about the probability of infinite-valued events. In this paper we will make the first steps towards establishing completeness for $\text{SFP}(\mathbf{L}, \mathbf{L})$ with respect to the class of standard SMV-algebras, a problem which has been left open since the first paper on SMV-algebras was published. More precisely we will prove that, if we restrict our attention to a particular, yet expressive, subclass of formulas, then theorems of $\text{SFP}(\mathbf{L}, \mathbf{L})$ are the same as tautologies of a class of SMV-algebras that can be reasonably called “standard”.

Keywords. MV-algebras, states, internal states, standard completeness, probabilistic logic.

1 Introduction

MV-algebras with an internal state, or SMV-algebras, have been introduced in [6] and they consist of an MV-algebra $\mathbf{A}$ (cf. [2]) and a unary map σ on $\mathbf{A}$ axiomatized so as to preserve some basic properties of a *state* of $\mathbf{A}$ (cf. [12, 5]), i.e., a $[0, 1]$ -valued, normalized and finitely additive function. These structures form a variety, denoted by SMV , which is the equivalent algebraic semantics for the probabilistic logic $\text{SFP}(\mathbf{L}, \mathbf{L})$ which permits to represent and reason about the probability of infinite-valued events. In that logic, in particular, events are described by formulas of the infinite-valued Łukasiewicz calculus (cf. [2]).

From the perspective of reasoning about uncertainty, the interest of Łukasiewicz events is twofold: whilst capturing properties of the world which are better described as *gradual* rather than *yes-or-no*, they also mimic bounded random variables. Indeed, any Łukasiewicz event e may be regarded as a $[0, 1]$ -valued continuous function e^* on a compact Hausdorff space (see [2, Theorem 9.1.5] and Section 2) and any state of e coincides with the expected value of e^* (see [4, Remark 2.8] and Section 3).

SMV-algebras provide a framework to treat a generalization of classical probability theory in a universal-algebraic setting. However, notwithstanding their meaningful expressive power, several problems concerning their universal-algebraic properties are still open. Here, we will concentrate on the following issue: determining a proper subclass of SMV-algebras which generates SMV and which

could be regarded as a class of “measure-theoretical” models. In other words, the above problem is to establishing a “standard completeness” for the logic $\text{SFP}(\mathbf{L}, \mathbf{L})$.

In particular we will show that, upon restricting our language to a special, yet sufficiently expressive subset of well-formed formulas, the logic $\text{SFP}(\mathbf{L}, \mathbf{L})$ turns out to be sound and complete with respect to a prominent subclass of SMV-algebras which can be reasonably called “standard”. Indeed, these standard SMV-algebras are grounded on the same MV-algebra of continuous real-valued functions and their internal states are obtained by integrating the continuous functions which correspond to Łukasiewicz events by a regular, Borel (and hence σ -additive) probability measure. Although our main result only offers a partial solution to the problem of establishing standard completeness for $\text{SFP}(\mathbf{L}, \mathbf{L})$, the idea behind its proof is promising and it suggests ways to tackle the general problem. Indeed, the class of formulas to which we will restrict our attention, called *purely probabilistic formulas*, form the ground probabilistic language upon which the all formulas of $\text{SFP}(\mathbf{L}, \mathbf{L})$ are inductively defined.

This paper is organized as follows: in the next section we will recall some basic definitions and needed results about Łukasiewicz logic and MV-algebras. Section 3 is dedicated to a brief introduction to generalized probability theory on Łukasiewicz logic, i.e., state theory, while in Section 4 we will focus on SMV-algebras and the probabilistic logic $\text{SFP}(\mathbf{L}, \mathbf{L})$. In Section 5 we will prove the main results of this paper and in particular that a purely probabilistic formula ϕ is a theorem of $\text{SFP}(\mathbf{L}, \mathbf{L})$ iff ϕ holds in all standard SMV-algebras.

2 Łukasiewicz logic and MV-algebras

The language of the infinite-valued Łukasiewicz calculus $\mathbf{L}$ is made of a countable set of propositional variables $\{q_1, q_2, \dots\}$, the binary connective $\oplus$, the unary connective $\neg$ and the constant $\perp$. Formulas are defined by the usual inductive rules. Further useful connectives are definable as follows:

$$\begin{aligned} \top &= \neg \perp; \varphi \odot \psi = \neg(\neg\varphi \oplus \neg\psi); \varphi \rightarrow \psi = \neg\varphi \oplus \psi; \varphi \wedge \psi = \varphi \odot (\varphi \rightarrow \psi); \\ \varphi \vee \psi &= \neg(\neg\varphi \wedge \neg\psi); \varphi \ominus \psi = \neg(\varphi \rightarrow \psi); d(\varphi, \psi) = (\varphi \rightarrow \psi) \oplus (\psi \rightarrow \varphi). \end{aligned}$$

