Doppler shifts and spectral line profile changes in the starlight scattered from an exoplanet

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ABSTRACT

Scattered starlight from an exoplanet, commonly called reflected light, can be used to characterize the planet including the determination of its albedo and inclination of orbit. The relatively low flux ratio between directly observed starlight and starlight scattered off hot Jupiters make these systems the prime candidates for detection of reflected light using high-resolution spectroscopy. The first detections have been claimed for 51 Peg b. In a first calculation, we derive the Doppler shift of reflected light measured by a remote observer for a planet modelled as a point-like particle in orbit around a star. We find that the Doppler shift of reflected light from planets have a different Doppler shift to that of light emitted directly from the planet with magnitude equivalent to the radial motion of the planet with respect to the star. This only occurs for non-circular orbits. Secondly, restricting our investigation to planets that are tidally locked and orbit in circular orbits we account for the finite size of the star and planet by integrating the contribution to a simulated spectral line across both their surfaces. Since exact analytical expressions cannot be easily derived as a function of all free parameters of the problem, we have developed a software tool called REFLECTED STARLIGHT (RESTART) that produces the resulting line profiles. By applying it to study cases found in the literature, we explicitly show that hot Jupiters such as WASP-19b and 51 Peg b should show substantial broadening and asymmetric distortions compared to the nominal stellar line.

Key words: techniques: spectroscopic – planet–star interactions – stars: rotation.

1 INTRODUCTION

Since the detection of the first exoplanet orbiting a main-sequence star in 1995 (Mayor & Queloz 1995), there are now over 4100 planet candidates with more than 400 of these being either hot or very hot Jupiters according to the Extrasolar Planet Encyclopaedia (Schneider et al. 2011). Hot Jupiters are defined as planets with mass approximately that of Jupiter (here, we assume greater than 100 Earth masses) and that orbit in a period of 3–9 d and have a semimajor axis less than 0.1 au. Very hot Jupiters orbit in period $P < 3$ d. (Perryman 2018).

Reflected light is the light scattered off the planet originating from its parent star. Reflected light is readily observed in the planets of the Solar system. The continuum of a typical spectrum of a Solar system planet has two peaks relating to where the reflected light dominates the spectrum at visible wavelengths and where thermal emission dominates in the near-infrared. Models derived for the spectrum of hot Jupiters also show the same peaks (Seager 2010).

Detecting reflected light is difficult. Even for planets in the solar neighbourhood there is a very small angle separating the planet from its parent star. This makes it difficult to separate the light from the planet and its parent star. This difficulty is compounded by the extremely small flux ratio between them (Claudi 2016). The largest flux ratios are expected for hot Jupiters and to be of the order of a few parts in 10^{-5} and is due to the large size of the planet and its proximity to its parent star. For an Earth twin orbiting a Sun-like star, a contrast ratio is expected of the order 10^{-10} (Martins et al. 2016).

High-resolution spectroscopy is seen as a potentially advantageous method of detecting reflected light, as although light scattered from the planet is expected to have a similar spectrum to the starlight (Seager & Sasselov 1998) it can be disentangled from the spectrum of the star in velocity space due to the large Doppler shift of the planet with respect to the star. Various techniques have been used to identify the signal of the reflected light including most recently using cross-correlation functions (CCFs) by Martins et al. (2015) or autocorrelation functions (ACFs) by Borra & Deschatelets (2018) on each spectrum. These techniques effectively combine the lines in the planet spectrum in order to increase the signal to noise. The CCFs (or ACFs) are then moved to the frame of the planet and combined

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in order to increase the signal to noise further to a level where the signal from the reflected light can be detected. If no detection can be made, then an upper bound on the strength of the reflected light signal can be given. In order to know the wavelength position of the reflected light in the spectrum, the required orbital parameters of the orbit need to be known in advance and come from the transit and radial velocity (RV) planet detection methods. In the case where the planet does not transit then the inclination parameter has to be fitted whilst performing the search for the reflected light. Indeed, detecting the reflected light for a system without a previously known inclination is important as once identified the mass of the planet can then be determined (Charbonneau et al. 1999).

Photometry can also be used to detect reflected light. Light curves with high enough signal to noise should be able to show the increase in light towards the superior conjunction and decrease towards inferior conjunction. The light curves of ν And b in the mid-infrared (Harrington et al. 2006) and HAT-P-7b in the visible (Borucki et al. 2009) provide examples of this. From the detection of reflected light in photometry, the flux ratio can be determined and provide input to the required signal to noise required to detect the reflected light in high-resolution spectroscopy.

The 51 Peg system is the only system where high-resolution spectroscopic detection of reflected light has been claimed in the optical, where Martins et al. (2015) used the CCF method and Borra & Deschatelets (2018) used the ACF method. Contrast levels were of the order of 6×10^{-5} and significant broadening was observed in the CCF line profile. High-precision photometry from *MOST* (Walker et al. 2006) of 51 Peg shows a light curve that peaks just before orbital phase 0.1 at a level of 2.5×10^{-4} greater than the minimum in the light curve.

In this paper, we determine the Doppler shift of starlight scattered off of exoplanets and show how for hot Jupiters the scattering can result in significant broadening of the spectral line profiles. We have produced a tool called Reflected STARlight (RESTART) that will produce a template spectrum of the reflected light from the planet based on the spectra from the star.

In Section 2 of this paper, we describe analytical models for reflected light and also describe the code RESTART that models reflected light spectra.

In Section 3, we provide results from running RESTART for the examples of the very hot Jupiter WASP-19b, the hot Jupiter 51 Peg b, and an Earth analogue system where the system is 10 pc from the observer. We compare our results with what we expect and from the detection paper for 51 Peg b.

In Section 4, we give the discussion and in Section 5 the conclusions to the paper. Appendices A–C provide derivations of some of the results that are referred to in Section 2.

2 ANALYTICAL MODEL AND THE RESTART CODE

We first determine the Doppler shift formula in a suitable form for starlight reflected off of a particle (Section 2.1). Then, using this equation we compute the Doppler shift of reflected light in a two-body star/planet system where the planet and the star are far enough away from each other that they can both be considered to be point-like (Section 2.2). We then model the flux of reflected starlight from a star and planet two-body system where the size of the planet and the star has to be taken into account due to their proximity. We restrict the orbits in this system to be circular and the planet to be tidally locked that are expected for hot Jupiter systems (Section 2.3).

The final expressions for the flux from Section 2.3 cannot be easily solved analytically as a function of all the parameters of the problem. Thus, we describe our software implementation called RESTART of these equations in Section 2.4, where we take as input the rotationally unbroadened spectrum from a star and have as output the spectrum of the reflected starlight as seen by a distant observer.

2.1 Doppler shift of light reflected from a point-like particle

In order to detect scattered starlight from an exoplanet, we need to be able to determine the Doppler shift of the reflected light as viewed by the observer on the Earth. The non-relativistic Doppler equation commonly used for reflection off of a mirror from Anglada-Escudé et al. (2007) is

$$\nu' = \nu \left[1 - 2 \frac{1}{c} \left((\hat{\sigma} \cdot \hat{n})(\mathbf{v}_m \cdot \hat{n}) \right) \right], \quad (1)$$

where c is the speed of light, ν' is the Doppler shifted frequency, ν is the original frequency, $\hat{\sigma}$ is the unit vector of the incoming light ray, $\hat{n}$ is the normal to the mirror, and $\mathbf{v}_m$ is the velocity the mirror.

In the case of light scattering off of a particle, we take the mirror to be perpendicular to the bisector between the incoming light ray and the outgoing scattered ray. Thus, $\hat{n}$ is

$$\hat{n} = \frac{\hat{\sigma}' - \hat{\sigma}}{|\hat{\sigma}' - \hat{\sigma}|}, \quad (2)$$

where $\hat{\sigma}'$ is the unit vector of the outgoing ray.

Substituting this value for $\hat{n}$ in equation (1) and denoting $\mathbf{v}_p$ as the velocity of the particle, we have

$$\nu' = \nu + \frac{\nu}{c} \mathbf{v}_p \cdot (\hat{\sigma}' - \hat{\sigma}). \quad (3)$$

Setting $\Delta\nu = \nu' - \nu$ and using the non-relativistic version of the Doppler shift equation

$$\frac{\Delta\nu}{\nu} = \frac{\Delta v}{c}, \quad (4)$$

where Δv is the Doppler shift of the outgoing ray in velocity space and then substituting for $\frac{\Delta v}{\nu}$ in equation (3) we have

$$\Delta\nu = \nu_p \cdot (\hat{\sigma}' - \hat{\sigma}). \quad (5)$$

The Doppler shift of scattered light caused by moving particles as a non-invasive way of determining the velocity of fluids has been studied extensively (see Charrett, James & Tatam 2012 for a review of laser Doppler velocimetry). Particles are inserted into the fluid (if the fluid itself is not reflective) and light rays usually using lasers are fired into the fluid. The light reflected is then captured on detectors and the frequency shift of the captured light is used to determine the velocity of the flows. The velocity flows studied range from small velocities to supersonic. In many cases, these particles are small and exhibit mainly Rayleigh scattering (Yeh & Cummins 1964), which is the same type of scattering we may expect from hot Jupiters (Seager & Sasselov 1998). It is very straightforward to show that the equations used to determine the velocity of the flows in the aforementioned papers are consistent with equation (3).

2.2 Doppler Shift of reflected light from exoplanets modelled as point-like particles

Here, we determine the Doppler shift of reflected light observed from the Earth of a star/planet system where the following assumptions hold:

Figure 1. The relationship between the plane of the orbit of the star (blue) to the plane of the sky (black) adapted from Murray & Correia (2010).

- (i) The only gravitational influence on the planet in the system is the parent star, so we can assume a simple two-body problem.
- (ii) The planet is far enough away from the star so that both objects can be treated as point masses and the starlight reaching the planet are in parallel rays.
- (iii) We ignore the effects of rotation of the planet and star.

