## Quasilocal conservation laws in cosmology: a first look\*

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Quasilocal definitions of stress-energy-momentum—that is, in the form of boundary densities (in lieu of local volume densities)—have proven generally very useful in formulating and applying conservation laws in general relativity. In this essay, we take a first basic look into applying these to cosmology, specifically using the Brown-York quasilocal stressenergy-momentum tensor for matter and gravity combined. We compute this tensor and present some simple results for a flat FLRW spacetime with a perfect fluid matter source.

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$$G_{ab} + \Lambda g_{ab} = 8\pi T_{ab} \,, \tag{1}$$

the typical interpretation given to the cosmological constant term follows by moving it to the RHS:  $G_{ab} = 8\pi (T_{ab} + T_{ab}^{(\Lambda)})$ , with  $T_{ab}^{(\Lambda)} = -\frac{1}{8\pi}\Lambda g_{ab}$  interpreted as playing the role of an effective local stress-energy-momentum of the "gravitational vacuum", with a constant local energy volume density  $\rho_{\Lambda} = T_{00}^{(\Lambda)}$  and equal but negative local pressure  $p_{\Lambda} = -\rho_{\Lambda}$ .

equation<sup>1</sup>,

Yet it has long been understood that, generally, gravitational energy-momentum cannot be treated fundamentally as a local concept in general relativity. There are many reasons for this, but the simplest explanation comes directly from the equivalence principle (see e.g. section 20.4 of [2]): if in any given locality one is free to transform to a frame of reference with a vanishing local "gravitational field" (connection coefficients), then any local definition of (changes in) the energy-momentum of that field would likewise vanish (even in situations where such changes are physically expected).

The problem of defining gravitational energy-momentum is a subtle one, and its resolution still lacks a general consensus among relativists today [3, 4]. It is nevertheless widely accepted that in the spatial infinity limit of an asymptotically-flat vacuum spacetime, any proposals for such definitions should recover the ADM definitions [5, 6]. For example, the ADM energy  $E_{ADM}$  of an asymptotically-flat vacuum spacetime is given by the integral over a closed two-surface  $\mathscr{S} \simeq \mathbb{S}_r^2$ (topologically a two-sphere  $\mathbb{S}_r^2$  of areal radius r) at spatial infinity ( $r \to \infty$ ) of an *energy surface density* (energy per unit area), given up to a factor by the trace k of the extrinsic curvature of that surface,

$$E_{\text{ADM}} = -\frac{1}{8\pi} \lim_{r \to \infty} \oint_{\mathscr{S}} \boldsymbol{\epsilon}_{\mathscr{S}} k \,. \tag{2}$$

The ADM definitions have proven widely useful in practice, but in principle are limited to determining the gravitational energy-momentum of an *entire* (asymptotically-flat vacuum) spacetime. Various current proposals exist [3, 4] for the gravitational energy-momentum of *arbitrary* spacetime regions within arbitrary spacetimes. These have generally retained the basic mathematical form (with an exact recovery in the appropriate limit) of the ADM definitions, i.e. that of closed two-surface integrals (of *surface densities*), and are for this reason referred to as *quasilocal* definitions.

In this essay, we assume and work with the quasilocal stress-energy-momentum tensor proposed by Brown and York [7]. Its definition can simply be motivated by the following argument [8]. Recall that the stress-energy-momentum tensor of matter *alone* is defined from the matter action  $S_{\text{matter}}$ , up to a factor, as  $T_{ab} \propto \delta S_{\text{matter}} / \delta g^{ab}$ , which is a local tensor (living in the bulk).

<sup>&</sup>lt;sup>1</sup> We work in the (-+++) signature of spacetime, in geometrized units (G = 1 = c), and follow the conventions of Wald [1]. In particular, Latin letters are used for abstract spacetime indices (a, b, c... = 0, 1, 2, 3). Furthermore we denote by  $\epsilon_{\mathscr{U}}$  the volume form of any manifold  $\mathscr{U}$ .