We invite the reader to consult [8, 2] for an axiomatization of Łukasiewicz logic. For the sake of the present paper it is important to recall that this calculus has an equivalent algebraic semantics: the variety $\mathbf{MV}$ of *MV-algebras*. These are structures of the form $\mathbf{A} = (A, \oplus, \neg, \perp)$ (the same language of Łukasiewicz logic), where $(A, \oplus, \perp)$ is a commutative monoid and, defining further operations as above, the following equations hold:

$$\begin{aligned} x \oplus \top &= \top \\ (x \rightarrow y) \rightarrow y &= (y \rightarrow x) \rightarrow x. \end{aligned}$$

The algebraizability of Łukasiewicz logic with respect to $\mathbf{MV}$ implies that $\mathbf{L}$ is sound and complete w.r.t. the class of MV-algebras. This means the following: for every MV-algebra $\mathbf{A}$, define an *$\mathbf{A}$ -valuation* as a map v from the propositional

variables q_i 's to A which extends to all formulas by truth functionality; then, a formula φ is a theorem of Łukasiewicz logic iff for every MV-algebra $\mathbf{A}$ and every $\mathbf{A}$ -valuation v , $v(\varphi) = \top$.

Furthermore, every formula φ on propositional variables $q_1, \dots, q_k$ can be regarded, on the algebraic side, as a term $t(x_1, \dots, x_k)$ on the same number of variables. For every MV-algebra $\mathbf{A}$ and for every term t , we denote by $t^{\mathbf{A}}$ the *interpretation* (or *semantics*) of t in $\mathbf{A}$.

In every MV-algebra the relation $x \leq y$ iff $x \rightarrow y = \top$ determines a lattice-order which coincides with the one given by the operations $\wedge$ and $\vee$. If in $\mathbf{A}$ the order $\leq$ is linear, we will say that $\mathbf{A}$ is an *MV-chain*.

The following collects two typical examples of MV-algebras which will play a central role in this paper.

Example 1. (1) The *standard* MV-algebra $[0, 1]_{MV}$ has support on the real unit interval $[0, 1]$ and for all $x, y \in [0, 1]$, $x \oplus y = \min\{1, x + y\}$, $\neg x = 1 - x$ and $\perp = 0$. The semantics, in $[0, 1]_{MV}$, of the Łukasiewicz connectives we defined above is as follows:

$$\begin{aligned} \top &= 1; \quad x \odot y = \max\{0, x + y - 1\}; \quad x \rightarrow y = \min\{1, 1 - x + y\}; \\ x \wedge y &= \min\{x, y\}; \quad x \vee y = \max\{x, y\}; \quad x \ominus y = \max\{0, x - y\}; \\ d(x, y) &= |x - y| \text{ (the usual Euclidean distance).} \end{aligned}$$

Chang's theorem [1] shows that $\mathbb{MV}$ is generated by $[0, 1]_{MV}$, whence Łukasiewicz logic is complete w.r.t. to $[0, 1]_{MV}$.

(2) For every $k \in \omega$, let $\mathcal{F}(k)$ be the set of functions $f : [0, 1]^k \rightarrow [0, 1]$ which are continuous, piecewise linear and such that each piece has integer coefficient, i.e., the set of *k-variable McNaughton functions* [2]. The point-by-point application of the operations of $[0, 1]_{MV}$ makes $\mathcal{F}(k)$ an MV-algebra which coincides, up to isomorphism, with the free MV-algebra over k -variables. For every formula ψ of Łukasiewicz language, we will denote by f_ψ its corresponding McNaughton function. The free MV-algebra over infinitely-many generators will be henceforth denoted by $\mathcal{F}(\omega)$.

Let $\mathbf{A}$ and $\mathbf{B}$ be two MV-algebras. An *MV-homomorphism* is a function $h : A \rightarrow B$ such that, adopting without danger of confusion the same symbols for the operations of both algebras: (1) $h(\perp) = \perp$; (2) $h(x \oplus y) = h(x) \oplus h(y)$; (3) $h(\neg x) = \neg h(x)$. If S is a subset of A , a map $h : S \rightarrow B$ is a *partial homomorphism* provided that the above conditions (1-3) holds for the elements of S . Injective (partial) homomorphisms are called *(partial) embeddings*. (Partial) embeddings of $\mathbf{A}$ to $\mathbf{B}$ will be denoted by $\mathbf{A} \hookrightarrow \mathbf{B}$ ($\mathbf{A} \hookrightarrow^p \mathbf{B}$, respectively).

Let $\mathbf{A}$ be an MV-algebra and let S be a subset of A . We denote $\langle S \rangle_{\mathbf{A}}$ the MV-subalgebra of $\mathbf{A}$ generated by S . In particular, if S is finite, for every $b \in \langle S \rangle_{\mathbf{A}}$, there are $a_1, \dots, a_n \in S$ and an n -ary term t such that $b = t^{\mathbf{A}}(a_1, \dots, a_n)$.

Lemma 1. (1) For every MV-chain $\mathbf{A}$ and for every finite subset S of A , there exists a partial embedding $h_S : S \hookrightarrow^p [0, 1]_{MV}$.

(2) Let $\mathbf{A}, \mathbf{B}$ be MV-algebras, let S be a finite subset of A . Every partial homomorphism $h_S : S \subseteq A \rightarrow B$ extends to a homomorphism $\hat{h}_S : \langle S \rangle_{\mathbf{A}} \rightarrow \mathbf{B}$.