We follow the two-body problem as described in Murray & Correia (2010).

In the two-body problem, the position of the planet with respect to the star is typically first determined in a planar Cartesian coordinate system xy , which is in the plane of the orbit centred on the star. The coordinates of the planet are given by

$$x = r \cos f, \quad (6)$$

$$y = r \sin f, \quad (7)$$

where r is the distance from the planet to the star and f is the true anomaly (see Fig. 1).

These equations are then transformed to the Cartesian coordinate system for the orbit in space where the X - and Y -axes are in the plane of the sky and the Z -axis points towards the observer. The position of the planet with respect to the star is then given by the following equations:

$$X = r(\cos \Omega \cos(\omega + f) - \sin \Omega \sin(\omega + f) \cos I), \quad (8)$$

$$Y = r(\sin \Omega \cos(\omega + f) + \cos \Omega \sin(\omega + f) \cos I), \quad (9)$$

$$Z = r \sin(\omega + f) \sin I, \quad (10)$$

where the three angles involved in the transformation are the longitude of ascending node Ω , argument of pericentre ω , and inclination I . These angles are used to rotate the xy axis, which

Figure 2. Position vectors to planet and star with respect to barycentre O' and observer O .

is in the plane of the orbit, to the XYZ axis, which is the orbit in space.

The above three equations can be used to give the position of the planet with respect to the barycentre of the system if their right-hand sides are multiplied by the factor $\frac{m_*}{m_* + m_p}$, where m_p and m_* are the mass of the planet and the star, respectively.

Similarly, the equations can be used to give the position of the star with respect to the centre of mass if their right-hand sides are multiplied by the factor $\frac{m_p}{m_* + m_p}$ and ω is replaced by ω_* , where $\omega_* = \omega + \pi$.

The equation for the velocity shift Δv due to the reflection is given in equation (5) where in this case $\hat{\sigma}$ is the

unit vector in the direction of the planet from the barycentre and $\hat{\sigma}'$ is the unit vector from the planet to the observer. In the XYZ coordinate frame $\hat{\sigma}$ and $\hat{\sigma}'$ are

$$\hat{\sigma} = \begin{bmatrix} \cos \Omega \cos(\omega + f) - \sin \Omega \sin(\omega + f) \cos I \\ \sin \Omega \cos(\omega + f) + \cos \Omega \sin(\omega + f) \cos I \\ \sin(\omega + f) \sin I \end{bmatrix}, \quad (11)$$

$$\hat{\sigma}' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \quad (12)$$

The position of the planet with respect to the observer is from Fig. 2:

$$\mathbf{r}_{po} = \mathbf{r}_{o'o} + \mathbf{r}_{po'}. \quad (13)$$

Differentiating, we get the velocity

$$\dot{\mathbf{r}}_{po} = \dot{\mathbf{r}}_{o'o} + \dot{\mathbf{r}}_{po'}. \quad (14)$$

The first term on the right is the barycentric contribution to the velocity with respect to the observer. The second term is the velocity of the planet with respect to the observer. We will work in the barycentric frame to determine the velocity shift due to the reflection in that frame and then move to the observer frame to get the shift due to the observer moving with respect to the barycentre.

The velocity of the planet with respect to the barycentre is

$$\mathbf{v}_{p,b} = \dot{\mathbf{r}}_{po'} = \frac{m_*}{m_* + m_p} \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix}, \quad (15)$$

where

$$\begin{aligned} \dot{X} &= \dot{r}[\cos \Omega \cos(\omega + f) - \sin \Omega \sin(\omega + f) \cos I] \\ &\quad - r \dot{f}[\cos \Omega \sin(\omega + f) + \sin \Omega \cos(\omega + f) \cos I], \end{aligned} \quad (16)$$

$$\begin{aligned} \dot{Y} &= \dot{r}[\sin \Omega \cos(\omega + f) + \cos \Omega \sin(\omega + f) \cos I] \\ &\quad + r \dot{f}[\cos \Omega \cos(\omega + f) \cos I - \sin \Omega \sin(\omega + f)], \end{aligned} \quad (17)$$

$$\dot{Z} = \dot{r} \sin(\omega + f) \sin I + r \dot{f} \cos(\omega + f) \sin I. \quad (18)$$

The Doppler shift in the barycentric frame of the reflected light is composed of two Doppler shifts. First, the Doppler shift when the light is emitted from the star and secondly from the reflection of the light off of the planet. The total Doppler shift is given by

$$\Delta v_{p,b} = -\frac{m_p}{m_* + m_p} \dot{r} + \mathbf{v}_{p,b} \cdot (\hat{\sigma}' - \hat{\sigma}). \quad (19)$$

From equations (11) and (12), we have

$$\hat{\sigma}' - \hat{\sigma} = \begin{bmatrix} -\cos \Omega \cos(\omega + f) + \sin \Omega \sin(\omega + f) \cos I \\ -\sin \Omega \cos(\omega + f) - \cos \Omega \sin(\omega + f) \cos I \\ 1 - \sin(\omega + f) \sin I \end{bmatrix}, \quad (20)$$

and thus the total Doppler shift is

$$\begin{aligned} \Delta v_{p,b} = & -\frac{m_p}{m_* + m_p} \dot{r} + \left(\frac{m_*}{m_* + m_p} \right) \\ & \times \left(\dot{r} [\sin(\omega + f) \sin I - 1] + r \dot{f} [\cos(\omega + f) \sin I] \right). \end{aligned} \quad (21)$$

Now, if we now consider the frame of the observer and assume the proper motion of the barycentre is V_z then we have

$$\begin{aligned} \Delta v_{p,o} = & V_z - \frac{m_p}{m_* + m_p} \dot{r} + \left(\frac{m_*}{m_* + m_p} \right) \\ & \times \left(\dot{r} [\sin(\omega + f) \sin I - 1] + r \dot{f} [\cos(\omega + f) \sin I] \right). \end{aligned} \quad (22)$$

From the point of view of the observer, the Doppler shift of light emitted from the planet is (using equation 14)

$$\Delta v_{r,o} = \dot{\mathbf{r}}_{p,o} \cdot \hat{\mathbf{z}} = \dot{\mathbf{r}}_{o'o} \cdot \hat{\mathbf{z}} + \dot{\mathbf{r}}_{p'o'} \cdot \hat{\mathbf{z}}, \quad (23)$$

where $\hat{\mathbf{z}} = (0, 0, 1)^T$ and expanding out this equation using equations (15) and (18) we have

$$\begin{aligned} \Delta v_{r,o} = & V_z + \frac{m_*}{m_* + m_p} \\ & \times \left(\dot{r} \sin(\omega + f) \sin I + r \dot{f} \cos(\omega + f) \sin I \right). \end{aligned} \quad (24)$$

Thus, the difference between the Doppler shift of light reflected and light originating directly from the planet is given by

$$\Delta v_{p,o} - \Delta v_{r,o} = -\dot{r}, \quad (25)$$

where $\dot{r}$ is given in terms of eccentricity e , semimajor axis a , and mean motion n where

$$n = \frac{2\pi}{P}, \quad (26)$$

and true anomaly f by

$$\dot{r} = \frac{na}{\sqrt{1-e^2}} e \sin f. \quad (27)$$

From the above equation, we can see that there will be a difference in observed Doppler shift between reflected starlight and light emitted directly from the planet when either the eccentricity is 0 – we have a circular orbit – or when the true anomaly is 0.

An order-of-magnitude estimate of the size of $\dot{r}$ can be made by first observing that

$$\dot{r} \leq \frac{nae}{\sqrt{1-e^2}}. \quad (28)$$

From Keplers third law, we have

$$\mu = a^3 n^2, \quad (29)$$

where

$$\mu = G(m_p + m_*) \approx Gm_*, \quad (30)$$

and thus

$$\dot{r} \leq \frac{n^{1/3} \mu^{1/3} e}{\sqrt{1-e^2}}. \quad (31)$$

The upper bound for $\dot{r}$ given above can be expressed in terms of the period and eccentricity of the planet, which we can obtain using the RV method and the mass of the star using equations (26), (29), and (30):

$$\dot{r} \leq (2\pi G)^{1/3} m_*^{1/3} P^{-1/3} \sqrt{\frac{e^2}{1-e^2}}. \quad (32)$$

It is useful to express this equation in terms of more practical units:

$$\dot{r} \leq 212.908 \text{ km s}^{-1} \left(\frac{m_*}{M_\odot} \right)^{1/3} \left(\frac{P}{\text{day}} \right)^{-1/3} \sqrt{\frac{e^2}{1-e^2}}. \quad (33)$$

When the eccentricity is small ($e \ll 1$), we can simplify the above equation to

$$\dot{r} \leq 212.908 \text{ km s}^{-1} \left(\frac{m_*}{M_\odot} \right)^{1/3} \left(\frac{P}{\text{day}} \right)^{-1/3} e. \quad (34)$$

It is useful to observe that $\dot{r}$ is independent of the inclination of the orbit. An important implication of this is that high-resolution spectroscopy could now be used to detect exoplanets, via detection of the Doppler shift of reflected light, in low inclination, even face on orbits, provided the orbits were eccentric.

2.3 Spectral line profile of reflected light from tidally locked hot Jupiters

In this subsection, we outline how we determine the flux and thus spectral line profile of reflected light arriving at the observer on the Earth from a tidally locked hot Jupiter in a circular orbit. Full details of the calculations are given in Appendix A. In this case, our main assumptions are as follows:

- (i) The only gravitational influence on the planet in the system is the parent star, so we can assume a simple two-body problem.
- (ii) The tidal interactions between planet and star have caused the orbits around the barycentre to be circularized. The planet is tidally locked with the same face always pointing towards the star. All other tidal effects are ignored.
- (iii) The planet and star are close enough to each other that the rays of starlight reaching the planet are not parallel.
- (iv) Particles exist in the atmosphere (or on the surface) across the entire planet, which reflect the starlight.

