**Three- and four-term approximate expansions of the Haverkamp
formulation to estimate soil hydraulic properties from disc
infiltrometer measurements**

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Abstract

Soil sorptivity, S , and saturated hydraulic conductivity, K_s , can be estimated from the inverse analysis of a disc infiltrometer cumulative infiltration curve using the quasi-exact implicit (QEI) analytical Haverkamp et al. (1994) equation, which is a function of K_s , S , the parameters β and γ , the disc radius, r_d , and the water content increase, $\Delta\theta$. Given the complexity of solving QEI, this paper presents three-term, 3T, and four-term, 4T, QEI expansions to estimate S and K_s . The interplays between β , γ , K_s and S employing S - K_s , γ - K_s , γ - S , γ - β , S - β and K_s - β error maps generated for a loam soil synthetic infiltration were analyzed with QEI. To reduce the number of variables, $\Delta\theta$, r_d and γ were packed into the $A = \frac{\gamma}{r_d \Delta\theta}$ term. Five different QEI expansion analyses were evaluated: (i and ii) optimization of 3T using A , $3T_A$, or β , $3T_\beta$, as fixed variables; (iii and iv) optimization of 3T, $3T_{A\beta}$, and 4T, $4T_{A\beta}$, using constant β and A values; and (v) optimization of 4T leaving all coefficients free, $4T_F$. The different methods were evaluated with synthetic infiltrations for homogeneous sand, loam and silt soils, and for an uniform loam soil with a contact sand layer. The QEI expansions were applied to experimental infiltrations, and the results were compared to those obtained with QEI, using $\beta = 0.6$ and $\gamma = 0.75$. Infiltrations were performed at zero cm of soil tension. Global optimization with QEI demonstrates that γ and β are extremely linked. This makes it difficult to estimate these two parameters. All procedures applied to synthetic homogeneous soils resulted in good K_s and S approaches. However, only $3T_A$, $3T_{A\beta}$ and $4T_{A\beta}$ obtained accurate estimates of K_s and S when applied to a synthetic homogeneous soil with a contact sand layer. Application of the different methods to the experimental infiltrations showed that $3T_{A\beta}$ and $4T_{A\beta}$ gave the most accurate and robust estimates of S and K_s .

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47 *Keywords:* Sorptivity; Hydraulic conductivity; Infiltration; Disc radius

48

49 1. INTRODUCTION

50 The tension disc infiltrometer (Perroux and White, 1988) has become a very popular
51 device for in situ estimates of the soil surface hydraulic properties, which can be
52 calculated from the cumulative transient water flow. Among all available water
53 infiltration models (e.g. Warrick and Broadbridge, 1992; Zhang, 1998), we have the
54 quasi-exact implicit (QEI) formulation of Haverkamp et al. (1994), which gives robust
55 descriptions of the 3D unsaturated cumulative infiltration curves measured with a disc
56 infiltrometer (Lassabatere et al., 2009; Latorre et al., 2015, Fernandez-Galvez et al.,
57 2019). This model can be applied to any water infiltration with a fixed negative or null
58 water pressure head over a disc source and an initial water pressure head profile
59 (Haverkamp et al., 1994). Later, Ross et al. (1996) extended the Haverkamp model to
60 the case of a positive water pressure head at the surface. The 3D Haverkamp et al.
61 (1994) equation, which is defined by the hydraulic conductivity, K , sorptivity, S , the
62 disc radius, r_d , the initial and final volumetric water contents, θ_i and θ_f , the shape
63 parameter β and proportionality constant γ , is the sum of the one-dimensional (1D)
64 infiltration model plus the $\frac{\gamma S^2}{r_d(\theta_f - \theta_i)} t$ term (Smettem et al., 1994). While S describes the
65 capacity of a porous medium to absorb or desorb liquid (Philip, 1957), K defines the
66 soil's ability to transmit water when it is submitted to a hydraulic gradient (Dane and
67 Hopmans, 2002). β is a shape parameter (Haverkamp et al., 1994) and γ is a
68 proportionality constant corrected for the use of a simplified wetting front, S and gravity

assumptions (Smettem et al., 1994). Although for normal conditions γ and β values range from 0.6 to 0.8 (Haverkamp et al., 1994) and 0.3 to 1.7 (Lassabatere et al., 2009), respectively, a $\beta = 0.6$ and $\gamma = 0.75$ are commonly employed by default (Angulo-Jaramillo et al., 2000). However, the interrelationship between S , K , γ and β is still unclear and there is no consistent explanation for why γ and β should be used as constant values. More recently, Latorre et al. (2018) demonstrated that β has a small influence on 1D infiltration. Given that direct equations are much more convenient than the implicit and complex QEI formulation, Haverkamp et al. (1994) proposed a simplified two-term (2T) approximation for three-dimensional (3D) disc infiltrometer geometries, valid only for early to intermediate times. Although this expansion has been successfully used to estimate K and S (Vandervaere et al., 2000; Lassabatere et al., 2006; Moret-Fernández et al., 2013), the term for “short to medium infiltration times” defined for 2T is ambiguous and makes its applicability quite subjective (Rahmati et al., 2019). To overcome this restriction, Latorre et al. (2015) developed a method for estimating K and S from the inverse analysis of a disc infiltrometer infiltration curve using the QEI formulation, which makes it possible to analyze the whole infiltration time. Although this procedure was found to be robust and accurate, the numerical analysis employed to solve QEI was complex and time consuming. These limitations, however, were in part solved by Rahmati et al. (2019), who proposed a 1D three-term (3T) approximation derived from the asymptotic development of QEI. Unlike the 2T expansion, the larger infiltration time that this new expansion allowed resulted in a more robust and realistic estimation of K and S . It should be noted that this expansion does not address the steady state, and thus should be restricted to a finite time interval. In addition, the 3T approximation relies on using three terms, which means that potentially three parameters may be estimated. However, as already observed in the QEI

model (Latorre et al., 2018), analysis of the 1D 3T demonstrates that β does not have a large effect on the 1D infiltration curve (Rahmati et al., 2019); this makes estimating it quite tricky. Although this 1D 3T expansion proved to be very efficient, its adaptation for 3D disc infiltrometer geometry has not yet been evaluated.