Proof. (1) Immediately follows from Gurevich-Kokorin theorem (see for instance [8, Theorem 1.6.17]).

(2) For every element $b \in \langle S \rangle_{\mathbf{A}}$, there is a term t and $a_1, \dots, a_n \in S$ such that $b = t^{\mathbf{A}}(a_1, \dots, a_n)$. Define $\hat{h}_S : \langle S \rangle_{\mathbf{A}} \rightarrow \mathbf{B}$ as follows:

$$\hat{h}(b) = \hat{h}(t^{\mathbf{A}}(a_1, \dots, a_n)) = t^{\mathbf{B}}(h(a_1), \dots, h(a_n)).$$

It is easy to see that $\hat{h}$ commutes with the operations of $\mathbf{A}$ and $\mathbf{B}$, whence it is an MV-homomorphism. $\square$

An *ideal* of an MV-algebra $\mathbf{A}$ is a subset I of A such that: (1) $\perp \in I$; (2) if $a \in I$ and $b \in I$, then $a \oplus b \in I$; (3) if $a \in I$ and $b \leq a$, then $b \in I$. Every ideal I of $\mathbf{A}$ determines a congruence Θ_I of $\mathbf{A}$:

$$(a, b) \in \Theta_I \text{ iff } d(a, b) \in I.$$

For every MV-algebra $\mathbf{A}$ and every ideal I of $\mathbf{A}$, we denote by $\mathbf{A}/I$ the quotient of $\mathbf{A}$ by the congruence Θ_I . Further, for every $a \in A$, we denote by a/I the equivalence class of a modulo Θ_I : $a/I = \{b \in A \mid d(a, b) \in I\}$. The set of ideals of any MV-algebra, ordered by the set-theoretical inclusion, forms a lattice whose maximal elements are called *maximal ideals*. We write $Max(\mathbf{A})$ for the set of maximal ideals of $\mathbf{A}$. The intersection of all maximal ideals of an MV-algebra $\mathbf{A}$ is an ideal, called the *radical* of $\mathbf{A}$ and denoted by $Rad(\mathbf{A})$.

In the rest of this paper two kinds of structures will be of particular interest: *simple* and *semisimple* MV-algebras. The latter are characterized as those MV-chains which, up to isomorphism, are MV-subalgebra of $[0, 1]_{MV}$ (and therefore can be regarded as algebras of real numbers) [2, Theorem 3.5.1]; the former are algebras of continuous $[0, 1]$ -valued functions defined on a compact Hausdorff space [2, Corollary 3.6.8]. It is worth to recall that semisimple MV-algebras are those structures whose radical coincides with $\{\perp\}$. Moreover, for every MV-algebra $\mathbf{A}$, the quotient $\mathbf{A}/Rad(\mathbf{A})$ is semisimple (see [2, Lemma 3.6.6]) and it will be called the *most general semisimple quotient* of $\mathbf{A}$.

3 Probability theory on Łukasiewicz events

States of MV-algebras were introduced by Mundici in [12] as averaging values of Łukasiewicz truth-valuations.

Definition 1. A *state* of an MV-algebra $\mathbf{A}$ is a map $s : A \rightarrow [0, 1]$ satisfying the following conditions:

- (1) $s(\top) = 1$,
- (2) for all $x, y \in A$ such that $x \odot y = \perp$, $s(x \oplus y) = s(x) + s(y)$.

While condition (1) says that every state is normalized, (2) is usually called *additivity* with respect to Łukasiewicz sum $\oplus$. Indeed, the requirement $x \odot y = 0$ is analogous to disjointness of a pair of elements in a Boolean algebra: if $\mathbf{A}$ is a

Boolean algebra, then $x \odot y = \perp$ iff $x \wedge y = \perp$. Thus states generalize finitely additive probabilities to the realm of MV-algebras: finitely additive probabilities on a Boolean algebra are states as a special case of the above definition.

The following results collect properties of states which will be used in the proof of the main result of this paper.

Proposition 1 ([5, Proposition 3.1.7(ii)]). *For every MV-algebra $\mathbf{A}$, for every state s of $\mathbf{A}$ and for every $a, b \in A$, if $a/\text{Rad}(\mathbf{A}) = b/\text{Rad}(\mathbf{A})$, then $s(a) = s(b)$.*

Proposition 1 above implies that every MV-algebra $\mathbf{A}$ and its most general semisimple quotient $\mathbf{A}/\text{Rad}(\mathbf{A})$ have the same class of states. Thus, every state s of any MV-algebra $\mathbf{A}$ assigns, for every $a, b \in A$ with $b \in a/\text{Rad}(\mathbf{A})$, the same value $s(a) \in [0, 1]$.

The following has been proved in [11, Theorem 6] and provides an MV-analogous of the well-known Horn-Tarski extension theorem [9].