The flux of the reflected light, $F_\oplus$, with frequency ν arriving at the distant Earth at time t is given by

$$\begin{aligned} F_\oplus(\nu, t) = & \left(\frac{R_p}{d_{p\oplus}} \right)^2 \int_0^{2\pi} \int_0^{\pi/2} I_{s\vartheta_p\phi_p}(\vartheta_p, \phi_p, \nu, t) \\ & \times \sin \vartheta_p \cos \vartheta_p d\vartheta_p d\phi_p, \end{aligned} \quad (35)$$

Table 1. Input parameters for the very hot Jupiter.

Parameter	Value	Units	References
t_0	2455168.96801	BJD	Hellier et al. (2011)
P	0.78884	d	Hellier et al. (2011)
m_p	1.168	m_{Jup}	Hellier et al. (2011)
R_p	1.386	R_{Jup}	Hellier et al. (2011)
m_*	0.97	$M_\odot$	Hellier et al. (2011)
R_*	0.99	$R_\odot$	Hellier et al. (2011)
Ω	0	deg	
i	79.4	deg	Hellier et al. (2011)
a	0.01655	au	Hellier et al. (2011)
d_{*p}	270.409129	pc	Gaia Collab. (2018)
$v \sin I$	4.63	km s^{-1}	Hellier et al. (2011)
$P_{*,\text{rot}}$	10.5	d	Hebb et al. (2010)
Resolution	40	—	—
ρ	$1/\pi$	—	—

Figure 4. In red the rotationally deconvolved absorption line from WASP-19 and in blue the reflected starlight as seen from an observer on the Earth. The reflected starlight has been shifted by 0.2 units.

Finally, we explore the impact on line depth and full width at half-maximum (FWHM) for a hot Jupiter-type system by changing a number of the key parameters of the system in turn: stellar radius, planet radius, stellar rotation, and then planet to star distance.

3.1 Results for a very hot Jupiter system WASP-19b

The main input to RESTART is typically the non-rotationally broadened spectrum of the star. This spectrum can be obtained by deconvolving the observed stellar spectrum with a rotational broadening kernel.

In this analysis, we use a simple single synthetic absorption line generated from a Gaussian function with $\text{FWHM} = 8.963 \text{ km s}^{-1}$ – taken from FWHM of HARPS CCFs and deconvolved using the $v \sin I$ in Table 1. The CCFs were taken from the HARPS data archive and were based on observations from the 3.6 m telescope at La Silla-Paranal Observatory as part of the ESO program 084-C-0185. The line is centred at 0 km s^{-1} as shown in Fig. 4 and has depth 1 relative to the continuum.

The input parameters for the system are given in Table 1. These parameters include the resolution number of 40 which means that the star and planet surface are broken into $40 \times 2 \times 40 = 3200$ elements for the integration. The bidirectional reflection distribution

coefficient was set to a constant $\frac{1}{\pi}$, so there would be isotropic scattering.

We ran RESTART with input time when the planet is at greatest elongation ($t = 2455169.16522 \text{ d}$). Fig. 4 shows the normalized spectrum of the reflected light as seen from the Earth. The asymmetry in the line is due to only one side of the planet being illuminated as viewed from the Earth and due to the planet observed as rotating from the Earth with period equal to that of the orbital period.

At greatest elongation, we would expect the RV of light reflected from the substellar point of the planet to be at the velocity semiamplitude where

$$K_{\text{planet}} = \frac{2\pi a}{P} \frac{\sin I}{1 + m_p/m_*} = 224 \text{ km s}^{-1}. \quad (42)$$

We fitted the reflected light to a Gaussian using the Levenberg–Marquardt algorithm and obtained an $\text{FWHM} = 80.3 \text{ km s}^{-1}$, which is over 10 times the FWHM of the line from the star. The Gaussian fit gave the centre of the line as -227.8 km s^{-1} and an amplitude of 0.1.

We checked the geometric albedo by running RESTART again this time setting the inclination to 90 deg and the date to that corresponding to when the phase angle $\alpha = 0$ (time of secondary transit).

The expected geometric albedo can be derived from the following equation where the flux ratio between the reflected light and the light from the star and the planet (Charbonneau et al. 1999; Collier Cameron et al. 1999, 2002) is

$$\epsilon(\alpha, \lambda) = \frac{F_{E,p}(\alpha, \lambda)}{F_{E,*}(\lambda)} = \Phi(\alpha, \lambda) p(\lambda) \frac{R_p^2}{d_{*p}^2}, \quad (43)$$

where $\Phi(\alpha, \lambda)$ is the the orbital phase function describing the amount of light reflected towards the observer at a given phase angle α and wavelength λ , $p(\lambda)$ is the geometric albedo, R_p is the radius of the planet, and d_{*p} is the distance from the planet to the star. All of the parameters on the right-hand side of the equation except the geometric albedo are in Table 1 and the fluxes ratio on the left-hand side comes from the continuum levels of the input spectrum file and the output spectrum file.

The value obtained was $p = 0.68$, which was only slightly higher than the $2/3$ that would be expected for an isotropically scattered Lambert disc.

We next explored the phase curve for the planet by running RESTART for orbital phase $\theta = \{0, \pi/4, \pi/2, 3\pi/2, \pi, 5\pi/4, 3\pi/2\}$, where $\theta = 0$ is at time of central transit and having reset the inclination back to 79.4 deg. The phase angle α is related to the orbital phase by

$$\cos \alpha = -\sin I \cos \theta, \quad (44)$$

from Mislis et al. (2012).

We normalized the flux values returned so that the flux at $\alpha = 0$ is 1 and plotted them against the curve of the Lambert phase function for this system (Fig. 5). If the planet reflects light isotropically according to Lambert’s law and we assume that the scattering phase function is grey – independent of wavelength – then we have from Sobolev (1975) that the Lambert phase function is

$$\Phi(\alpha) = \frac{1}{\pi} [\sin \alpha + (\pi - \alpha) \cos \alpha]. \quad (45)$$

The figure shows that the reflected light from our model closely follows the Lambert phase function.

Finally, we examined the line formed at time of secondary transit (superior conjunction). Fig. 6 shows the line. It is as expected

Figure 5. Phase curve for reflected light of WASP-19b along with Lambert Phase Curve for comparison.

Figure 6. Absorption line for reflected light for a very hot Jupiter at time of secondary transit.

symmetric about velocity 0 km s^{-1} and when fitted with a Gaussian has an FWHM of 90.5 km s^{-1} and a line depth of 0.09.

3.2 Results for 51 Peg b system

As for WASP-19, we use a simple single synthetic absorption line generated from a Gaussian function with $\text{FWHM} = 7.36 \text{ km s}^{-1}$ taken from FWHM of HARPS CCFs and deconvolved using the $v \sin I$ in Table 2. The CCFs were taken from the HARPS data archive and were based on observations from the 3.6 m telescope at La Silla-Paranal Observatory as part of the ESO program 091.C-0271. The line is centred at 0 km s^{-1} as shown in Fig. 7 and has depth 1 relative to the continuum.

The input parameters for the system are given in Table 2. We used a planet radius of $1.6 R_{\text{Jup}}$ as this according to the detection paper corresponded to a geometric albedo of $2/3$.

We ran RESTART with input time when the planet is at greatest elongation ($t = 2456022.3137 \text{ d}$). Fig. 7 shows the normalized spectrum of the reflected light as seen from the Earth. The asymmetry in the line is due to only one side of the planet being illuminated as viewed from the Earth and due to the planet being observed

Table 2. Input parameters for 51 Peg system.

Parameter	Value	Units	References
t_0	2456021.256	BJD	Martins et al. (2015)
P	4.231	d	Martins et al. (2015)
m_p	0.457	m_{Jup}	Martins et al. (2015)
R_p	1.6	R_{Jup}	Martins et al. (2015)
m_*	1.04	$M_{\odot}$	Santos et al. (2003)
R_*	1.237	$R_{\odot}$	van Belle & von Braun (2009)
Ω	0.0	deg	
i	80	deg	Martins et al. (2015)
a	0.052	au	Martins et al. (2015)
$d_{* \oplus}$	14.7	pc	Gaia Collab. (2018)
$v \sin I$	2.8	km s^{-1}	Mayor & Queloz (1995)
$P_{*, \text{rot}}$	37.0	d	Mittag, Schmitt & Schröder (2018)
Resolution	30	—	—
ρ	$1/\pi$	—	—

Figure 7. In red the rotationally deconvolved absorption line from 51 Peg and in blue the reflected starlight from 51 Peg b. The reflected starlight has been shifted by 0.4 units.

as rotating from the Earth with period equal to that of the orbital period.

At greatest elongation, we would expect the RV of light reflected from the substellar point of the planet to be at the velocity semiamplitude where

$$K_{\text{planet}} = \frac{2\pi a}{P} \frac{\sin I}{1 + m_p/m_*} = 131 \text{ km s}^{-1}. \quad (46)$$

We fitted the reflected light to a Gaussian using the Levenberg–Marquardt algorithm and obtained an $\text{FWHM} = 19.0 \text{ km s}^{-1}$. The Gaussian fit gave the centre of the line as -131.2 km s^{-1} and an amplitude of 0.349.

Fig. 7 shows in detail the reflected light absorption line. The line is slightly asymmetric with the deepest part of the line occurring at a velocity of -132 km s^{-1} . The asymmetry in the line is due to only one side of the rotating planet being illuminated as seen from the Earth.

We examined the phase curve for the planet by running RESTART for a number of phases. We normalized the flux values returned so that the flux at $\alpha = 0$ is 1 and plotted them against the curve of the Lambert phase function (Fig. 8). This figure shows that the reflected light from our model closely follows the Lambert phase function.