In addition to these mathematical aspects, experimental concerns also have a large impact on the estimates of the hydraulic parameters. For instance, the correct use of the tension disc infiltrometer requires the membrane of the disc base to be completely in contact with the soil surface. To achieve this contact, a thick layer of sand is usually placed between the bare soil surface and the disc base (Perroux and White, 1988). Although this procedure makes it possible to take infiltration measurements in most field situations, the water stored in the sand layer modifies the cumulative infiltration curve, which can have a large effect on the K and S estimates (Angulo-Jaramillo et al., 2000). In this case, the influence of the contact layer should be removed. Vandervaere et al. (2000) proposed removing the effect of the contact sand layer by applying a differentiated linearization method to the 2T expansion. However, Latorre et al. (2015) demonstrated that this method was very inaccurate when applied to “noisy” experimental infiltration curves. As an alternative, these authors used a layered flow model where water does not infiltrate into the soil until the sand layer is completely saturated. The sand layer effect is then omitted by finding the sand infiltration time and shifting the experimental data to the origin. Applying this method to the QEI model achieved very satisfactory results (Latorre et al., 2015), and solved the limitations found in the Vandevaere et al. (2000) procedure.

Given that a four-dimensional (S, K, β, γ) global optimization for 3D infiltration using QEI is complex and time consuming, it is necessary to find alternative, simpler models for accurately estimating the soil hydraulic properties from the best fit of a cumulative

infiltration curve measured with a disc infiltrometer. As a continuation of the work carried out by Rahmati et al. (2019), the objective of this paper is to present and evaluate the 3T and a new four-term (4T) QEI expansion to estimate S and K from 3D disc infiltrometer measurements. In order to understand the inter-relationship between all soil hydraulic properties, an initial theoretical sensitivity analysis of QEI for the S , K , β and γ variables was performed. As reported by Lassabatere et al. (2006), γ , the disc radius and $\Delta\theta$ were packed in a more global term, A , that relates 1D and 3D infiltrations curves. We evaluated five different procedures, which consisted in calculating S and K by optimizing 3T and 4T leaving all coefficients free or using fixed β and/or A values.. The 3T and 4T expansions were validated on infiltration curves generated for homogeneous synthetic soils with and without a contact sand layer. The results were confirmed using real experimental data from three experimental fields with different characteristics and soil managements.

2. THEORY

The 1D cumulative infiltration at saturation conditions can be described with the QEI model, I_{1D} , derived from the Richards equation (Haverkamp et al., 1994):

$$\frac{2(K_s - K_i)^2}{S^2} t = \frac{2}{1 - \beta} \frac{(K_s - K_i)(I_{1D} - K_i t)}{S^2} - \frac{1}{1 - \beta} \ln \left[\frac{1}{\beta} \exp(2\beta(K_s - K_i)(I_{1D} - K_i t)/S^2) + \frac{\beta - 1}{\beta} \right] \quad (1)$$

where t is time (T), K_s and K_i ($L \ T^{-1}$) are the hydraulic conductivity values corresponding to the saturated, θ_s , and initial, θ_i , volumetric water content ($L^{-3} \ L^{-3}$), respectively; S ($L \ T^{-0.5}$) is the sorptivity; and β is an integral shape parameter. Smettem et al. (1994) extended the 1D QEI model to 3D water infiltration through a disc source as

$$I_{3D} = I_{1D} + \frac{\gamma S^2}{r_d(\theta_s - \theta_i)} t \quad (2a)$$

$$I_{3D} = I_{1D} + A t \quad (2b)$$

$$A = \frac{\gamma S^2}{r_d(\theta_s - \theta_i)} \quad (2c)$$

where r_d is the radius of the disc (L) and γ is a proportionality constant corrected for the use of a simplified wetting front, sorptivity and gravity assumptions. The parameter A , which combines γ , the radius of the source and the difference in water content, $\Delta\theta$, quantifies the contribution of the additional lateral cumulative infiltration (Lassabatere et al., 2006). For regular working conditions, γ varies between 0.6 and 0.8 (Haverkamp et al., 1994) and β between 0.3 and 1.7 (Lassabatere et al., 2009).

As reported by Haverkamp et al. (1994), the cumulative 1D infiltration curve can be written as a power series in $t^{1/2}$, as

$$I_{1D_nT} = c_1 t^{\frac{1}{2}} + c_2 t + c_3 t^{\frac{3}{2}} + c_4 t^2 + c_5 t^{\frac{5}{2}} + \dots \quad (3)$$

where c_i ($i=1:n$) ($L T^{-i/2}$) are coefficients that depend on the soil hydraulic properties and the initial conditions (Angulo-Jaramillo et al., 2016). Given that we have four unknown parameters (S , K_s , β and A), it is only necessary to have a power series up to t^2 for parameter predictions. Based on the work in Rahmati et al. (2019) on 1D infiltrations, and thus considering that the contribution of the initial hydraulic conductivity can be neglected, i.e., $\Delta K \approx K_s$, application of the Taylor series to the QEI formulation up to the fourth order in powers of $t^{0.5}$ results in

$$I_{1D_4T}(t) = S t^{\frac{1}{2}} + \frac{2-\beta}{3} K_s t + \frac{K_s^2}{9S} (\beta^2 - \beta + 1) t^{\frac{3}{2}} + 2(\beta - 2)(\beta + 1) \frac{(1-2\beta) K_s^3}{135 S^2} t^2 \quad (4)$$