Proposition 2. *Let $\mathbf{A}$ and $\mathbf{B}$ be MV-algebras and let $\mathbf{B}$ be an MV-subalgebra of $\mathbf{A}$. Every state $s_B : B \rightarrow [0, 1]$ extends (not uniquely) to a state $s_A : A \rightarrow [0, 1]$.*

Remark 1. Every homomorphism h of an MV-algebra $\mathbf{B}$ to $[0, 1]_{MV}$ is a state. Thus, if $\mathbf{B}$ is an MV-subalgebra of $\mathbf{A}$, Proposition 2 shows that h extends (not uniquely) to a state of $\mathbf{A}$. In other words, for every pairs of MV-algebras $\mathbf{B}$ and $\mathbf{A}$ such that $\mathbf{B}$ is an MV-subalgebra of $\mathbf{A}$ and for every homomorphism $h : \mathbf{B} \rightarrow [0, 1]_{MV}$, there exists a state s of $\mathbf{A}$ such that for every $b \in B$, $s(b) = h(b)$.

The following theorem, independently proved by Kroupa [10] and Panti [14] (see also [5, §4]), represents every state s of an MV-algebra $\mathbf{A}$ as the Lebesgue integral given by a unique regular, Borel probability measure. More precisely, every state of an MV-algebra $\mathbf{A}$ is the Lebesgue integral on the continuous functions a^* 's of its most general semisimple quotient, defined on the compact Hausdorff space $\text{Max}(\mathbf{A})$ of maximal ideals of $\mathbf{A}$.

Theorem 1. *For every MV-algebra $\mathbf{A}$ and for every state s of $\mathbf{A}$ there exists a unique regular, Borel probability measure μ on the Borel subsets of $\text{Max}(\mathbf{A})$ such that, for every $a \in A$,*

$$s(a) = \int_{\text{Max}(\mathbf{A})} a^* d\mu$$

4 A probabilistic logic on Łukasiewicz events

In this section we will recall basic definitions and properties for the logics $\text{SFP}(\mathbf{L}, \mathbf{L})$ and its algebraic semantics¹. The language of $\text{SFP}(\mathbf{L}, \mathbf{L})$ is obtained by expanding that of Łukasiewicz infinite-valued calculus with a unary modality P . The set $\mathbf{SFPm}$ of well-formed formulas in this language is defined by

¹ We invite the reader to consult [6, 7] and [5, §6] for a more exhaustive introduction to SMV-algebras, fuzzy probabilistic logics and their relation with uncertain reasoning.

induction as usual. Axioms and rules of $\text{SFP}(\mathbf{L}, \mathbf{L})$ are as follows: all instances of axioms and rules of Łukasiewicz logic, plus the following axioms and rules for the modality P

- $P(\perp) \rightarrow \perp$ (Normalization);
- $P(\varphi \rightarrow \psi) \rightarrow (P(\varphi) \rightarrow P(\psi))$ (Monotonicity);
- $P(P(\varphi) \oplus P(\psi)) \leftrightarrow P(\varphi) \oplus P(\psi)$ (Idempotency);
- $P(\varphi \oplus \psi) \leftrightarrow [P(\varphi) \oplus P(\psi \ominus (\varphi \odot \psi))]$ (Additivity);
- from ϕ derive $P(\phi)$ (Necessitation).

Proofs are defined with no modification from the classical definition and $\vdash_{\text{SFP}} \phi$ will be used to denote that ϕ is a *theorem* of $\text{SFP}(\mathbf{L}, \mathbf{L})$. If $\Gamma \cup \{\phi\}$ is a subset of **SPFm**, we will write $\Gamma \vdash_{\text{SFP}} \phi$ if there is a proof of ϕ from Γ .

The logic $\text{SFP}(\mathbf{L}, \mathbf{L})$ is algebraizable and its equivalent algebraic semantics is the variety **SMV** of *MV-algebras with internal state* (or *SMV-algebras* for short).

Definition 2 ([6]). An *SMV-algebra* is a pair $(\mathbf{A}, \sigma)$ where $\mathbf{A}$ is an *MV-algebra* and the internal state $\sigma : A \rightarrow A$ satisfies the following equations:

- ($\sigma 1$) $\sigma(\perp) = \perp$;
- ($\sigma 2$) $\sigma(x \rightarrow y) \leq \sigma(x) \rightarrow \sigma(y)$;
- ($\sigma 3$) $\sigma(\sigma(x \oplus y)) = \sigma(x \oplus y)$;
- ($\sigma 5$) $\sigma(x \oplus y) = \sigma(x) \oplus \sigma(y \ominus (x \odot y))$.

Example 2. (1) Every idempotent endomorphism σ of an MV-algebra $\mathbf{A}$ makes $(\mathbf{A}, \sigma)$ an SMV-algebra.