Finally, we checked the FWHM and amplitude of the line RESTART predicts against the detected signal from Martins et al.

Figure 8. Phase curve for reflected light of 51 Peg b along with Lambert phase curve for comparison.

Table 3. Input parameters for the Earth analogue.

Parameter	Value	Units	References
t_0	100.0	BJD	—
P	365.25	d	—
m_p	0.00315	m_{Jup}	—
R_p	0.08921	R_{Jup}	—
m_*	1.0	$M_{\odot}$	—
R_*	1.0	$R_{\odot}$	—
Ω	0	deg	—
i	90	deg	—
a	1.0	au	—
$d_{* \oplus}$	10	pc	—
$P_{*, \text{rot}}$	24	d	—
resolution	30	—	—
ρ	$1/\pi$	—	—

(2015) at $\Phi(\alpha) = 0.87$. The FWHM from RESTART was 19.6 km s^{-1} , which is within 1σ of the detected value.

The flux ratio we obtained between the reflected light and the star at the same phase was 1.3×10^{-4} . However, this is only the fractional difference in the level of the continuum but taking into account the reduction of the line amplitude due to the broadening (line amplitude was 0.41) we have a signal level of 5.4×10^{-5} , which is within 1.5σ of the detection result reported by Martins et al. (2015). Increasing the radius of the planet to $1.7 R_{\text{Jup}}$ would bring the flux ratio to the detected value of 6.0×10^{-5} .

3.3 Results for a tidally locked Earth analogue

We used the absorption line generated for 51 Peg as the input stellar spectrum as 51 Peg has a similar spectral type to the Sun. The input parameters for the system are given in Table 3.

We ran RESTART with input time when the planet is at greatest elongation. Fig. 9 Tab (b) shows the normalized spectrum of the observed reflected light.

At greatest elongation, we would expect the RV of light reflected from the substellar point of the planet to be at the velocity semiamplitude where

$$K_{\text{planet}} = \frac{2\pi a}{P} \frac{\sin I}{1 + m_p/m_*} = 29.8 \text{ km s}^{-1}. \quad (47)$$

Fitting a Gaussian using the Levenberg–Marquardt algorithm to the line gave its centre at -29.8 km s^{-1} as expected. Its FWHM of 6.8 km s^{-1} and depth of 0.93 showed broadening compared to the FWHM of 6.4 km s^{-1} and depth of 1.0 of the line before it was reflected. This is again due to light originating from opposing limbs of the star having different angles of incidence on the planet. In order to confirm this, we reduced the radius of the Sun by a factor of 10 and reran RESTART. This resulted in a line of depth 0.99 and FWHM of 6.4 km s^{-1} , which were within 1 per cent of the figures for the original line.

We checked the geometric albedo by running RESTART again this time setting the date to that corresponding to when the phase angle $\alpha = 0$ (time of secondary transit). The value for the geometric albedo obtained was $p = 0.668$, which was consistent with the $2/3$ that would be expected for a Lambert disc.

We next explored the phase curve for the planet by running RESTART for orbital phase $\theta = \{0, \pi/4, \pi/2, 3\pi/2, \pi, 5\pi/4, 3\pi/2\}$, where $\theta = 0$ is at time of centre of primary transit.

We normalized the flux values returned so that the flux at $\alpha = 0$ is 1 and plotted them against the curve of the Lambert phase function (Fig. 10). This figure shows that the reflected light from our model follows the Lambert phase function.

3.4 Impact of system parameter changes to line depth and FWHM

Here, we report on the impact on line depth and FWHM for a hot Jupiter-type system by changing a number of the parameters of the system including star radius, planet radius, planet-to-star distance, and stellar rotation in turn.

We use the parameters based on those for 51 Peg b as the base for the system except that we set the inclination to 90 deg and the time of observation to be when the planet is fully illuminated with respect to the observer. This permits us to look at the system when the broadened line is symmetric. The parameters are specified in Table 4.

The line depths and FWHM for the broadened lines output from RESTART are determined by fitting a Gaussian using the Levenberg–Marquardt algorithm as before.

The stellar radius was varied between 0.1 and 3 solar radii covering the domain of main-sequence stars from M dwarfs to A stars. The results are shown in Fig. 11. The FWHM is correlated with the stellar radius. We can also see that from approximately $0.5R_{\odot}$ upwards in stellar radius the increase in FWHM is approximately linear. In terms of line amplitude against stellar radius, below $0.5R_{\odot}$ the relationship is linear as it heads asymptotically towards the level of no broadening and a line amplitude of 1.

Next, the planet radius was varied between 0.09 and 13 Jupiter radii covering the domain of the Earth-sized planets up towards the largest planets which approach 2 Jupiter radii and then beyond that in order to show the asymptotic relationship. The results are shown in Fig. 12. The FWHM of the spectral line for the planet calculated from RESTART increases as we increase the planet radius. In particular, this relationship when the planet radius is large (>8 Jupiter radii) is close to linear. As the planet radius decreases the FWHM asymptotically approaches a value of just over 19 km s^{-1} , where the broadening is primarily due to the size of the star.

The line amplitude is inversely correlated with the planet radius. At planet radius 5–13 Jupiter radii the relationship with line amplitude is close to linear and as the planet radius decreases towards zero the line amplitude asymptotically approaches a value of just over 0.41.

Figure 9. Tab (a) shows the rotationally deconvolved absorption line representing a solar line. Tab (b) shows the reflected light as seen from the Earth. Both tabs have had the continuum normalized to 1.

Figure 10. Phase curve for reflected light from the Earth analogue along with Lambert phase curve for comparison.

Table 4. Parameters set for hot Jupiter type system based on 51 Peg used for exploring the impact of system parameter changes to line depth and FWHM.

Parameter	Value	Units	References
t_0	56020.756	MJD	Martins et al. (2015)
P	4.231	d	Martins et al. (2015)
m_p	0.457	m_{Jup}	Martins et al. (2015)
R_p	1.6	R_{Jup}	Martins et al. (2015)
m_*	1.04	$M_{\odot}$	Santos et al. (2003)
R_*	1.237	$R_{\odot}$	van Belle & von Braun (2009)
Ω	0.0	deg	—
i	90	deg	—
a	0.052	au	Martins et al. (2015)
$d_{* \oplus}$	14.7	pc	Gaia Collab. (2018)
$v \sin i$	2.8	km s^{-1}	Mayor & Queloz (1995)
$P_{*, \text{rot}}$	37.0	d	Mittag et al. (2018)
resolution	30	—	—
ρ	$1/\pi$	—	—
FWHM	7.36	km s^{-1}	—

The observed period of rotation of the star ($P_{*, \text{rot}}$) was varied between 0.1 and 100 d next. The results are shown in Fig. 13. As expected the minimum FWHM and maximum line amplitude occurs when the planet and star are in synchronous orbit with each other, which is when the observed stellar rotation period is at the planet orbital period of 4.231 d. As the observed stellar rotation period decreases below the orbital period then the broadening increases and the line amplitude decreases rapidly. For increasing stellar rotation period (as the star spins down), the broadening increases but then flattens out to an FWHM of just over 20 km s^{-1} .

Next, we vary the distance between the star and the planet. In doing this, we also modify the orbital period so that the system is consistent with Kepler's laws and we modify the observed stellar rotation so that it is always 30 d longer than the orbital period of the system. The results are shown in Fig. 14. Here, we see that the broadening is small with $\text{FWHM} < 10 \text{ km s}^{-1}$ until the distance between planet and star is less than 0.25 au and then as we get closer to the star the level of broadening increases and the line amplitude decreases very rapidly.

Finally, we vary the distance between the star and the planet again but this time we also vary the observed stellar rotation, so that its rate is that of the orbital period of the planet and thus the orbits are synchronized. The results are shown in Fig. 15. The broadening is minimal until 0.05 au where the broadening increases steeply with decreasing distance between planet and star. The broadening is due to the large range of incident angles for starlight being reflected from the planet due to the proximity of the star to the planet.

4 DISCUSSION

In the development of the model for reflected starlight, we have deliberately excluded a number of mainly lower order effects from the model. Here, we describe some of them and detail how future versions of the model could include them.

We have only including isotropic scattering $\rho = \text{constant}$ in our model calculations resulting in the planet phase curves closely resembling those for a Lambert phase curve. For non-isotropic and multiple scattering of reflected light from exoplanets (Seager, Whitney & Sasselov 2000), we would need to identify types and sizes of scatterers in the atmosphere of the planet along with their relevant phase functions. Models for these scatterers already

Figure 11. Impact of change of stellar radius on FWHM (left) and line amplitude (right) for a hot Jupiter-type system based on the 51 Peg system. The input parameters for the system are specified in Table 4.

Figure 12. Exoplanet system as for Fig. 11 except showing the impact of change of planet radius on FWHM (left) and line amplitude (right).

Figure 13. Exoplanet system as for Fig. 11 except showing the impact of change of stellar rotation on FWHM (left) and line amplitude (right).

Figure 14. Exoplanet system as for Fig. 11 except showing the impact of change of distance between planet and star on FWHM (left) and line amplitude (right).

Figure 15. Exoplanet system as for Fig. 11 except showing the impact of change of distance between planet and star on FWHM (left) and Line Amplitude (right) for a synchronized hot Jupiter system.

exist, for example, see the aforementioned paper and could be incorporated into the model.

We have also not taken account of the velocities of the particles/winds in particular those due to recirculation from day to night side, which can be important in particular for hot Jupiters. These winds have been measured (e.g. Loudon & Wheatley 2015) and are typically of the order of a few km s^{-1} .