Combining Eqs. (2b) and (4), the 3D cumulative infiltration is

$$I_{3D_4T}(c_1, c_2, c_3, c_4, t) = St^{\frac{1}{2}} + \left(\frac{2-\beta}{3}K_s + A\right)t + \frac{K_s^2}{9S}(\beta^2 - \beta + 1)t^{\frac{3}{2}} + 2(\beta - 2)(\beta + 1)\frac{(1-2\beta)K_s^3}{135S^2}t^2 \quad (5)$$

or

$$I_{3D_4T}(c_1, c_2, c_3, c_4, t) = c_1t^{\frac{1}{2}} + c_2t + c_3t^{\frac{3}{2}} + c_4t^2 \quad (6)$$

Similarly, the application of the Taylor series to the 3D QEI formulation up to the third order in powers of $t^{0.5}$ results in

$$I_{3D_3T}(t) = St^{\frac{1}{2}} + \left(\frac{2-\beta}{3}K_s + A\right)t + \frac{K_s^2}{9S}(\beta^2 - \beta + 1)t^{\frac{3}{2}} \quad (7)$$

or

$$I_{3D_3T}(c_1, c_2, c_3, t) = c_1t^{\frac{1}{2}} + c_2t + c_3t^{\frac{3}{2}} \quad (8)$$

The input variables, S , K_s , β and A , are retrieved from the c_i coefficients, whose values are calculated from the best fit between QEI expansions and the experimental data. The c_i coefficients involved in Eqs. (6) and (8) can be expressed as

$$c_1 = S \quad (9)$$

$$c_2 = \frac{2-\beta}{3}K_s + A \quad (10)$$

$$c_3 = \frac{K_s^2}{9c_1}(\beta^2 - \beta + 1) \quad (11)$$

$$c_4 = 2(\beta - 2)(\beta + 1)\frac{(1-2\beta)K_s^3}{135c_1^2} \quad (12)$$

If β and/or A are considered as fixed and known values, the c_3 and c_4 coefficients should be expressed as

$$c_3 = \frac{1}{9c_1}(\beta^2 - \beta + 1)\left((c_2 - Ac_1^2)\frac{3}{2-\beta}\right)^2 \quad (13)$$

$$c_4 = 2(\beta - 2)(\beta + 1) \frac{(1-2\beta) \left((c_2 - Ac_1^2)^{\frac{3}{2-\beta}} \right)^3}{135c_1^2} \quad (14)$$

3. MATERIAL AND METHODS

3.1. Synthetic soils

Synthetic infiltration experiments were simulated with the HYDRUS-3D model (Šimunek et al., 1999). Sand, loam and silt soils were considered (Table 1). We used the van Genuchten (1980) model for water retention curves and unsaturated hydraulic conductivity functions. The soil volume was discretized as a cylinder (25 cm radius and depth), covering the axisymmetric plane with a 2-D rectangular mesh of 100 x 900 cells. A disc infiltrometer with a 10 cm radius base was used as a constant pressure head boundary on the corresponding top cells. A soil tension of -0.001 cm was defined in the radius disc base, and the null pressure head was considered as the bottom boundary. The initial soil water content corresponded to the residual water content. The synthetic data were computed for time datasets ranging from 300 s for sand to 10000 s for silt (Table 1). According to Rahmati et al. (2019), these values are within the infiltration time for which 3T is valid. Indeed, the 4T approximation is valid over larger times than the approximation for 3T, 2T or 1T. More details about the cumulative infiltration curve simulations can be found in Latorre et al. (2015).

The simulation was repeated for a homogenous loam soil with a contact sand layer with a thickness of 5 mm (with the same radius as the disc base) placed on the column surface. This new numerical data were used to evaluate the influence of the contact sand layer on the K_s and S estimates using 3T and 4T expansions. The hydraulic properties of the contact sand layer are shown in Table 1.

206

207 3.2. Sensitivity analysis of the QEI model

208 Considering $\Delta\theta$ and r_d as known values, the β , γ , K_s and S relationships were
209 analyzed using the 3D QEI model (Eq. 2a). The sensitivity analysis consisted in
210 minimizing the objective function, Q , which quantifies the difference between the
211 modeled $I(S, K_s, \beta, \gamma, t)$ and the experimental data (I_i):

$$212 \quad Q(S, K, \beta, \gamma) = \sum_{i=1}^N [(I_i - I(S, K_s, \beta, \gamma, t_i)) \Delta t]^2 \quad (15)$$

213 The existence and the uniqueness of the solution were studied with Q error maps
214 generated for the S - K_s , γ - K_s , γ - S , γ - β , S - β and K_s - β planes, and a global optimization
215 (Pardalos and Romeijn, 2002) was employed. In each 2D-map, the two remaining
216 parameters were fixed to the known values. The response surfaces were calculated on a
217 rectangular grid with K_s from 0.001 to 0.1 mm s⁻¹, S from 0.1 to 1.2, β from 0.5 to 1.6
218 (Lassabatere et al., 2009) and γ from 0.6 to 0.8 (Haverkamp et al., 1994). More details
219 of the global analysis procedure can be found in Latorre et al. (2018).

220

221 3.3. Fitting of QEI expansions

222 The QEI approximate expansions (Eq. 3) were limited to the third (3T) (Eq. 8) and
223 fourth (4T) (Eq. 6) order in powers of $t^{0.5}$. The approximate expansions were fitted to
224 the synthetic or experimental curves, and the c_i coefficients were estimated using a
225 nonlinear (weighted) least-squares model implemented in the R software (R Foundation
226 for Statistical Computing, version 3.5.0). The accuracy of the fit was evaluated with the
227 RMSE according to

$$RMSE = \sqrt{\frac{\sum_i^t \Delta t (I_i - I_o)^2}{\sum (\Delta t - 1)}} \quad (16)$$

where I_o is the reference infiltration curve.