(2) Let $\mathcal{F}_{\mathbb{R}}(\omega)$ be the set of all continuous and piecewise linear functions with *real* coefficients. The pointwise application of $\oplus$ and $\neg$ of $[0, 1]_{MV}$ makes $\mathcal{F}_{\mathbb{R}}(\omega)$ into an MV-algebra which contains all constant functions $\bar{\alpha}$ for every $\alpha \in [0, 1]$. These functions are continuous and hence Riemann-integrable. Let $\sigma : \mathcal{F}_{\mathbb{R}}(\omega) \rightarrow \mathcal{F}_{\mathbb{R}}(\omega)$ be defined as follows: for every $f \in \mathcal{F}_{\mathbb{R}}(\omega)$ with k variables,

$$\sigma(f) = \overline{\int_{[0,1]^k} f \, dx}.$$

That is, $\sigma(f)$ is the function which is constantly equal to the Riemann integral of f . This map σ is an internal state of $\mathcal{F}_{\mathbb{R}}(\omega)$.

Proposition 3 ([6]). For every SMV-algebra $(\mathbf{A}, \sigma)$ the following properties hold:

- (1) the image $\sigma(A)$ of A under σ , endowed with the operations inherited from $\mathbf{A}$ forms an MV-subalgebra $\sigma(\mathbf{A})$ of $\mathbf{A}$;
- (2) if $(\mathbf{A}, \sigma)$ is a subdirectly irreducible, then $\sigma(\mathbf{A})$ is totally ordered. As a consequence the variety **SMV** is generated by its elements $(\mathbf{A}, \sigma)$ such that $\sigma(\mathbf{A})$ is an MV-chain.

Valuations to an SMV-algebra $(\mathbf{A}, \sigma)$ are defined in the usual way. A valuation v is said to be a *model* of a formula ϕ , if $v(\phi) = \top$. A formula $\phi \in \mathbf{SPFm}$ is said to be an *SMV-tautology* ($\models_{SMV} \phi$ in symbols), if $v(\phi) = \top$ for every SMV-algebra and every valuation to it. Further, for every (finite or infinite) subset $\Gamma \cup \{\phi\}$ of $\mathbf{SPFm}$, we will write $\Gamma \models_{SMV} \phi$ if for every SMV-algebra $(\mathbf{A}, \sigma)$ and every valuation v to $(\mathbf{A}, \sigma)$ which is a model of each formulas in Γ , then v is a model of ϕ as well.

Proposition 4 ([6, Theorem 4.5]). *For every (finite or infinite) subset $\Gamma \cup \phi$ of $\mathbf{SPFm}$, $\Gamma \vdash_{SFP} \phi$ iff $\Gamma \models_{SMV} \phi$.*

We end this section with the following observation.

Remark 2. One of the most important properties for a t-norm based fuzzy logic, is completeness with respect to the class of its *standard* algebras. By “standard” we usually mean linearly ordered structures, order-embeddable into the real unit interval. Although $SFP(\mathbf{L}, \mathbf{L})$ is grounded on Lukasiewicz calculus which enjoys a standard completeness theorem, $SFP(\mathbf{L}, \mathbf{L})$ is not *standard complete* in the usual sense. In fact, for instance, the equation $\sigma(x \wedge y) = \sigma(x) \wedge \sigma(y)$ holds in every SMV-chain, but it does not hold in every SMV-algebra and in particular it does not hold in the algebra $(\mathcal{F}_{\mathbb{R}}(\omega), \sigma)$ of Example 2(2).

5 Toward a standard completeness for $SFP(\mathbf{L}, \mathbf{L})$

In this section we will present a class of SMV-algebras that, in the light of Remark 2, it is reasonable to call *standard*. Our main result will show that, for a restricted, yet quite expressive, subset of $\mathbf{SPFm}$, a formula is provable in $SFP(\mathbf{L}, \mathbf{L})$ iff it is true in these standard algebras.

Definition 3. *The set $\mathbf{PFm}$ of purely probabilistic formulas is the smallest subset of $\mathbf{SPFm}$ which contains all the modal formulas $P(\psi)$ (for ψ a Lukasiewicz formula) and which is closed under the connectives of Lukasiewicz logic.*

Notice that each formula of $\mathbf{PFm}$ is in the form $t[P(\psi_1), \dots, P(\psi_k)]$ for t being a(n MV-)term. Thus, for instance $P(\varphi) \rightarrow (P(\psi) \oplus P(\gamma)) \in \mathbf{PFm}$, but neither $P(\psi \rightarrow P(\varphi))$, nor $P(\psi) \rightarrow \gamma$ belong to $\mathbf{PFm}$. It is also worth to point out that $\mathbf{PFm}$ forms the language for the weaker logic, with respect to $SFP(\mathbf{L}, \mathbf{L})$, introduced in [3] and denoted by $FP(\mathbf{L}, \mathbf{L})$.

First of all, we will recall the definition of a class of SMV-algebras which was introduced in [7].

Definition 4. *An SMV-algebra $(\mathbf{A}, \sigma)$ is said to be σ -simple if $\mathbf{A}$ is semisimple and $\sigma(\mathbf{A})$ is simple.*

In the light of the last comment of Subsection 2 a σ -simple SMV-algebra $(\mathbf{A}, \sigma)$ can be regarded as an algebra of continuous functions on a compact Hausdorff space endowed with a normalized, idempotent, finitely additive and real-valued internal state σ .