In terms of tides, we have not taken account of the deformed shapes of the planet and star in our model. For instance, Correia (2014) estimated the level of deformation of close in exoplanets assuming the deformed shape was in the form of a triaxial ellipsoid. For WASP-19b, they estimated the length of the three axis to be 17.3, 15.4, and 14.8 $R_{\oplus}$ under the assumption that if the planet were spherical its radius would be 15.2 $R_{\oplus}$. In terms of our model, this means that we would underestimate the flux and line broadening at greatest elongation and overestimate it at time of secondary transit. The size of the deformations is based on assumptions on the Love number used and to date no evidence of the deformations has been observed. We could similarly update the model to include the ellipticity of the star and gravitational darkening due to tides using a model such as that outlined in Jackson et al. (2012). For

WASP-19b, the maximum increase in radius due to tides would be $3.32 \times 10^{-4} R_{*}$.

Although it is expected that most hot Jupiters orbit their parent star in circular orbits and are expected to be tidally locked, we could also lift these assumptions from our model.

5 CONCLUSIONS

In this work, we have shown using a simple model of scattering that the Doppler shift of reflected light from planets whose parent star is at a distance where the light rays arriving at the planet can be considered to be parallel, have a different Doppler shift to that of light emitted directly from the planet. The magnitude of this difference in Doppler shift is equivalent to the radial motion of the planet with respect to the star.

We also showed that in the case of planets close to the parent star and in particular hot Jupiters the reflected starlight observed should be broadened due to the difference in angles of incidence of starlight arriving at the planet from different limbs of the star. We gave estimates for this broadening and developed a model that describes

the scattering that takes place for tidally locked hot Jupiters in circular orbits.

We implemented the model in software (RESTART) and then ran this software tool for three exoplanets: We found that significant broadening occurred for the WASP-19b very hot Jupiter system and significant broadening also took place for the 51 Peg b system. Our results for 51 Peg were consistent with the detection results provided we assumed the planet had size of $1.7R_{\text{Jup}}$. Very minimal broadening took place for our Earth analogue.

Light curves produced for all three planets showed no significant deviations from a Lambertian phase curve as was expected for isotropic scattering.

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APPENDIX A: FLUX AND DOPPLER SHIFT OF REFLECTED LIGHT FROM HOT JUPITERS

Here, we determine the flux and Doppler shift of the reflected light arriving at the Earth from a tidally locked hot Jupiter in a circular orbit. The geometry we use to determine this is shown in Fig. 3.

We use two main coordinate systems. The first is the Cartesian coordinate system $X_p Y_p Z_p$ centred on the planet, where the Z_p axis is pointing towards the Earth and the X_p and Y_p axis in the plane of the sky as viewed from the Earth. We use a colatitude coordinate system $(\vartheta_p, \phi_p, R_p)$ to describe the location of points on the surface of the planet.

In order to determine the reflected starlight arriving at the Earth, we first determine the starlight incident at each point (ϑ_p, ϕ_p) on the surface of the planet.

For this, we use a second stellar Cartesian coordinate system $X_* Y_* Z_*$ centred on the star and that has Z_* axis pointing towards (ϑ_*, ϕ_*) as shown in Fig. 3. This stellar coordinate system also has its associated colatitude system $(\vartheta_*, \phi_*, R_*)$ in order to describe points on the surface of the star. Let d be the distance between C_p and the centre of the star. C_p is the point $(R_p \cos \phi_p \sin \vartheta_p, R_p \sin \phi_p \sin \vartheta_p, R_p \cos \vartheta_p)$ in the planet frame. In this frame, we assume we have that $O_* = (x_*, y_*, z_*)$ and we can determine d as follows:

$$\text{Angle } C_p \hat{O}_p O_* = \gamma \text{ is}$$

$$\cos(\gamma) = \cos \phi_p \sin \vartheta_p \frac{x_*}{a} + \sin \phi_p \sin \vartheta_p \frac{y_*}{a} + \cos \vartheta_p \frac{z_*}{a}. \quad (\text{A1})$$

Using the cosine rule, we have

$$d^2 = a^2 + R_p^2 - 2aR_p \cos \gamma, \quad (\text{A2})$$

and we can now determine $\tan \omega$ as

$$\tan \omega = \frac{R_* \sin \vartheta_*}{d - R_* \cos \vartheta_*}. \quad (\text{A3})$$

$\sin \omega$ and $\cos \omega$ and their implicit differentials are

$$\sin \omega = \frac{R_* \sin \vartheta_*}{\sqrt{R_*^2 + d^2 - 2R_* d \cos \vartheta_*}}, \quad (\text{A4})$$

$$\cos \omega \, d\omega = \left(\frac{R_* \cos \vartheta_*}{\sqrt{R_*^2 + d^2 - 2dR_* \cos \vartheta_*}} - \frac{R_*^2 d \sin^2 \vartheta_*}{(R_*^2 + d^2 - 2dR_* \cos \vartheta_*)^{3/2}} \right) d\vartheta_*, \quad (\text{A5})$$

$$\cos \omega = \frac{d - R_* \cos \vartheta_*}{\sqrt{R_*^2 + d^2 - 2R_* d \cos \vartheta_*}}, \quad (\text{A6})$$

$$\sin \omega \, d\omega = \frac{-R_* \sin \vartheta_*}{\sqrt{R_*^2 + d^2 - 2R_* d \cos \vartheta_*}} + \frac{R_* d \sin \vartheta_* (d - R_* \cos \vartheta_*)}{(R_*^2 + d^2 - 2dR_* \cos \vartheta_*)^{3/2}} d\vartheta_*. \quad (\text{A7})$$

The solid angle of the star subtended by the planet cell is

$$\Omega_{\theta_p \phi_p} = \int_0^{2\pi} \int_0^{\tan^{-1}(\frac{R_*}{d})} \sin \omega \, d\omega \, d\phi_* - \text{blocked portion.} \quad (\text{A8})$$

The blocked portion of the solid angle is the portion where light cannot travel to the planet cell due to some part of the star or planet being in its path. In order to take account of this, when we calculate the integral numerically for each element we check whether

$$0 \leq \hat{\sigma} \cdot \hat{n}_* < 1, \quad (\text{A9})$$

and

$$0 \leq -\hat{\sigma} \cdot \hat{n}_p < 1. \quad (\text{A10})$$

If the light is blocked, then at least one of the two inequalities will not hold and instead of using the $\sin \omega$ factor in the integral we use 0 instead.

The flux at the point (ϑ_p, ϕ_p) of the planet from the star is

$$F_{\vartheta_p \phi_p}(\nu) = \int_0^{2\pi} \int_0^{\tan^{-1}(\frac{R_*}{d})} I_*(\vartheta_*, \phi_*, \nu - \nu_*, t) \times (-\hat{n}_p \cdot \sigma) \sin \omega \, d\omega \, d\phi_*, \quad (\text{A11})$$

where ν_* is the frequency Doppler shift due to the motion of the star cell.

We have to work out the dot product $(\hat{n}_p \cdot \sigma)$, which is complicated by the fact that currently we only know the position of the cell C_* in the coordinates of the star $(R_* \cos \phi_* \sin \vartheta_*, R_* \sin \phi_* \sin \vartheta_*, R_* \cos \vartheta_*)$.

In order to move to the planet frame from the star frame, we have two rotations in order to align the Z_* axis and a translation to perform (see Fig. A1).

In terms of vectors, we have

$$\overrightarrow{O_p C_*} = \overrightarrow{O_p O_*} + \overrightarrow{O_* C_*} = \overrightarrow{O_* C_*} + \mathbf{R}_x(-\Lambda_x) \mathbf{R}_y(-\Lambda_y) \overrightarrow{O_* C'_*}, \quad (\text{A12})$$

where $\mathbf{R}_x$ and $\mathbf{R}_y$ are rotation matrices about the X - and Y -axes, respectively, and the angles Λ_x and Λ_y are the angles so that the rotations $\mathbf{R}_y(\Lambda_y) \mathbf{R}_x(\Lambda_x)$ move the Z_p axis to be in the same direction as the Z_* axis.

In component form, we have

$$\begin{bmatrix} C_{*x} \\ C_{*y} \\ C_{*z} \end{bmatrix}_p = [\mathbf{R}_x(-\Lambda_x) \mathbf{R}_y(-\Lambda_y)] \begin{bmatrix} R_* \cos \phi_* \sin \vartheta_* \\ R_* \sin \phi_* \sin \vartheta_* \\ R_* \cos \vartheta_* \end{bmatrix}_* + \begin{bmatrix} x_* \\ y_* \\ z_* \end{bmatrix}_p. \quad (\text{A13})$$

The subscripts on the matrix correspond to the Cartesian coordinate frame the components are in (p for planet and * for star).