Following a similar logic, consider a total (matter plus gravitational) action,

$$S_{\text{total}} = S_{\text{matter}} + S_{\text{gravity}} \,. \tag{3}$$

The gravitational action  $S_{\text{gravity}}$  is, in any spacetime region  $\mathscr{V}$ —which for simplicity henceforth we take to be a worldtube<sup>2</sup>, i.e. the history of a finite spatial three-volume, see Fig. 1—as a sum,

$$S_{\text{gravity}} = S_{\text{EH}} + S_{\text{GHY}} \,. \tag{4}$$

The first term is the Einstein-Hilbert (bulk) term,

$$S_{\rm EH} = \frac{1}{16\pi} \int_{\mathscr{V}} \boldsymbol{\epsilon}_{\mathscr{V}} R \,, \tag{5}$$

and the second is the Gibbons-Hawking-York (boundary) term,

$$S_{\rm GHY} = -\frac{1}{8\pi} \int_{\mathscr{B}} \boldsymbol{\epsilon}_{\mathscr{B}} K \,, \tag{6}$$

where K is the trace of the extrinsic curvature of the boundary  $\mathscr{B} = \partial \mathscr{V} \simeq \mathbb{R} \times \mathbb{S}^2$ . From the total action (3), a *total* stress-energy-momentum tensor  $\tau_{ab}$  can be defined (analogously to the matteronly  $T_{ab}$ ), as  $\tau_{ab} \propto \delta S_{\text{total}}/\delta g^{ab}$ . Assuming the Einstein equation (1) holds in  $\mathscr{V}$ , the bulk term in the functional derivative now vanishes, and the result evaluates to a tensor *living on the boundary*  $\mathscr{B}$ , i.e. a quasilocal tensor, known as the *Brown-York tensor*, and given (with the appropriate proportionality factor restored) by

$$\tau_{ab} = -\frac{1}{8\pi} \Pi_{ab} \,, \tag{7}$$

where  $\Pi_{ab}$  is the canonical momentum (defined in the usual way from the extrinsic curvature) of  $\mathscr{B}$ . This expresses boundary densities of the total (matter plus gravitational) energy-momentum which, when integrated over the two-surface intersection of a spacelike Cauchy slice and  $\mathscr{B}$ , yield the total values thereof contained in the part of the Cauchy slice (three-volume) within  $\mathscr{B}$ .

Conservation laws for energy, momentum and angular momentum using the Brown-York tensor have been formulated with the use of a concept called *quasilocal frames* [8, 10]. Essentially, the idea is that additional structure is required on  $\mathscr{B}$  in order to specify the components of stressenergy-momentum seen be a particular set of observers on  $\mathscr{B}$ . In particular, what is required is a two-parameter congruence with timelike observer four-velocity  $u^a \in T\mathscr{B}$ , the integral curves of which constitute  $\mathscr{B}$ . Such a pair  $(\mathscr{B}, u^a)$  is referred to as a quasilocal frame. See Fig. 2. We thus have, e.g., the following general expression for the quasilocal energy surface density (and similar expressions for the momentum and stress):

$$\mathcal{E} = \tau_{ab} u^a u^b = -\frac{1}{8\pi} k \,, \tag{8}$$

<sup>&</sup>lt;sup>2</sup> This assumption is made here only to simplify our discussion. For a full analysis for a completely arbitrary  $\mathcal{V}$ , see e.g. [9].



Figure 1. A (2+1) picture of a worldtube  $\mathscr{V}$  with boundary  $\mathscr{B} = \partial \mathscr{V}$ . A spatial slice of the latter is denoted by  $\mathscr{S}$  (a closed two-surface, topologically a two-sphere).

where k is the observers' (two-dimensional) spatial trace of the extrinsic curvature of  $\mathscr{B}$ . In the simplest case of  $u^a$  being orthogonal to a spatial slice  $\mathscr{S}$  of  $\mathscr{B}$  (which is sufficient for our purposes here), the total energy contained in the spatial volume inside  $\mathscr{S}$  is:

$$\mathbf{E} = \int_{\mathscr{S}} \boldsymbol{\epsilon}_{\mathscr{S}} \, \mathcal{E} \,. \tag{9}$$