The notions of *valuation* in a σ -simple SMV-algebra and *tautology* are as in Section 4. We will write $\models_\sigma \phi$ to denote that the formula ϕ is a tautology in this setting.

In the proof of the following result we will make use of the construction of *MV-tensor product* which was introduced by Mundici in [13] and heavily used in [6] to *internalize* a state of an MV-algebra. In what follows we will restrict the attention to those MV-algebras $[0, 1]_{MV} \otimes \mathbf{A}$ defined by taking the tensor product of the standard MV-algebra $[0, 1]_{MV}$ with an MV-algebra $\mathbf{A}$. For the sake of the present paper, it is important to recall the following properties: (1) the generators of $[0, 1]_{MV} \otimes \mathbf{A}$ are of the form $\alpha \otimes a$ for $\alpha \in [0, 1]$ and $a \in A$; (2) the maps $a \in A \mapsto 1 \otimes a$ and $\alpha \mapsto \alpha \otimes \top$ respectively are embeddings of $\mathbf{A}$ and $[0, 1]_{MV}$ into $[0, 1]_{MV} \otimes \mathbf{A}$; (3) the top element of $[0, 1]_{MV} \otimes \mathbf{A}$ is $1 \otimes \top$. By (2), $[0, 1]_{MV} \otimes \mathbf{A}$ contains isomorphic copies of both $[0, 1]_{MV}$ and $\mathbf{A}$.²

Theorem 2. *For every formula $\phi \in \mathbf{PFm}$, $\vdash_{\text{SFP}} \phi$ iff $\models_\sigma \phi$.*

Proof. The left-to-right direction follows from Proposition 4.

In order to prove the right-to-left direction, assume $\not\models_{\text{SFP}} \phi$. Thus, from Proposition 4 there exists an SMV-algebra $(\mathbf{A}, \sigma)$ and a valuation v in A such that $v(\phi) < \top$. Without loss of generality, by Proposition 3, we can assume $(\mathbf{A}, \sigma)$ to be subdirectly irreducible and hence $\sigma(\mathbf{A})$ linearly ordered. Since $\phi \in \mathbf{PFm}$, it will be in the form

$$t[P(\psi_1), \dots, P(\psi_k)]$$

where t is a term in the language of MV-algebras. Let us denote by S_ϕ the following subset of A :

$$S_\phi = \{v(\phi), v(P(\psi_1)), \dots, v(P(\psi_k))\} = \{v(\phi), \sigma(v(\psi_1)), \dots, \sigma(v(\psi_k))\}.$$

Therefore S_ϕ is a finite subset of $\sigma(A)$ and since $\sigma(\mathbf{A})$ is an MV-chain, there is a partial embedding h_ϕ of S_ϕ into the standard MV-algebra $[0, 1]_{MV}$ (see Lemma 1 (1)):

$$h_\phi : S_\phi \hookrightarrow^p [0, 1]_{MV}.$$

By Lemma 1 (2), h_ϕ extends to a homomorphism, that we still denote by h_ϕ , of the MV-subalgebra $\mathbf{S} = \langle S_\phi \rangle_{\sigma(\mathbf{A})}$ of $\sigma(\mathbf{A})$, generated by S_ϕ , to $[0, 1]_{MV}$:

$$h_\phi : \mathbf{S} \hookrightarrow [0, 1]_{MV}.$$

Notice that $h_\phi(v(\phi)) < 1$. By Remark 1, h_ϕ extends to a state $s_\phi : \sigma(\mathbf{A}) \rightarrow [0, 1]$. Thus, let $s : A \rightarrow [0, 1]$ be the map

$$s : a \in A \mapsto s_\phi(\sigma(a)).$$

Notice that s and s_ϕ agree on $\sigma(A)$. Indeed by the idempotency of σ (equation ($\sigma 3$) of Definition 2), if $a = \sigma(b) \in \sigma(A)$, $s(a) = s(\sigma(b)) = s_\phi(\sigma\sigma(b)) =$

² We invite the reader to consult [13] for an exhaustive description of the MV-tensor product construction and [6, 7] for its application to the theory of SMV-algebras.

$s_\phi(\sigma(b)) = s_\phi(a)$. Therefore, for every subformula $P(\psi_i)$ of ϕ , since s_ϕ extends h_ϕ ,

$$s(v(\psi_i)) = s_\phi(\sigma(v(\psi_i))) = h_\phi(v(P(\psi_i))), \quad (1)$$

which leads to the following fact.

Fact 1 $t^{[0,1]}[s(v(\psi_1)), \dots, s(v(\psi_k))] < 1$.