Expanding out the rotation matrices, we have

$$\begin{bmatrix} C_{*x} \\ C_{*y} \\ C_{*z} \end{bmatrix}_p = \begin{bmatrix} \cos \Lambda_y & 0 & -\sin \Lambda_y \\ \sin \Lambda_x \sin \Lambda_y & \cos \Lambda_x & \sin \Lambda_x \cos \Lambda_y \\ \cos \Lambda_x \sin \Lambda_y & -\sin \Lambda_x & \cos \Lambda_x \cos \Lambda_y \end{bmatrix} \begin{bmatrix} R_* \cos \phi_* \sin \vartheta_* \\ R_* \sin \phi_* \sin \vartheta_* \\ R_* \cos \vartheta_* \end{bmatrix}_* + \begin{bmatrix} x_* \\ y_* \\ z_* \end{bmatrix}_p. \quad (\text{A14})$$

Expanding out the components, we have

$$C_{*x} = \cos \Lambda_y R_* \cos \phi_* \sin \vartheta_* - \sin \Lambda_y R_* \cos \vartheta_* + x_*, \quad (\text{A15})$$

$$C_{*y} = \sin \Lambda_x \sin \Lambda_y R_* \cos \phi_* \sin \vartheta_* + \cos \Lambda_x R_* \sin \phi_* \sin \vartheta_* + \sin \Lambda_x \cos \Lambda_y R_* \cos \vartheta_* + y_*, \quad (\text{A16})$$

$$C_{*z} = \cos \Lambda_x \sin \Lambda_y R_* \cos \phi_* \sin \vartheta_* - \sin \Lambda_x R_* \sin \phi_* \sin \vartheta_* + \cos \Lambda_x \cos \Lambda_y R_* \cos \vartheta_* + z_*. \quad (\text{A17})$$

We determine the components of $\sigma = (\sigma_x, \sigma_y, \sigma_z)^T$ as

$$\sigma_x = R_p \cos \phi_p \sin \vartheta_p - (\cos \Lambda_y R_* \cos \phi_* \sin \vartheta_* - \sin \Lambda_y R_* \cos \vartheta_* + x_*), \quad (\text{A18})$$

$$\sigma_y = R_p \sin \phi_p \sin \vartheta_p - (\sin \Lambda_x \sin \Lambda_y R_* \cos \phi_* \sin \vartheta_* + \cos \Lambda_x R_* \sin \phi_* \sin \vartheta_* + \sin \Lambda_x \cos \Lambda_y R_* \cos \vartheta_* + y_*), \quad (\text{A19})$$

$$\sigma_z = R_p \cos \vartheta_p - (\cos \Lambda_x \sin \Lambda_y R_* \cos \phi_* \sin \vartheta_* - \sin \Lambda_x R_* \sin \phi_* \sin \vartheta_* + \cos \Lambda_x \cos \Lambda_y R_* \cos \vartheta_* + z_*). \quad (\text{A20})$$

The unit vector in the direction of σ is defined as

$$\hat{\sigma} = \frac{1}{s} \sigma, \quad (\text{A21})$$

where s is the distance between the cells C_p and C_* (the square root of the sum of squares of the components of σ given above).

The unit vector normal to the planet cell $\hat{n}_p$ is

$$\hat{n}_p = \begin{bmatrix} \cos \phi_p \sin \vartheta_p \\ \sin \phi_p \sin \vartheta_p \\ \cos \vartheta_p \end{bmatrix}_p. \quad (\text{A22})$$

The dot product of the two unit vectors is

$$\begin{aligned} \hat{\sigma} \cdot \hat{n}_p = \frac{1}{s} \{ & \cos \phi_p \sin \vartheta_p [R_p \cos \phi_p \sin \vartheta_p \\ & - (\cos \Lambda_y R_* \cos \phi_* \sin \vartheta_* - \sin \Lambda_y R_* \cos \vartheta_* + x_*)] \\ & + \sin \phi_p \sin \vartheta_p [R_p \sin \phi_p \sin \vartheta_p \\ & - (\sin \Lambda_x \sin \Lambda_y R_* \cos \phi_* \sin \vartheta_* \\ & + \cos \Lambda_x R_* \sin \phi_* \sin \vartheta_* + \sin \Lambda_x \cos \Lambda_y R_* \cos \vartheta_* + y_*)] \\ & + \cos \vartheta_p [R_p \cos \vartheta_p - (\cos \Lambda_x \sin \Lambda_y R_* \cos \phi_* \sin \vartheta_* \\ & - \sin \Lambda_x R_* \sin \phi_* \sin \vartheta_* + \cos \Lambda_x \cos \Lambda_y R_* \cos \vartheta_* + z_*)] \}. \end{aligned} \quad (\text{A23})$$

Now, we determine the angles Λ_x and Λ_y . We use equation (A14) and set the position of the star cell to be in the direction of the Z'_* axis and we set it to unit length. When this axis is rotated, it must

Figure A1. Geometry showing the rotation from the original star frame $X'_*Y'_*Z'_*$ to the frame $X_*Y_*Z_*$ where the Z_* axis points towards the planet cell C_p .

be pointing towards the planet cell so we have

$$\frac{1}{q} \begin{bmatrix} C_{px} - x_* \\ C_{py} - y_* \\ C_{pz} - z_* \end{bmatrix}_p = \begin{bmatrix} \cos \Lambda_y & 0 & -\sin \Lambda_y \\ \sin \Lambda_x \sin \Lambda_y & \cos \Lambda_x & \sin \Lambda_x \cos \Lambda_y \\ \cos \Lambda_x \sin \Lambda_y & -\sin \Lambda_x & \cos \Lambda_x \cos \Lambda_y \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_* \quad (\text{A24})$$

where q is the normalizing factor

$$q^2 = (R_p \cos \phi_p \sin \vartheta_p - x_*)^2 + (R_p \sin \phi_p \sin \vartheta_p - y_*)^2 + (R_p \cos \vartheta_p - z_*)^2. \quad (\text{A25})$$

Simplifying the equation, we get

$$\frac{1}{q} \begin{bmatrix} R_p \cos \phi_p \sin \vartheta_p - x_* \\ R_p \sin \phi_p \sin \vartheta_p - y_* \\ R_p \cos \vartheta_p - z_* \end{bmatrix} = \begin{bmatrix} -\sin \Lambda_y \\ \sin \Lambda_x \cos \Lambda_y \\ \cos \Lambda_x \cos \Lambda_y \end{bmatrix}, \quad (\text{A26})$$

and from this we have

$$\Lambda_y = \sin^{-1} \left(\frac{x_* - R_p \cos \phi_p \sin \vartheta_p}{q} \right), \quad (\text{A27})$$

$$\Lambda_x = \sin^{-1} \left(\frac{R_p \sin \phi_p \sin \vartheta_p - y_*}{q \cos \Lambda_y} \right). \quad (\text{A28})$$

The flux arriving at the surface of the planet at the point $(\vartheta_p, \phi_p, R_p)$ is

$$F_{\vartheta_p \phi_p}(\nu) = \int_0^{2\pi} \int_0^{\pi/2} H(\vartheta_p, \phi_p, \vartheta_*, \phi_*, \nu - \Delta\nu_*, t) d\vartheta_* d\phi_*, \quad (\text{A29})$$

where $H(\vartheta_p, \phi_p, \vartheta_*, \phi_*, \nu, t)$ is zero when starlight is blocked from reaching the planet cell

$$H(\vartheta_p, \phi_p, \vartheta_*, \phi_*, \nu, t) = \begin{cases} 0, & \text{if } \hat{\sigma} \cdot \hat{n}_* < 0 \\ 0, & \text{if } -\hat{\sigma} \cdot \hat{n}_p < 0, \end{cases} \quad (\text{A30})$$

otherwise non-zero when

$$H(\vartheta_p, \phi_p, \vartheta_*, \phi_*, \nu, t) = I_*(\vartheta_*, \phi_*, \nu, t)(-\hat{n}_p \cdot \hat{\sigma}) \times \left(\frac{-R_* \sin \vartheta_*}{\sqrt{R_*^2 + d^2 - 2R_*d \cos \vartheta_*}} + \frac{R_*d \sin \vartheta_*(d - R_* \cos \vartheta_*)}{(R_*^2 + d^2 - 2dR_* \cos \vartheta_*)^{3/2}} \right). \quad (\text{A31})$$

The intensity of the reflected light leaving the surface from the planet cell (ϑ_p, ϕ_p) in the direction of the Earth is

$$I_{s\vartheta_p \phi_p}(\nu) = \int_0^{2\pi} \int_0^{\pi/2} \rho(\vartheta_p, \phi_p, \vartheta_*, \phi_*) \times H(\vartheta_p, \phi_p, \vartheta_*, \phi_*, \nu - \Delta\nu, t)(\hat{n}_p \cdot \hat{k}) d\vartheta_* d\phi_*, \quad (\text{A32})$$

where $\rho(\vartheta_p, \phi_p, \vartheta_*, \phi_*)$ is the bidirectional reflectance distribution function, $\hat{k}$ is the unit vector in the direction of the Earth, and the frequency Doppler shift $\Delta\nu$ is

$$\frac{\Delta\nu}{\nu} = \frac{\Delta v}{c}, \quad (\text{A33})$$

where Δv is the velocity shift due to the reflection of the light on the cell C_p and the shift due to the radial motion of the star cell with respect to the planet cell $\Delta v_{r, C_* C_p}$. This velocity shift is

$$\Delta v = v_p \cdot (\hat{k} - \hat{\sigma}) + \Delta v_{r, C_* C_p}. \quad (\text{A34})$$

The surface velocity of the planet cell v_p is determined in Appendix B and $\Delta v_{r, C_* C_p}$ is determined in Appendix C.

We now work out the reflected starlight reaching the observer at the Earth. The viewpoint of the observer is given in Fig. A2 and from this we can see that $d\Omega_p$ and $\tan \omega$ is

$$d\Omega_p = \frac{dA}{r^2} = \sin \omega d\omega d\phi_p, \quad (\text{A35})$$

$$\tan \omega = \frac{\rho}{d_{p\oplus}}. \quad (\text{A36})$$

Assuming the Earth is very far from the planet we can treat ω as a small angle and Taylor expanding only to first order we have

Figure A2. Geometry showing the view of the planet with respect to the observer.