The weights (W_i) of each c_i coefficient (%) were calculated as follows

$$W = \frac{c_i t^{\frac{i}{2}}}{I_t} 100 \quad (17)$$

where I_t is the cumulative infiltration at time t . Since this paper is focused on analyzing transient infiltration curves measured with a disc infiltrometer, infiltration times of 500 s and 2000 s, commonly used in disc infiltrometer measurements, were selected for synthetic sand, loam and silt soils.

3.4. QEI expansion analysis for homogeneous synthetic soils

S and K_s were considered as target variables. Since $\Delta\theta$ and r_d can be subjected to experimental errors and γ is an unknown variable, $\Delta\theta$, r_d and γ were combined in the $A = \frac{\gamma}{r_d(\theta_f - \theta_i)}$ term (Lassabatere et al. 2006). This made it possible to reduce the number of unknown variables to four: K_s , S , β and A . However, considering that S is estimated directly from Eq. (9), only three unknown variables K_s , β and A could be estimated. The A term was calculated from the actual r_d and $\Delta\theta$ (Table 1) and an average γ of 0.75 (Haverkamp et al., 1994). The following analyses were considered:

(i and ii) Estimate of K_s with 3T, but using a fixed A , $3T_A$, or β , $3T_\beta$, respectively. In this case, the estimated c_2 and c_3 represent a system of two equations with three unknown variables. This means that A or β were fixed to a default value and then

the remaining parameters estimated, either (K_s, β) or (K_s, A) . While a β value of 0.6 and 1.6 was employed in $3T_\beta$, $A \pm 5\%$ was considered in $3T_A$.

(iii) Calculation of K_s with $3T$, but using constant β and A values, $3T_{A\beta}$. In this case, c_3 was defined according to Eq.(13), and K_s was estimated from c_2 . S and K_s were estimated for β equal to 0.6 and 1.6 and $A \pm 5\%$. This means there are four $\beta \times A$ combinations: $[0.6, A + 5\%]$, $[0.6, A - 0.5\%]$, $[1.6, A + 5\%]$ and $[1.6, A - 5\%]$.

(iv) Determination of K_s with $4T$ and constant β and A values, $4T_{A\beta}$. c_3 and c_4 were defined with Eqs.(13) and (14), respectively, and K_s was estimated from c_2 . Four $\beta \times A$ combinations, $[0.6, A + 5\%]$, $[0.6, A - 0.5\%]$, $[1.6, A + 5\%]$ and $[1.6, A - 5\%]$, were considered

(v) Estimate of K_s , S , β and A with $4T$, but leaving free the fitting of all c_i coefficients, $4T_F$.

3.5. *QEI expansion analysis for synthetic soils with a contact sand layer*

Given that experimental infiltration with a disc infiltrometer commonly uses a contact sand layer between the soil surface and the base disc, special attention was paid to the effect of the sand layer on the infiltration curve analysis using the $3T_A$, $3T_\beta$, $3T_{A\beta}$, $4T_{A\beta}$ and $4T_F$ procedures. In all scenarios, the influence of the contact sand layer on the infiltration curve was detected and removed using the procedure of Latorre et al. (2015). This consists of a layered flow model that assumes that water does not infiltrate into the soil until the sand layer is completely saturated. The sand effect can be then considered as a time and volume delay before water infiltrates into the soil. The contact sand layer effect is removed by finding the sand infiltration time (and its corresponding water

volume) and shifting the experimental data to the origin. The infiltration time due to the sand layer was fixed between 0 and 10 s, and the theoretical analysis was performed on a synthetic cumulative infiltration curve generated for a loam soil plus a 5 mm thickness surface sand layer (see Section 3.1).

3.6. *Experimental sites*

The infiltration measurements were performed on three fields with different soils and managements. The first site corresponded to a cherry field (Ch) located in the municipality of Belver de Cinca (41°41' N, 0°10' E, 196 m alt.), in the Huesca province in Aragon (NE Spain). The average annual rainfall and air temperature are 361 mm and 14.6 °C, respectively. The soil is silt loam, with a low organic matter content (Table 2). The second site is located in the municipality of Poveda de la Sierra (40°38' N, 2°1' W, 1198 m alt.), in the Guadalajara (G) province in Castilla-La Mancha (Alto Tajo Natural Park, Central Spain). Mean annual precipitation and air temperature are 783 mm and 10 °C, respectively. This site is a constructed ecosystem after the reclamation of a kaolin quarry (El Machorro mine). The soil in the new topography consists in a 1 m depth layer of clay-loam textured geological materials coming from the Escucha cretacic formation of Albian age. This type of geological substratum is called overburden in mining reclamation terminology. The third site is located in the municipality of Mediana de Aragón (M1) (41°25'N, 0°44'W), in the province of Zaragoza in Aragón (NE Spain). Average annual temperature and precipitation are 14.9 °C and 350 mm yr⁻¹, respectively. The lithology in the area is mainly gypsum outcrops. Soils are Leptosols in the hills and Gypsisols in the flat-bottomed valleys (Navas, 1991). These are poorly developed soils, with high gypsum (> 600 g kg⁻¹) and low organic matter (< 15 g kg⁻¹)

content (Navas, 1991). Land use in the area is based on a traditional agro-pastoral system involving dry cereal croplands and extensive sheep production. All infiltrations were performed on bare soil with low (M1L), medium (M1M) and high (M1H) grazing intensity.

3.7. Field measurements

All soil samples for soil texture, chemical properties and organic matter content were taken from the 1 to 10 cm depth soil layer. The gypsum content was estimated from the sulphur content, calculated using a LECO 144DR elemental analyzer, and the soil texture was measured with the laser diffraction technique (COULTER LS230). The organic carbon was determined by an improved chromic-acid digestion and spectrophotometric procedure (Heanes, 1984) and the results were transformed to organic matter (OM) by multiplying by 1.724.