As a matter of fact, by (1)

$$\begin{aligned} t^{[0,1]}[s(v(\psi_1)), \dots, s(v(\psi_k))] &= t^{[0,1]}[h_\phi(v(P(\psi_1))), \dots, h_\phi(v(P(\psi_k)))] \\ &= h_\phi(t^{\mathbf{A}}[v(P(\psi_1)), \dots, v(P(\psi_k))]) \\ &= h_\phi(v(t[P(\psi_1), \dots, P(\psi_k)])) \\ &= h_\phi(v(\phi)) < 1. \end{aligned}$$

In order to get the claim, we have now to define a σ -simple SMV-algebra $(\mathbf{M}, \sigma_M)$ and a valuation w to M such that $w(\phi) < 1$. To this aim, let $\mathbf{N} = [0, 1]_{MV} \otimes \mathbf{A}$ and $s_N : N \rightarrow [0, 1]$ be such that $s_N(\alpha \otimes a) = \alpha \cdot s(a)$. Then s_N is a state of $\mathbf{N}$ which extends s (indeed, $\mathbf{A}$ is a subalgebra of $\mathbf{N}$). Regardless of the semisimplicity of $\mathbf{A}$, $\mathbf{N}$ needs not to be semisimple (see [13, Theorem 3.3]). Thus, let $\mathbf{M}$ its most general semisimple quotient:

$$\mathbf{M} = \mathbf{N}/\text{Rad}(\mathbf{N}) \text{ and } \sigma_M : m/\text{Rad}(\mathbf{N}) \in M \mapsto s_N(m).$$

Notice that σ_M is well-defined because of Proposition 1. Moreover, $\mathbf{M}$ is semisimple, σ_M is an internal state of $\mathbf{M}$ (because the range $[0, 1]_{MV}$ of σ_M is a subalgebra of $\mathbf{M}$) and $\sigma_M(\mathbf{M})$ is clearly simple. Thus, $(\mathbf{M}, \sigma_M)$ is σ -simple.

Let w be the following valuation to M : for each propositional variable q , $w(q) = 1 \otimes v(q)$ and $w(P(\gamma)) = \sigma_M(v(\gamma)) \otimes \top$. Notice that, for every subformula $P(\psi_i)$ of ϕ , one has

$$w(P(\psi_i)) = \sigma_M(v(\psi_i)) \otimes \top = s_N(v(\psi_i)) \otimes \top = s(v(\psi_i)) \otimes \top,$$

and hence, by Fact 1, $w(\phi) = t^{[0,1]}[s(v(\psi_1)), \dots, s(v(\psi_k))] \otimes \top < 1 \otimes \top$. $\square$

We now introduce a further refined semantics for SFP(L, L). To this end let $\mathcal{F}_{\mathbb{R}}(\omega)$ be the MV-algebra of Example 2(2). It is worth to notice that $\mathcal{F}_{\mathbb{R}}(\omega)$ is semisimple [7, Proposition 2.4]. Further, every state $s : \mathcal{F}_{\mathbb{R}}(\omega) \rightarrow [0, 1]$ can be easily internalized by considering the map $\sigma_s : \mathcal{F}_{\mathbb{R}}(\omega) \rightarrow \mathcal{F}_{\mathbb{R}}(\omega)$ defined as follows: for every $a \in \mathcal{F}_{\mathbb{R}}(\omega)$

$$\sigma_s(a) = \overline{s(a)}. \quad (2)$$

Then $(\mathcal{F}_{\mathbb{R}}(\omega), \sigma_s)$ is a σ -simple SMV-algebra. Notice that, thanks to Theorem 1 the internal state σ_s assigns to every $f \in \mathcal{F}_{\mathbb{R}}(\omega)$, the real number $\int f \, d\mu$: the Lebesgue integral of f by a regular, Borel probability measure.

Definition 5. A standard SMV-algebra is a pair $(\mathcal{F}_{\mathbb{R}}(\omega), \sigma_s)$ where $s : \mathcal{F}_{\mathbb{R}}(\omega) \rightarrow [0, 1]$ is a state and σ_s is defined as in (2).

Taking into account that the free ω -generated MV-algebra of Example 1(2) is a subalgebra of $\mathcal{F}_{\mathbb{R}}(\omega)$ (see [7, Proposition 2.4]), a *valuation* v of $\text{SFP}(\mathbb{L}, \mathbb{L})$ -formulas in $(\mathcal{F}_{\mathbb{R}}(\omega), \sigma_s)$ can be inductively defined as follows:

- $v(\psi) = f_\psi$ for ψ a Łukasiewicz formula;
- v commutes with Łukasiewicz connectives;
- $v(P(\psi)) = \sigma_s(v(\psi))$.

We will write $\models_{St} \phi$ to denote that ϕ is a tautology for this class of algebras.

Theorem 3. *For every formula $\phi \in \mathbf{PFm}$, $\models_{\sigma} \phi$ iff $\models_{St} \phi$. In other words, σ -simple SMV-algebras and standard SMV-algebras share the same $\mathbf{PFm}$ -tautologies.*

Proof. Since every standard SMV-algebra is σ -simple, $\models_{\sigma} \phi$ implies $\models_{St} \phi$. Let hence assume that $\not\models_{\sigma} \phi$ and let v be a valuation in a σ -simple SMV-algebra $(\mathbf{A}, \sigma)$ such that $v(\phi) < 1$. Define $s : \mathcal{F}(\omega) \rightarrow [0, 1]$ by the following stipulation: for every Łukasiewicz formula ψ , $s(f_\psi) = \sigma(v(\psi))$.