$\sin \omega = \omega$ and $\cos \omega = 1$ and thus

$$\omega = \frac{R_p \sin \vartheta_p}{d_{p\oplus}}, \quad (\text{A37})$$

where $d_{p\oplus}$ is distance from the Earth to the planet. Differentiating, we have

$$d\omega = (R_p \cos \vartheta_p / d_{p\oplus}) d\vartheta_p. \quad (\text{A38})$$

The flux arriving at the distant Earth is

$$\begin{aligned} F_{\oplus}(\nu) &= \int_0^{2\pi} \int_0^{R_p/d_{p\oplus}} I_{s\vartheta_p\phi_p}(\vartheta_p, \phi_p, \nu, t) \cos \omega \sin \omega \, d\omega \, d\phi_p \\ &= \int_0^{2\pi} \int_0^{R_p/d_{p\oplus}} I_{s\vartheta_p\phi_p}(\vartheta_p, \phi_p, \nu, t) \omega \, d\omega \, d\phi_p, \end{aligned} \quad (\text{A39})$$

and thus

$$\begin{aligned} F_{\oplus}(\nu, t) &= \left(\frac{R_p}{d_{p\oplus}} \right)^2 \int_0^{2\pi} \int_0^{\pi/2} I_{s\vartheta_p\phi_p}(\vartheta_p, \phi_p, \nu, t) \\ &\quad \times \sin \vartheta_p \cos \vartheta_p \, d\vartheta_p \, d\phi_p. \end{aligned} \quad (\text{A40})$$

APPENDIX B: POSITION AND VELOCITY OF POINTS ON SURFACE OF THE PLANET

B1 Introduction

Initially, we work in the frame centred on the barycentre and with xy plane coincident with the plane of the orbit and measure position and velocity of points on the surface of the planet relative to the barycentre. We then move on to determining these velocities in the reference frame as is typically viewed from the Earth using circularized forms of the elliptical two-body equations in order to transform from the barycentric frame to the Earth frame.

The planet moves round the barycentre in a circle of radius a in the xy plane with coordinates

$$x = a \frac{m_*}{m_* + m_p} \cos \theta, \quad (\text{B1})$$

$$y = a \frac{m_*}{m_* + m_p} \sin \theta. \quad (\text{B2})$$

Fig. B1 shows the path of the orbit of the planet and star and that of the surface cells around the barycentre in the xy plane.

The coordinates of the planet are $P = (x_p, y_p, 0)$ and surface cell coordinates are $C_p = (x_{pc}, y_{pc}, z_{pc})$.

From Fig. 14, it is clear that the cell C_p moves in a circular orbit with radius

$$d_{pc} = \sqrt{x_{pc}^2 + y_{pc}^2}, \quad (\text{B3})$$

at a height z_{pc} above the xy plane. We can write the x and y coordinates of the planet in terms of θ and d_{pc} as

$$x_{pc} = d_{pc} \cos(\theta + \theta'_{pc}), \quad (\text{B4})$$

$$y_{pc} = d_{pc} \sin(\theta + \theta'_{pc}), \quad (\text{B5})$$

where

$$\sin \theta'_{pc} = \frac{\mathbf{BP} \times (x_{pc}, y_{pc}, 0)^T}{|\mathbf{BP}| |(x_{pc}, y_{pc}, 0)|}, \quad (\text{B6})$$

and θ'_{pc} is in range $\{-\pi/2, \pi/2\}$.

Differentiating, we get the velocity of the cell in the frame to be

$$\dot{x}_{pc} = -d_{pc} \dot{\theta} \sin(\theta + \theta'_{pc}), \quad (\text{B7})$$

$$\dot{y}_{pc} = d_{pc} \dot{\theta} \cos(\theta + \theta'_{pc}), \quad (\text{B8})$$

$$\dot{z}_{pc} = 0. \quad (\text{B9})$$

We can now move to the barycentric frame with Z -axis pointing to the Earth and the X - and Y -axes in the plane of the sky

$$\begin{aligned} \begin{bmatrix} X_{pc} \\ Y_{pc} \\ Z_{pc} \end{bmatrix} &= \begin{bmatrix} \cos \Omega & -\sin \Omega & 0 \\ \sin \Omega & \cos \Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos I & -\sin I \\ 0 & \sin I & \cos I \end{bmatrix} \\ &\quad \times \begin{bmatrix} d_{pc} \cos(\theta + \theta'_{pc}) \\ d_{pc} \sin(\theta + \theta'_{pc}) \\ z_{pc} \end{bmatrix}, \end{aligned} \quad (\text{B10})$$

$$\begin{aligned} \begin{bmatrix} X_{pc} \\ Y_{pc} \\ Z_{pc} \end{bmatrix} &= \begin{bmatrix} \cos \Omega & -\sin \Omega & 0 \\ \sin \Omega & \cos \Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &\quad \times \begin{bmatrix} d_{pc} \cos(\theta + \theta'_{pc}) \\ d_{pc} \sin(\theta + \theta'_{pc}) \cos I - z_{pc} \sin I \\ d_{pc} \sin(\theta + \theta'_{pc}) \sin I + z_{pc} \cos I \end{bmatrix}, \end{aligned} \quad (\text{B11})$$

$$\begin{bmatrix} X_{pc} \\ Y_{pc} \\ Z_{pc} \end{bmatrix}$$

$$= \begin{bmatrix} d_{pc} (\cos \Omega \cos(\theta + \theta'_{pc}) - \sin \Omega \sin(\theta + \theta'_{pc}) \cos I) + z_{pc} \sin \Omega \sin I \\ d_{pc} (\sin \Omega \cos(\theta + \theta'_{pc}) + \cos \Omega \sin(\theta + \theta'_{pc}) \cos I) - z_{pc} \cos \Omega \sin I \\ d_{pc} \sin(\theta + \theta'_{pc}) \sin I + z_{pc} \cos I \end{bmatrix}. \quad (\text{B12})$$

Figure B1. Geometry looking down on xy plane of planet and star orbiting the barycentre B along with the paths and motion of cells on the surface of the planet and the star.

Differentiating, we get the velocity

follows:

$$\begin{bmatrix} \dot{X}_{pc} \\ \dot{Y}_{pc} \\ \dot{Z}_{pc} \end{bmatrix} = \begin{bmatrix} -\dot{\theta} d_{pc} \left(\cos \Omega \sin(\theta + \theta'_{pc}) + \sin \Omega \cos(\theta + \theta'_{pc}) \cos I \right) \\ -\dot{\theta} d_{pc} \left(\sin \Omega \sin(\theta + \theta'_{pc}) - \cos \Omega \cos(\theta + \theta'_{pc}) \cos I \right) \\ \dot{\theta} d_{pc} \cos(\theta + \theta'_{pc}) \sin I \end{bmatrix}. \quad (\text{B13})$$

We typically know Ω , I and $\{X_{pc}, Y_{pc}, Z_{pc}\}^T$ and θ and need to determine d_{pc} , z_{pc} , and θ'_{pc} from the above equations. Noting that the above equations are linear in $d_{pc} \cos(\theta + \theta'_{pc})$, $d_{pc} \sin(\theta + \theta'_{pc})$ and z_{pc} , we can arrange the matrices as

$$\begin{bmatrix} X_{pc} \\ Y_{pc} \\ Z_{pc} \end{bmatrix} = \begin{bmatrix} \cos \Omega & -\sin \Omega \cos I & \sin \Omega \sin I \\ \sin \Omega & \cos \Omega \cos I & -\cos \Omega \sin I \\ 0 & \sin I & \cos I \end{bmatrix} \times \begin{bmatrix} d_{pc} \cos(\theta + \theta'_{pc}) \\ d_{pc} \sin(\theta + \theta'_{pc}) \\ z_{pc} \end{bmatrix}. \quad (\text{B14})$$

We can invert the above matrix (determinant is 1 so it is non-singular) and solve for our three parameters:

$$\begin{bmatrix} d_{pc} \cos(\theta + \theta'_{pc}) \\ d_{pc} \sin(\theta + \theta'_{pc}) \\ z_{pc} \end{bmatrix} = \begin{bmatrix} \cos \Omega & \sin \Omega & 0 \\ -\sin \Omega \cos I & \cos \Omega \cos I & \sin I \\ \sin \Omega \sin I & -\cos \Omega \sin I & \cos I \end{bmatrix} \times \begin{bmatrix} X_{pc} \\ Y_{pc} \\ Z_{pc} \end{bmatrix} = \begin{bmatrix} l \\ m \\ n \end{bmatrix}. \quad (B15)$$

From this, we can see that

$$d_{pc} = \sqrt{l^2 + m^2}, \quad (B16)$$

and

$$\sin \theta' = \frac{m \cos \theta - l \sin \theta}{d_{pc}}. \quad (B17)$$

APPENDIX C: POSITION AND VELOCITY OF POINT ON SURFACE OF THE STAR

We determine the velocity and position of a point on the surface of the star – a star cell – with respect to the barycentre of the system.

The velocity of a star cell v_{*c} is formed of two components due to the motion of the orbit of the star around the centre of mass of the system $v_{*,o}$ and the spin of the star $v_{*,sp}$

$$v_{*c} = v_{*,o} + v_{*,sp}. \quad (C1)$$

In our determination of the Doppler shift of reflected light, we need to know the component of the surface velocity in the direction of a given planet cell. If we denote $(\hat{\sigma})$ as the unit vector in that direction then the component velocity is given by

$$\Delta v_{r,c*cp} = v_{*c} \cdot \hat{\sigma}. \quad (C2)$$

The next two subsections determine each of the terms on the right-hand side of equation (C1) in turn.

C1 Orbital component of velocity of star cell

The star moves round the barycentre in a circle of radius a in the xy plane with coordinates

$$x = a \frac{m_p}{m_* + m_p} \cos \theta, \quad (C3)$$

$$y = a \frac{m_p}{m_* + m_p} \sin \theta. \quad (C4)$$

Fig. B1 shows the path of the orbit of the planet and star and that of the surface cells around the barycentre in the xy plane. The coordinates of the star are $P = (x_*, y_*, 0)$ and surface cell coords is $C_p = (x_{*c}, y_{*c}, z_{*c})$. From B1, it is clear that the cell C_* moves in a circular orbit with radius

$$d_{*c} = \sqrt{x_{*c}^2 + y_{*c}^2}, \quad (C5)$$

at a height z_{*c} above the xy plane. We can write the x and y coordinates of the planet in terms of θ and d_{*c} as

$$x_{*c} = d_{*c} \cos(\theta + \theta'_{*c} + \pi) = -d_{*c} \cos(\theta + \theta'_{*c}), \quad (C6)$$

$$y_{*c} = d_{*c} \sin(\theta + \theta'_{*c} + \pi) = -d_{*c} \sin(\theta + \theta'_{*c}), \quad (C7)$$

$$z_{*c} = z_{*c}, \quad (C8)$$

where

$$\sin \theta'_{*c} = \frac{BS \times (x_{*c}, y_{*c}, 0)^T}{|BS| |(x_{*c}, y_{*c}, 0)|}, \quad (C9)$$

and θ'_{*c} is in range $\{-\pi/2, \pi/2\}$.