The advantage of this construction is that it permits the computation of (changes in) these various quantities for any region  $\mathcal{V}$  on the boundary of which such a congruence (quasilocal frame) can be defined. For any small spatial region, that is, one contained inside a topological two-sphere having an areal radius r much smaller than the spacetime curvature and scale of matter density variation, the quasilocal energy density (8) evaluates in general to [8]:

$$\mathcal{E} = \mathcal{E}_{\text{vac}} + \mathcal{O}\left(r\right) \,, \tag{10}$$

where

$$\mathcal{E}_{\rm vac} = -\frac{1}{4\pi r} \tag{11}$$

is known as the *vacuum* quasilocal energy density. Matter contributions to  $\mathcal{E}$  begin possibly from  $\mathcal{O}(r)$  and gravitational contributions possibly from  $\mathcal{O}(r^3)$ . This vacuum energy is a geometrical term, simply accountable from the fact that the extrinsic curvature trace of a round two-sphere in flat space is k = 2/r, and it is often regarded in the literature as unphysical [3, 9]. Yet, analyses and applications of the quasilocal conservation laws have shown how this term is in fact needed for a proper accounting of gravitational energy-momentum transfer [8]. In particular, it is intimately linked to and logically self-consistent with the existence of a *vacuum pressure* (similarly,



Figure 2. A (2+1) picture of a quasilocal frame  $(\mathscr{B}, u^a)$ , where  $\mathscr{B} = \partial \mathscr{V}$  is the boundary of a worldtube  $\mathscr{V}$  and  $u^a$  is the timelike four-velocity of a two-parameter family of observers the integral curves of which constitute  $\mathscr{B}$ .

the leading term in an expansion in r of the quasilocal pressure P, defined from the observers' spatial trace of  $\tau_{ab}$ ),

$$P_{\rm vac} = -\frac{1}{8\pi r} \,. \tag{12}$$

Physically, a negative vacuum pressure acts as a positive surface tension on the boundary  $\mathscr{S}$ , resulting in a "P<sub>vac</sub>dA" work term, which exactly accounts for the change in negative vacuum energy. These vacuum terms have been shown to be necessary in applications, including recently in the gravitational self-force problem [11], where they play a key role in accounting for the perturbative correction to the motion of a point particle due to gravitational back-reaction.

Thus far, to our knowledge, these quasilocal quantities and their conservation laws have not been investigated in detail in the context of cosmology. In what follows, we present a basic application of these to a flat FLRW spacetime with a perfect-fluid matter  $T_{ab}$ . We begin by writing the metric  $g_{ab}$  in spherical coordinates  $\{X^a\} = \{T, R, \Theta, \Phi\}$  as

$$g_{ab} \mathrm{d}X^a \mathrm{d}X^b = -\mathrm{d}T^2 + a^2(T) \left[\mathrm{d}R^2 + R^2 \mathrm{d}\Omega^2\right],$$
 (13)

where  $d\Omega$  is the line element on the unit two-sphere with coordinates  $\{\Theta, \Phi\}$ . We transform this now into coordinates  $\{x^a\} = \{t, r, \theta, \phi\}$  adapted to a time-dependent spherically-symmetric quasilocal frame  $(\mathscr{B}, u^a)$ , such that  $\mathscr{B} = \{r = \text{const.}\}$ , by applying the transformation

$$\begin{cases} T = t, \\ R = rf(t), \\ \Theta = \theta, \\ \Phi = \phi, \end{cases}$$
(14)

where f(t) specifies the boundary time-dependence. Thus  $\mathscr{S} = \mathbb{S}_r^2$  at any t = const. In these coordinates, for any a(t) and f(t), the four-velocity of the quasilocal observers is

$$u^{a} = \frac{1}{\sqrt{1 - a^{2} \dot{f}^{2} r^{2}}} \delta^{a}{}_{0}, \qquad (15)$$

and the quasilocal energy density (8), evaluates to

$$\mathcal{E} = -\frac{1 + a\dot{a}ffr^2}{4\pi a fr\sqrt{1 - a^2\dot{f}^2r^2}}\,.$$
(16)