The soil dry bulk density (ρ_b) was determined with the core method (Grossman and Reinsch, 2002), using cores of 50 mm in diameter and height. To determine the dry weight of the soil, the samples were dried at 105 °C for 24 h. Soil with high gypsum content was dried at 50 °C during 48 h. One replication was performed per sampling site. This sampling was also used to determine the initial volumetric water content (θ_i). The cumulative infiltration curves were measured with a 10 cm diameter Perroux and White (1988) model tension disc infiltrometer. A thin layer (< 1 cm thickness) of commercial sand (80–160 μ m grain size) was placed between the base disc and soil surface. A zero cm pressure head was applied to the base disc. The water infiltration was monitored with a ± 35.2 cm differential pressure transducer (Microswitch, Honeywell) at 1 s time intervals. Infiltration measurements continued between 500 and

1800 s (Table 2). At the end of the experiment and after removing the sand layer, a saturated soil sample was taken under the base disc. θ_s was calculated as the product between saturated gravimetric water content and ρ_b . A total of 15 infiltrations were measured (Table 2): 3 replications in Ch and G and 9 replications in M1. The influence of the contact sand layer on the K_s and S estimates was removed according to the Latorre et al. (2015) procedure.

The K_s and S estimated with $3T_A$, $3T_\beta$, $3T_{A\beta}$, $4T_{A\beta}$ and $4T_F$ were compared with those obtained with 3D QEI, using β and γ equal to 0.6 and 0.75, respectively (Latorre et al., 2015), and the actual disc radius and measured $\Delta\theta$ (Table 2).

4. RESULTS

4.1. Synthetic soils

The 3D QEI sensitivity analysis performed on the infiltration curve for a synthetic loam soil showed error maps with a unique minimum in all cases (Fig. 1). However, while the K_s - S plane had concentric pit-shaped contour lines, the other maps (γ - K_s , γ - S , γ - β , S - β and β - K_s) showed valley shapes with a very slight slope, which makes it difficult to locate the actual minimum. Overall, S and K_s predictions were affected very little by β and γ . However, while K_s and S slightly increased with β (see S - β and K_s - β subpanels) these same parameters decreased with increasing γ (see S - γ and K_s - γ subpanels).

The weights of the c_i coefficients (Eq. 17) calculated with 3T and 4T for the infiltration curves generated for synthetic sand, loam and silt soils increased with infiltration time, but decreased with the coefficient series order (Fig. 2). Similarly, as observed in QEI,

the relationship between A , β , and K_s obtained with $3T_A$ or $3T_\beta$ on a 5000 s infiltration generated for a synthetic loam soil showed that A had a large influence on β and vice versa (Fig. 3a). This means that large variations of β led to large changes in A . A completely different behavior was observed for the relationship of A and β vs K_s , where A and β showed less influence on K_s predictions (Fig. 3b and c). The estimated K_s varied from 2.26×10^{-3} to 4.11×10^{-3} mm s⁻¹ ($1.54 < Error_{K_0} < 6.06\%$) (Fig. 3b) and from 2.7×10^{-3} to 3.9×10^{-3} mm s⁻¹ ($0.2 < Error_{K_s} < 7\%$) as a function of A and β , respectively (Fig. 3c).

Overall, satisfactory relationships were found between the theoretical K_s and S (Table 1) and the corresponding values estimated with the $3T_A$, $3T_\beta$, $3T_{A\beta}$, $4T_{A\beta}$ and $4T_F$ methods applied to the infiltration curves generated for synthetic sand, and loam and silt soils without a contact sand layer: $R^2 = 0.93$, $y = 1.129x + 2 \cdot 10^{-4}$, $p < 0.0001$ for K_s and $R^2 = 0.99$, $y = 0.987x + 0.01$, $p < 0.0001$ for S (Fig. 4a and b). Different behavior was observed for the β and A estimates, as large dispersions were found in $4T_F$, $3T_A$ and $3T_\beta$ (Fig. 4c and d).

Different behavior was also observed when the $3T_A$, $3T_\beta$, $3T_{A\beta}$, $4T_{A\beta}$ and $4T_F$ procedures together with the Latorre et al. (2015) method for removing the effect of the sand contact were applied to a synthetic loam soil that included a surface sand layer. In all cases, the minimum RMSE indicates the time at which cumulative infiltration due to the sand layer was completely removed (Fig. 5). Overall, the estimated hydraulic parameters and RMSE were affected by the additional cumulative infiltration promoted by the sand layer (Fig. 5), and the behavior of the hydraulic properties varied depending on the different 3T and 4T analyses. For instance, β and K_s estimated with $3T_A$ showed opposite behaviors, with extreme values at the earliest times (Fig. 5a). However K_s

tended to stabilize close to the $[0.6, 1.6]$ β interval (blue horizontal lines in Fig. 5a, around time = 1.5 s), which corresponds to the β range defined by Lassabatere et al. (2009) for soils. Reasonably accurate estimates of S and K_s were obtained with $3T_A$ (Table 3). Unlike $4T_F$, for which the contact sand layer had a large influence on the behavior of the hydraulic parameters (Fig. 5e), $3T_{A\beta}$ and $4T_{A\beta}$ seemed to be less affected by the sand layer (Fig. 5c and d). This means that $3T_{A\beta}$ and $4T_{A\beta}$ resulted in the most consistent estimates of S and K_s (Table 3). The minimum RMSE calculated by the different approaches followed the gradient $3T_{A\beta} > 4T_{A\beta} > 3T_A = 3T_\beta > 4T_F$ (Table 2). The S behavior also depended on the method employed. Thus, while S was observed to be quasi-constant in $3T_{A\beta}$ and $4T_{A\beta}$, the S values calculated with $3T_A$, $3T_\beta$ and $4T_F$ decreased with increasing time (Fig. 5).