Differentiating, we get the velocity of the cell in the frame to be

$$\dot{x}_{*c} = d_{*c} \dot{\theta} \sin(\theta + \theta'_{*c}), \quad (C10)$$

$$\dot{y}_{*c} = -d_{*c} \dot{\theta} \cos(\theta + \theta'_{*c}), \quad (C11)$$

$$\dot{z}_{*c} = 0. \quad (C12)$$

We can now move to the barycentric frame with Z -axis pointing to the Earth and the X and Y axis in the plane of the sky:

$$\begin{bmatrix} X_{*c} \\ Y_{*c} \\ Z_{*c} \end{bmatrix} = \begin{bmatrix} \cos \Omega & -\sin \Omega & 0 \\ \sin \Omega & \cos \Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos I & -\sin I \\ 0 & \sin I & \cos I \end{bmatrix} \begin{bmatrix} -d_{*c} \cos(\theta + \theta'_{pc}) \\ -d_{*c} \sin(\theta + \theta'_{pc}) \\ z_{*c} \end{bmatrix}, \quad (C13)$$

$$\begin{bmatrix} X_{*c} \\ Y_{*c} \\ Z_{*c} \end{bmatrix} = \begin{bmatrix} \cos \Omega & -\sin \Omega & 0 \\ \sin \Omega & \cos \Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} -d_{*c} \cos(\theta + \theta'_{*c}) \\ -d_{*c} \sin(\theta + \theta'_{*c}) \cos I - z_{*c} \sin I \\ -d_{*c} \sin(\theta + \theta'_{*c}) \sin I + z_{*c} \cos I \end{bmatrix}, \quad (C14)$$

$$\begin{bmatrix} X_{*c} \\ Y_{*c} \\ Z_{*c} \end{bmatrix} = \begin{bmatrix} -d_{*c} (\cos \Omega \cos(\theta + \theta'_{*c}) - \sin \Omega \sin(\theta + \theta'_{*c}) \cos I) + z_{*c} \sin \Omega \sin I \\ -d_{*c} (\sin \Omega \cos(\theta + \theta'_{*c}) + \cos \Omega \sin(\theta + \theta'_{*c}) \cos I) - z_{*c} \cos \Omega \sin I \\ -d_{*c} \sin(\theta + \theta'_{*c}) \sin I + z_{*c} \cos I \end{bmatrix}. \quad (C15)$$

Differentiating, we get the velocity

$$v_{*,o} = \begin{bmatrix} \dot{X}_{*c} \\ \dot{Y}_{*c} \\ \dot{Z}_{*c} \end{bmatrix}, \quad (C16)$$

$$v_{*,o} = \begin{bmatrix} \dot{\theta} d_{*c} (\cos \Omega \sin(\theta + \theta'_{*c}) + \sin \Omega \cos(\theta + \theta'_{*c}) \cos I) \\ \dot{\theta} d_{*c} (\sin \Omega \sin(\theta + \theta'_{*c}) - \cos \Omega \cos(\theta + \theta'_{*c}) \cos I) \\ -\dot{\theta} d_{*c} \cos(\theta + \theta'_{*c}) \sin I \end{bmatrix}. \quad (C17)$$

We typically know Ω , I and $\{X_{*c}, Y_{*c}, Z_{*c}\}^T$ and θ and need to determine d_{*c} , z_{*c} and θ'_{*c} from the above equations. Noting that the

above equations are linear in $d_{*c}\cos(\theta + \theta'_{*c})$, $d_{*c}\sin(\theta + \theta'_{*c})$ and z_{*c} , we can arrange the matrices as follows:

$$\begin{bmatrix} X_{*c} \\ Y_{*c} \\ Z_{*c} \end{bmatrix} = \begin{bmatrix} \cos\Omega & -\sin\Omega\cos I & \sin\Omega\sin I \\ \sin\Omega & \cos\Omega\cos I & -\cos\Omega\sin I \\ 0 & \sin I & \cos I \end{bmatrix} \times \begin{bmatrix} -d_{*c}\cos(\theta + \theta'_{*c}) \\ -d_{*c}\sin(\theta + \theta'_{*c}) \\ z_{*c} \end{bmatrix}. \quad (C18)$$

We can invert the above matrix (determinant is 1 so it is non-singular) and solve for our three parameters:

$$\begin{bmatrix} -d_{*c}\cos(\theta + \theta'_{*c}) \\ -d_{*c}\sin(\theta + \theta'_{*c}) \\ z_{*c} \end{bmatrix} = \begin{bmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega\cos I & \cos\Omega\cos I & \sin I \\ \sin\Omega\sin I & -\cos\Omega\sin I & \cos I \end{bmatrix} \times \begin{bmatrix} X_{*c} \\ Y_{*c} \\ Z_{*c} \end{bmatrix} = \begin{bmatrix} l \\ m \\ n \end{bmatrix}. \quad (C19)$$

From this, we can see that

$$d_{*c} = \sqrt{l^2 + m^2}, \quad (C20)$$

and

$$\sin\theta' = \frac{l\sin\theta - m\cos\theta}{d_{*c}}. \quad (C21)$$

C2 Star cell velocity component due to spin of star

Now, we determine the velocity of the star cell due to the spin of the star $v_{*c, \text{sp}}$. We make the assumption that the stellar rotation is aligned to its orbit and in particular that there is no obliquity. We also assume that the observed rotation period of the star has been determined from photometry $P_{*, \text{obs}}$ and is related to the observed angular velocity $\Omega_{*, \text{obs}}$ as

$$\Omega_{*, \text{obs}} = \frac{2\pi}{P_{*, \text{obs}}}. \quad (C22)$$

The observed angular velocity is made up of two components due to the spin of the star and the orbit of the system:

$$\Omega_{*, \text{obs}} = \Omega_* + \Omega_{*, \text{orb}}. \quad (C23)$$

The magnitude of $\Omega_{*, \text{orb}}$ is known from the period of the exoplanet and star from the RV measurements, so we can determine the spin from the star Ω_* . We determine the spin axis unit vector by noting that it is in the same direction out of the plane of orbit of the two-body model centred on the planet. Thus, we have

$$\hat{\Omega}_* = \frac{\mathbf{r} \times \dot{\mathbf{r}}}{|\mathbf{r} \times \dot{\mathbf{r}}|}. \quad (C24)$$

Components of $\mathbf{r}$ and $\dot{\mathbf{r}}$ with respect to the planet are

$$x = a\cos(\theta + \pi) = -a\cos\theta, \quad (C25)$$

$$y = a\sin(\theta + \pi) = -a\sin\theta, \quad (C26)$$

$$z = 0. \quad (C27)$$

Transforming to viewpoint of the Earth in XYZ frame we have

$$\mathbf{r} = \begin{bmatrix} X_{*p} \\ Y_{*p} \\ Z_{*p} \end{bmatrix} = \begin{bmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos I & -\sin I \\ 0 & \sin I & \cos I \end{bmatrix} \times \begin{bmatrix} -a\cos\theta \\ -a\sin\theta \\ 0 \end{bmatrix}, \quad (C28)$$

$$\mathbf{r} = \begin{bmatrix} X_{*c} \\ Y_{*c} \\ Z_{*c} \end{bmatrix} = \begin{bmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -a\cos\theta \\ -a\sin\theta\cos I \\ -a\sin\theta\sin I \end{bmatrix}, \quad (C29)$$

$$\mathbf{r} = \begin{bmatrix} X_{*p} \\ Y_{*p} \\ Z_{*p} \end{bmatrix} = \begin{bmatrix} -a(\cos\Omega\cos\theta - \sin\Omega\sin\theta\cos I) \\ -a(\sin\Omega\cos\theta + \cos\Omega\sin\theta\cos I) \\ -a\sin\theta\sin I \end{bmatrix}. \quad (C30)$$

Differentiating, we get the velocity

$$\dot{\mathbf{r}} = \begin{bmatrix} \dot{X}_{*p} \\ \dot{Y}_{*p} \\ \dot{Z}_{*p} \end{bmatrix} = \begin{bmatrix} \dot{\theta}a(\cos\Omega\sin\theta + \sin\Omega\cos\theta\cos I) \\ \dot{\theta}a(\sin\Omega\sin\theta - \cos\Omega\cos\theta\cos I) \\ -\dot{\theta}a\cos\theta\sin I \end{bmatrix}. \quad (C31)$$

Now, the cross product is

$$\mathbf{r} \times \dot{\mathbf{r}} = a^2\dot{\theta} \begin{bmatrix} \sin I \sin\Omega \\ -\sin I \cos\Omega \\ \cos I \end{bmatrix}. \quad (C32)$$

Dividing out the leading term $a^2\dot{\theta}$, we get the unit vector for $\hat{\Omega}_*$ to be

$$\hat{\Omega}_* = \begin{bmatrix} \sin I \sin\Omega \\ -\sin I \cos\Omega \\ \cos I \end{bmatrix}. \quad (C33)$$

If we let $\mathbf{r}_{*c}$ denote the vector $\overrightarrow{O_*C_*}$ from the centre of the star to the star cell, then we have that the velocity of the star cell $\mathbf{v}_{*c}$ is just

$$\mathbf{v}_{*c, \text{sp}} = \Omega_* \times \mathbf{r}_{*c}. \quad (C34)$$

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