Two cases of interest are when the quasilocal observers are *co-moving*, i.e. at a fixed coordinate radius (and so the spacetime physics is essentially encoded in their motion), and when they are *rigid*, i.e. at a fixed proper radius (and thus the physics is essentially encoded in the energy-momentum boundary fluxes), respectively given by the choices:

$$f_{\rm C}(t) = 1$$
, (17)

$$f_{\rm R}(t) = 1/a(t)$$
. (18)

Let us denote these quasilocal frames respectively as  $(\mathscr{B}_{C}, u_{C}^{a})$  and  $(\mathscr{B}_{R}, u_{R}^{a})$ . See Fig. 3. The co-moving observers see only the energy of the vacuum,

$$\mathcal{E}_{\rm C} = -\frac{1}{4\pi a r}\,,\tag{19}$$

while for rigid observers, the quasilocal energy is

$$\mathcal{E}_{\rm R} = -\frac{\sqrt{1 - H^2 r^2}}{4\pi r} \,, \tag{20}$$

where  $H = \dot{a}/a$ .

We can now calculate from these the total energies, given by (9). For the co-moving energy, we find the conservation expression  $\dot{E}_{C} = HE_{C}$ , telling us that this energy changes at an exponential rate of the integral of H(t). Moreover, the total rigid and co-moving energies are simply related by a time dilation factor,  $E_{R} = E_{C}\sqrt{1-v^{2}}$ , where v = -rH is the relative radial velocity of rigid quasilocal observers as seen by co-moving observers.



Figure 3. A (2 + 1) picture in a flat FLRW spacetime showing, in red, a co-moving quasilocal frame  $(\mathscr{B}_{\mathsf{C}}, u^a_{\mathsf{C}})$  (with the boundary at a fixed co-moving radius) and, in green, a rigid quasilocal frame  $(\mathscr{B}_{\mathsf{R}}, u^a_{\mathsf{R}})$  (with the boundary at a fixed proper radius). The black dotted line is the center of the spatial FLRW coordinates  $\{X^a\}$ , depicted (as a Cartesian system) on each Cauchy slice by dotted blue lines.

It is interesting to consider  $E_R$  for small r (in a "small locality"). Denoting the total (flat-space) vacuum energy as  $E_{vac} = \int_{\mathscr{S}} \epsilon_{\mathscr{S}} \mathcal{E}_{vac} = -r$ , this evaluates to  $E_R = E_{vac} + \frac{1}{2}H^2r^3 + \mathcal{O}(r^5)$ . Inserting the Friedmann equation  $H^2 = \frac{8\pi}{3}\rho + \frac{1}{3}\Lambda$ , this can be written as:

$$E_{\rm R} = E_{\rm vac} + (\rho + \rho_{\Lambda}) \frac{4\pi r^3}{3} + \mathcal{O}(r^5).$$
 (21)

Thus at leading order beyond the vacuum, we recover the usual mass of matter (as the three-ball volume multiplying the matter energy density  $\rho$ ), along with the cosmological constant term which can thus be seen to behave, locally, similarly to matter with a (volume) energy density  $\rho_{\Lambda} = \frac{\Lambda}{8\pi}$ . Concordantly, the quasilocal pressure after using both Friedmann equations has the expansion

$$P_{\mathbf{R}} = P_{\mathbf{vac}} - \frac{1}{2} \left( p + p_{\Lambda} \right) r + \mathcal{O}(r^3) , \qquad (22)$$

where  $p_{\Lambda} = -\frac{\Lambda}{8\pi} = -\rho_{\Lambda}$ , so the cosmological constant term also acts locally like matter with a pressure opposite in sign to its effective volume energy density.

These results offer a basic initial confirmation that these quasilocal methods yield reasonable results when applied to cosmology, and good encouragement to develop this application further. In particular, the perspective that may be gained upon the nature of the quasilocal vacuum energy and its relation to the cosmological constant warrants further conceptual investigation. While we have presented a simple computation in this essay to illustrate the proof of concept, we are working on

also including and deriving conservation laws for cosmological perturbations using these methods [12].

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