4.2. Experimental soils

Table 2 and Figure 6 show the soil bulk density and the initial and final volumetric water content measured at the experimental sites, and the corresponding infiltration curves. Given the difficulties for estimating A and β from the inverse analysis of a single infiltration curve (see Section 3.1), only S and K_s estimates were considered in this section. The range of A and β employed in $3T_A$, $3T_\beta$, $3T_{A\beta}$ and $4T_{A\beta}$ were $[0.6, 1.6]$ and $A \pm 5\%$, respectively, where A values were calculated from the actual r_d and $\Delta\theta$ (Table 1) and an average $\gamma = 0.75$.

The comparison between $3T_A$, $3T_\beta$, $3T_{A\beta}$ and $4T_{A\beta}$ and the QEI model with constant A and β values showed that the $3T_{A\beta}$ and $4T_{A\beta}$ methods had, for both S and K_s , the best correlations with QEI (Fig. 7c and d). For the remaining methods, the gradient of correlations with QEI was $3T_A > 3T_\beta > 4T_F$. The differences between the average S and

K_s estimated with $3T_A$ and $3T_\beta$ and the corresponding values calculated with QEI were 3, 12, 34 and 13%, respectively. On average, S and K_s estimated with $4T_F$ were 31 and 15% larger than those obtained with QEI, respectively.

The differences between the average S and K_s estimated with $3T_{A\beta}$ and $4T_{A\beta}$ and those calculated with QEI were lower than 2% in all cases. Overall, the β and A intervals employed in the $3T_{A\beta}$ and $4T_{A\beta}$ had a small effect on S and K_s (Fig. 7c and d). No apparent differences were observed between $3T_{A\beta}$ and $4T_{A\beta}$ (Fig. 8c and d). A good correlation was also found between S and K_s estimated with $3T_{A\beta}$ and the corresponding values obtained with $4T_{A\beta}$ ($K_{s_4T_{A\beta}} = 0.9551 K_{s_3T_{A\beta}} + 0.0002$, $R^2 = 0.99$, $p < 0.0001$; $S_{s_3T_{A\beta}} = 1.0178 S_{s_4T_{A\beta}} + 0.002$, $R^2 = 0.99$, $p < 0.0001$).

5. DISCUSSION

5.1. Synthetic soils

The sensitivity analysis of the QEI model suggests that the inverse analysis of a long enough and perfect synthetic infiltration curve may in principle make it possible to estimate all Haverkamp et al. (1994) parameters. However, we should be careful about this statement, since the elongated and highly flattened valley shape observed in the γ - β error map indicates that γ and β are strongly linked. This could mean that during the inverse analysis these parameters compensate each other, which could make it difficult to estimate their exact values. As Latorre et al. (2018) found in 1D infiltration, S and K_s predictions were affected very little by β .

The small weight of the c_4 coefficient (Eq. 3) obtained from infiltration curves generated for synthetic sand, loam and silt (Fig. 2), with a value ranging between

0.002% and 0.194% for silt and sand at 500 s and 2000 s, respectively, suggests that a fourth power series in $t^{1/2}$ is enough for accurate estimates of K_s and S within a commonly employed infiltration time. The theoretical analysis performed on the infiltrations curves simulated for homogeneous synthetic soils without a contact sand layer indicated that, within a representative β and A interval, all $3T_A$, $3T_\beta$, $3T_{A\beta}$, $4T_{A\beta}$ and $4T_F$ procedures could be used to make consistent K_s and S estimates. The worst results obtained for β and A predictions (Fig.4c and d) are due to β and A being extremely linked, which is described in the QEI sensitivity analysis (Fig. 3a). In fact, the effects of β and A may compensate each other, allowing similar fits for different values. This is particularly evident regarding the expression $\left(\frac{2-\beta}{3}K_s + A\right)$ (Eq. 10). These results suggest that A and β cannot be simultaneously estimated from the inverse analysis of a single infiltration curve using the QEI expansions, and they should be used within its validity.

Application of the Latorre et al. (2015) method for removing the effect of the sand contact layer on all QEI expansions showed that the sand layer has a large effect on the RMSE and the hydraulic parameters (Vandervaere et al. 2000) (Fig. 5). Its effect was different depending on the QEI expansion. For instance, the unclear conduct observed in $4T_F$ (Fig. e5) can be explained because of the large flexibility of this model, which means that the best fitted c_i coefficients do not always correspond with their actual values. Note that the small weight c_4 (Fig. 2) can mean that little variations in the infiltration curve may affect this coefficient, and hence, the K_s estimate. The inaccurate K_s obtained with $3T_\beta$ is explained by small variations in c_3 within the $[0.6, 1.6]$ β interval that led to large variations in K_s (Eq. 11) (Table 3). These results indicate that $3T_\beta$ and $4T_F$ are not optimal for predicting K_s when a contact sand layer is applied to the soil surface. The robust S and K_s values obtained with $3T_{A\beta}$ and $4T_{A\beta}$ (Table 3) are in

accordance with Latorre et al. (2015), who found that accurate K_s values are obtained with QEI when constant β and A values are employed. Finally, compared to $3T_A$, $3T_\beta$ and $4T_F$, the quasi-constant S values obtained with $3T_{A\beta}$ and $4T_{A\beta}$ indicate that these last methods also obtain better estimates of S . The results indicate that $3T_{A\beta}$ and $4T_{A\beta}$, and in a second term $3T_A$, obtain the most realistic estimates of both K_s and S when applied to synthetic soils with a contact sand layer.

5.2. Experimental soils

The worst estimates obtained with $4T_F$ in the experimental infiltrations (Fig. 7e) can be attributed to the large flexibility of this model. This means that the c_i coefficients obtained from the best fitting do not always correspond with their required values. For instance, this is the case of G5 soil, for which the minimum RMSE was beyond the maximum infiltration time defined for the contact sand layer (10 s) (Fig. 8e). This resulted in unrealistic fittings, and hence, in erroneous estimates of the hydraulic properties.