Clearly s is a state of $\mathcal{F}(\omega)$ which, since $\mathcal{F}(\omega)$ is an MV-subalgebra of $\mathcal{F}_{\mathbb{R}}(\omega)$, extends to a state $\hat{s}$ of $\mathcal{F}_{\mathbb{R}}(\omega)$ by Proposition 2. Define $\sigma_{\hat{s}}$ by (2) and the valuation w on $(\mathcal{F}_{\mathbb{R}}(\omega), \sigma_{\hat{s}})$ by $w(q) = f_q$. Let us prove that w is not a model of ϕ . Since $\phi \in \mathbf{PFm}$, it is in the form $t[P(\psi_1), \dots, P(\psi_k)]$ and because s and $\hat{s}$ agree on $\mathcal{F}(\omega)$, so do σ_s and $\sigma_{\hat{s}}$ by definition. Thus, by (2) one has:

$$\begin{aligned} w(\phi) &= t^{[0,1]}[\sigma_{\hat{s}}(w(\psi_1)), \dots, \sigma_{\hat{s}}(w(\psi_k))] \\ &= t^{[0,1]}[\sigma_{\hat{s}}(f_{\psi_1}), \dots, \sigma_{\hat{s}}(f_{\psi_k})] \\ &= t^{[0,1]}[\sigma_s(f_{\psi_1}), \dots, \sigma_s(f_{\psi_k})] \\ &= t^{[0,1]}[s(f_{\psi_1}), \dots, s(f_{\psi_k})] \\ &= v(\phi) < 1 \end{aligned}$$

Therefore, $\not\models_{St} \phi$ which settles the claim. $\square$

The following result is a direct consequence of Theorem 2 and Theorem 3.

Corollary 1. *For every formula $\phi \in \mathbf{PFm}$, $\vdash_{\text{SFP}} \phi$ iff $\models_{St} \phi$.*

6 Conclusion and future work

In this paper we presented a partial solution to the problem of establishing a standard completeness theorem for the probabilistic logic $\text{SFP}(\mathbb{L}, \mathbb{L})$ introduced in [6]. In particular we proved that, for a restricted class of formulas, theorems of $\text{SFP}(\mathbb{L}, \mathbb{L})$ are tautologies for a class of SMV-algebras which are defined from real-valued metric spaces. In our future work on this argument we plan to extend the ideas and constructions which led to the proof of our main theorems to provide a standard completeness theorem for the whole language of $\text{SFP}(\mathbb{L}, \mathbb{L})$.

Acknowledgments

The author acknowledges partial support by the Spanish Ramon y Cajal research program RYC-2016-19799; the Spanish FEDER/MINECO project TIN2015-71799-C2-1-P and the SYSMICS project (EU H2020-MSCA-RISE-2015, Project 689176).

References

1. C. C. Chang. *Algebraic Analysis of Many-valued Logics*. Trans. Am. Math. Soc. 88, 467–490, 1958.
2. R. Cignoli, I. M. L. D'Ottaviano, D. Mundici. *Algebraic Foundations of Many-valued Reasoning*. Kluwer, 2000.
3. T. Flaminio, L. Godo. A logic for reasoning about the probability of fuzzy events. *Fuzzy Sets and Systems* 158(6): 625–638, 2007.
4. T. Flaminio, H. Hosni, S. Lapenta, Convex MV-algebras: Many-valued logics meet decision theory. *Studia Logica* 106(5): 913–945, 2018.
5. T. Flaminio, T. Kroupa. States of MV-algebras. Chapter XVII of *Handbook of Mathematical Fuzzy Logic - volume 3*, C. Fermüller, P. Cintula and C. Noguera (Eds.), Studies in Logic, Mathematical Logic and Foundations, vol. 58, College Publications, London, 2015.
6. T. Flaminio, F. Montagna. MV-algebras with internal states and probabilistic fuzzy logics. *International Journal of Approximate Reasoning*, 50(1): 138–152, 2009.
7. T. Flaminio, F. Montagna. Models for Many-Valued Probabilistic Reasoning. *Journal of Logic and Computation*, 21(3): 447–464, 2011.
8. P. Hájek. *Metamathematics of Fuzzy Logics*, Kluwer Academic Publishers, Dordrecht, 1998.
9. A. Horn, A. Tarski. Measures on Boolean algebras. *Trans. Amer. Math. Soc.*, 64:467–497, 1948.
10. T. Kroupa. Every state on semisimple MV-algebra is integral. *Fuzzy Sets and Systems* 157 (20), 2771–2787, 2006.
11. T. Kroupa. Representation and extension of states on MV-algebras. *Archive for Mathematical Logic*, 45: 381–392, 2006.
12. D. Mundici. Averaging the Truth-Value in Łukasiewicz Logic. *Studia Logica* 55: 113–127, 1995.
13. D. Mundici, Tensor Products and the Loomis-Sikorski Theorem for MV-Algebras. *Advances in Applied Mathematics*, 22: 227–248, 1999.
14. G. Panti. Invariant measures on free MV-algebras. *Communications in Algebra*, 36(8):2849–2861, 2009.