As observed in the analysis of the synthetic soil plus contact sand layer, $3T_A$ allowed slightly better estimates of S and K_s than $3T_\beta$ (Fig. 7a and b). However, although $3T_A$ and $3T_\beta$ gave better results than $4T_F$, the three freedom degrees of these models resulted, in some cases, in unrealistic S and K_s estimates (Fig. 7). For instance, this is the case of G5 soil, for which the best fitting (or smaller RSME) with $3T_A$ resulted in an atypically high β ($\beta = 5.0$) (Fig. 8a) or an unrealistic negative A ($A = - 0.065$) when $3T_\beta$ was employed (Fig. 8b). In this example, both $3T_A$ and $3T_\beta$ were not suitable for estimating the soil hydraulic properties.

The most robust S and K_s estimations observed with both $3T_{A\beta}$ and $4T_{A\beta}$ (Fig. 7c and d) suggest that the QEI expansions require limiting the β and A parameters, because otherwise unrealistic S and K_s values can be obtained. These results are in accordance with Angulo-Jaramillo et al. (2000), who suggest using fixed β and γ , and also with Latorre et al. (2015), who observed that accurate K_s estimates are obtained when constant β and A are applied to the QEI model. The good correlation obtained between S and K_s estimated with $3T_{A\beta}$ and $4T_{A\beta}$ indicates that, for the employed infiltration time, both models were suitable for estimating the soil hydraulic properties. However, $4T_{A\beta}$ could be more appropriate if, for instance, much longer infiltration times were required. The contradictory results obtained between synthetic homogeneous and experimental soils could be probably explained because of the soil profiles corresponding to the experimental infiltration curves would not meet the soil homogeneity conditions required by the Richards model. Thus, in the cases where the soil homogeneity conditions cannot be ensured, constant β and A values are proposed to be employed.

6. CONCLUSIONS

Given the complexity of solving the quasi-analytical solution of Haverkamp et al. (1994), this work presents and evaluates alternative QEI approximate expansions (three-term, $3T$ and four-term, $4T$) for estimating S and K_s from disc infiltrometer measurements. Using the 3D QEI formulation, we analyzed the relationships among β , γ , K_s and S for synthetic loam soil infiltration curves. Five different analyses of QEI expansions were studied: optimization of $3T$ fixing A , $3T_A$, or β , $3T_\beta$; $3T$, $3T_{A\beta}$, and $4T$, $4T_{A\beta}$, with constant β and A values; and $4T$ leaving all coefficients free, $4T_F$. The methods were evaluated on synthetic infiltration curves with and without a contact sand

layer, and with experimental infiltration data. Global optimization with QEI demonstrated that inverse analysis of a cumulative infiltration curve only obtains accurate estimates of K_s and S . Overall, the application of the different 3T and 4T procedures to experimental data showed that 3T_{Aβ} and 4T_{Aβ} obtained the most robust estimates of S and K_s . These results suggest that realistic estimates of S and K_s are obtained when the number of independent variables is reduced to two. In conclusion, this work presents robust 3T and 4T procedures for estimating S and K_s . Compared to the 2-terms and QEI formulation, the new analyses make it possible to enlarge the infiltration time to be analyzed and simplify the calculus for estimating the hydraulic properties, respectively. However, more research into the influence of the infiltration time and the disc radius on the estimation of S and K_s is necessary.

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Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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Figures captions

Figure 1. Error maps for the S - K_s , γ - K_s , γ - S , γ - β , S - β and K_s - β planes calculated using the QEI model for a 2000 s 3D loam soil synthetic infiltration curve. The remaining parameters of each error map were fixed at the theoretical values. Red lines indicate the 0.02 mm contour line and the blue symbols denote the theoretical S , K_s , β and γ values (see Table 2).

Figure 2. Coefficient weights (W) of the 3T and 4T approximate expansions of QEI calculated for synthetic infiltration curves at 500 and 2000 s for sand, loam and silt soils.

Figure 3. Relationship between (a) A and β , (b) A and K_s and (c) β and K_s estimated with the 3T model, leaving the c_1 to c_3 coefficients free on a 5000 s infiltration curve generated with a 100 mm radius disc infiltrometer for a synthetic loam soil. The red lines denote the corresponding theoretical hydraulic parameters.

Figure 4. Relationship between reference theoretical (a) S , (b) K_s , (c) β and (a) A parameters and the corresponding values estimated by applying the 3T_A, 3T_β, 3T_{Aβ}, 4T_{Aβ} and 4T_F procedures to synthetic infiltration curves for sand, loam and silt soils, respectively. The bars indicate the variation of estimates when β or A are varied around their averaged values.

Figure 5. Time evolution of the soil hydraulic parameters after applying the (a) 3T_A, (b) 3T_β, (c) 3T_{Aβ}, (d) 4T_{Aβ} and (e) 4T_F procedures to a synthetic loam soil with a contact sand layer. K_{teor} denotes the theoretical hydraulic conductivity

Figure 6. Experimental infiltration curves (symbols) and the best fitting 3T model (black lines) after removing the cumulative infiltration due to the contact sand layer.

Figure 7. Relationship between S and K_s calculated from the experimental infiltration curves with QEI using constant $\beta = 0.6$, $\gamma = 0.75$, and the measured r_d and $\Delta\theta$ and the corresponding values calculated with the (a) 3T_A, (b) 3T_β, (c) 3T_{Aβ}, (d) 4T_{Aβ} and (e) 4T_F procedures. The bars indicate the variation of estimates when β or A are varied around their averaged values.

Figure 8. Time evolution of the soil hydraulic parameters after applying the (a) 3T_A, (b) 3T_β, (c) 3T_{Aβ}, (d) 4T_{Aβ} and (e) 4T_F procedures to the G_5 soil experimental infiltration curve for soil G5.

Figure 1. Error maps for the S - K_s , γ - K_s , γ - S , γ - β , S - β and K_s - β planes calculated using the QEI model for a 2000 s 3D loam soil synthetic infiltration curve. The remaining parameters of each error map were fixed at the theoretical values. Red lines indicates the 0.02 mm contour line and blue symbol denotes the theoretical S , K_s , β and γ values (see Table 2).

Figure 2. Coefficients weights (W) of the 3T and 4T approximate expansion of QEI calculated for synthetic infiltration curves at 500 and 2000 s for sand, loam and silt soils

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Figure 4. Relationship between reference theoretical (a) S , (b) K_s , (c) β and (a) A parameter and the corresponding values estimated by applying the 3T_A, 3T_b, 3T_{Aβ}, 4T_{Aβ} and 4T_F procedures to synthetic infiltration curves for a sand, loam and silt soils, respectively. The bars indicate the variation of estimates when β or A are varied around their averaged values.

Figure 6. Experimental infiltration curves (symbols) and the 3T model best fitting (black lines) after removing the cumulative infiltration due to the contact sand layer.

Figure 7. Relationship between S and K_s calculated from the experimental infiltration curves with QEI using constant $\beta = 0.6$, $\gamma = 0.75$, and the measured r_d and $\Delta\theta$ and the corresponding values calculated with (a) $3T_A$, (b) $3T_\beta$, (c) $3T_{A\beta}$, (d) $4T_{A\beta}$ and (e) $4T_F$ procedures. The bars indicate the variation of estimates when β or A are varied around their averaged values.

Table 1. Maximum infiltration time, t , disc radius, r_d , initial, θ_i , residual, θ_r , and final, θ_s , soil volumetric water contents, sorptivity, S , saturated hydraulic conductivity, K_s , α and n parameters of van Genuchten (1980) model, and β , γ and A (Eq. 9) parameters of synthetic sand, loam and silt soils.

Soil	t s	r_d mm	θ_s cm ³ cm ⁻³	θ_i, θ_r cm ³ cm ⁻³	α cm ⁻³	n	S mm s ^{-0.5}	K_s mm s ⁻¹	β	γ	A
Sand	300	100	0.43	0.045	0.145	2.68	1.521	0.08250	0.63	1.030	0.0267
Loam	5000	100	0.43	0.078	0.036	1.56	0.367	0.00288	1.25	0.756	0.0215
Silt	10000	100	0.46	0.034	0.016	1.37	0.238	0.00069	1.56	0.748	0.0176

Table 2. Maximum infiltration time, t , disc radius, r_d , initial, θ_i , and final, θ_s , soil volumetric water contents, organic matter, OM, and gypsum content, soil bulk density, ρ_b , textural characteristics and soil management.

#soil	Soil	t s	r_d mm	θ_i	θ_s	ρ_b	OM	$Gypsum$	$Sand$	$Silt$	$Clay$	$Soil\ use$
				cm ³ cm ⁻³	cm ³ cm ⁻³	cm ³ cm ⁻³	g kg ⁻¹	g kg ⁻¹				
1	Ch_3	1200	50	0.15	0.42	1.32	31.5	373	271	572	157	Cherry field
2	Ch_7	1300	50	0.11	0.50	1.10	34.6	359	267	548	185	Cherry field
3	Ch_8	370	50	0.12	0.50	1.07	29.0	335	297	535	168	Cherry field
4	G_1	1800	50	0.05	0.38	1.65	5.0	-	647	228	125	Restored mine
5	G_3B	900	50	0.04	0.29	1.78	5.2	-	548	294	158	Restored mine
6	G_5	1400	50	0.05	0.56	1.76	4.8	-	540	331	129	Restored mine
7	M1H1R1	900	50	0.03	0.54	1.04	22.3	943	744	238	18	Highly grazed
8	M1H2R1	1200	50	0.01	0.52	1.18	13.7	724	678	251	71	Highly grazed
9	M1H5R1	900	50	0.01	0.49	1.15	8.9	684	648	279	73	Highly grazed
10	M1L1R1	1000	50	0.01	0.51	0.96	30.1	726	708	233	59	Ungrazed
11	M1L2R1	900	50	0.01	0.51	1.26	21.7	774	604	286	110	Ungrazed
12	M1L4R1	900	50	0.01	0.51	1.05	15.3	761	506	400	94	Ungrazed
13	M1M1R1	600	50	0.01	0.52	1.27	17.8	624	458	404	138	Moderate grazed
14	M1M2R1	900	50	0.01	0.52	1.03	12.6	888	422	462	117	Moderate grazed
15	M1M2R2	750	50	0.01	0.52	1.13	48.5	921	541	327	133	Moderate grazed

Table 3. Theoretical sorptivity, S , saturated hydraulic conductivity, K_s , β and A parameters and the corresponding values optimized with $3T_A$, $3T_\beta$, $3T_{A\beta}$, $4T_{A\beta}$ and $4T_F$ procedures on a 5000 s disc infiltrometer infiltration curve generated for a synthetic loam soil (see Table 1) with contact sand layer. t_{sand} denotes the estimated infiltration time of the contact sand layer. Within the square brackets, estimated maximum and minimum values for each β and A range.

	<i>Theoretical</i>	$3T_A$	$3T_\beta$	$3T_{A\beta}$	$4T_{A\beta}$	$4T_F$
S	0.367	0.3708	0.3708	0.3583	0.371	0.3599
K_s	0.00288	[0.00291, 0.00337]	[0.00333, 0.00506]	[0.00262, 0.00306]	0.00194	[0.00268, 0.00313]
β	1.27	[1.48, 1.71]	-	-	3.04	-
A	0.0215	-	[0.0074, 0.025]	-	0.028	-
t_{sand}	-	1.6	1.6	[1.5, 1.9]	[1.5, 2.0]	1.6
$RMSE$		0.0029	0.0029	[0.0053, 0.0227]	[0.0051, 0.0221]	0.0016