SOME CHARACTERIZATIONS OF c-PARACOMPACT AND c-COLLECTIONWISE NORMAL SPACES BY CONTINUOUS SELECTIONS

by

JUAN MARGALEF ROIG and ENRIQUE OUTERELO DOMINGUEZ

(Conclusión)

3. A CHARACTERIZATION OF THE *c*-COLLECTIONWISE NORMAL SPACES BY MEANS OF SELECTIONS

A topological space X is said to be *c-collectionwise normal,* where c is an infinite cardinal, if for every discret family $\{C_i\}_{i \in I}$ of closed subsets of X with card (I) $\leqslant c$, there exists a family $\{G_i\}_{i \in I}$ of pairwise disjoint open subsets of X such that $C_i \subset G_i$ for all $i \in I$.

It is well-known that the $\aleph_0$-collectionwise normal spaces are the normal spaces ([0], Thm. 1.12). Hence, every c-collectionwise normal space is normal.

LEMMA 8.—Let X be a topological space, c an infinite cardinal. The following are equivalent:

a) X is c-collectionwise normal.

b) For every discrete family $\{C_i\}_{i \in I}$ of closed subsets of X with By the hypothesis, there is a family $\{G'_i\}_{i \in I}$ of open subsets of X such that $C_i \subset G_i$ for all $i \in I$.

PROOF.—a) $\Longrightarrow$ b).

By the hypothesis, there is a family $\{G'_i\}_{i \in I}$ of open subsets of X such that $C_i \subset G'_i$ for all $i \in I$ and $G'_i \cap G'_j = \varnothing$ for all $i, j \in I$ with $i \neq j$.

Now $C = \bigcup_{i \in I} C_i$ is a closed subset of X since $\{C_i\}_{i \in I}$ is a locally finite family of closed subsets of X. So $C \subset \bigcup_{i \in I} G'_i$ and as X is normal, there exists an open $A \subset X$ such that.

$$C \subset A \subset \bar{A} \subset \bigcup_{i \in I} G'_i$$

It is enough to take $G_i = G'_i \cap A$ for all $i \in I$.

b) $\Longrightarrow$ a).

Trivial.

LEMMA 9.—Let F be a closed subset of a c-collectionwise normal space X and $\mathcal{U} = \{U_i \mid i \in I\}$ a locally finite open cover of F such that card (I) $\leqslant c$. Then there exists a locally finite open cover

$$\mathcal{V} = \{V_i \mid i \in I\}$$

of X such that $F \cap V_i \subset U_i$ for all $i \in I$.

PROOF.—As F is closed in X, F is normal. Hence there exists $\phi = \{\phi_i\}_{i \in I}$ continuous partition of the unity subordinated to $\mathcal{U}$. Then $\mathcal{W} = \{W_i\}_{i \in I}$, where $W_i = \sup_A \phi_i$ for all $i \in I$, is an open cover of F. For every $i \in I$, W_i is an F_σ in F and then in X. Also $W_i \subset U_i$.

Let $\leqslant$ be a well-ordering of I. For every $i \in I$, let

$$C_i = W_i - \bigcup_{j < i} W_j.$$

We have:

a) For all $i \in I$, C_i is an F_σ in F, and therefore in X.

b) $\mathcal{C} = \{C_i\}_{i \in I}$ is a cover of F.

c) For all $i, j \in I$, $i \neq j$, $C_i \cap C_j = \varnothing$.

For each $i \in I$, by a),

$$C_i = \bigcup_{n \in N} C_i^n$$

where for every $n \in N$, C_i^n is closed in X.

It follows from c) that $\{C_i^n \mid i \in I\}$ is a discrete family of closed subsets of X, for all $n \in N$, since $C_i^n \subset C_i \subset U_i$ and $\mathcal{U}$ is locally finite in F. By the above lemma and X being *c*-collectionwise normal, for every $n \in N$ there is a discrete family of open subsets of X $\{G_i^n \mid i \in I\}$ such that $C_i^n \subset G_i^n$. Then

$$C_i^n \subset G_i^n \cap (X - (F - U_i)).$$

So, by the Urysohn lemma, there exists an open and F_σ set in X, M_i^n such that

$$C'_i \subset M_i^n \subset G_i^n \cap (X - (F - U_i))$$

for all $i \in I$ and $n \in N$.

Let

$$M = \bigcup_{\substack{i \in I \\ n \in N}} M_i.$$

It is clear that M is an open subset of X which includes F. So, by the Urysohn lemma again, there is an open and F_σ set M_0 in X such that

$$X - M \subset M_0 \subset X - F.$$

Let i_0 be the first member of $(I, \leqslant)$. We consider

$$P_{i_0}^1 = M_0 \cup M_{i_0}^1 \quad \text{and} \quad P_i^n = M_i^n \quad \text{if} \quad (i, n) \neq (i_0, 1).$$

Then $\{P_i^n \mid i \in I, n \in N\}$ is an open cover of X, its elements are F_σ and such that

$$C_i^n \subset P_i^n \subset X - (F - U_i).$$

Also $\{P_i^n \mid i \in I\}$ is locally finite in X for all $n \in N$. For every $n \in N$, consider

$$P_n = \bigcup_{i \in I} P_i^n.$$

Then $\{P_n \mid n \in N\}$ is an open cover of X and P_n is F_σ for all $n \in N$. So $\{P_n \mid n \in N\}$ has a locally finite open refinement $\{Q_n \mid n \in N\}$.

Let $V_i^n = P_i^n \cap Q_n$ and

$$V_i = \bigcup_{n \in N} V_i^n.$$

Then:

1) $\mathcal{V} = \{V_i \mid i \in I\}$ is an open cover of X.

2)

$$F \cap V_i \subset F \cap \left(\bigcup_{m \in N} P_i^m\right) \subset F \cap (X - (F - U_i)) \subset U_i.$$

3) $\mathcal{V}$ is locally finite.

Lemma 10 ([12]).—Let X be a normal space and $\mathcal{U} = \{U_n \mid n \in N\}$ a pointwise finite open cover of X. Then $\mathcal{U}$ admits a locally finite open refinement

$$\mathcal{V} = \{V_n \mid n \in N\} \quad \text{such that} \quad V_n \subset U_n \quad \text{for all} \quad n \in N.$$

Lemma 11. — Let X be a c-collectionwise normal space and $\mathcal{U} = \{U_i\}_{i \in I}$ a pointwise finite open cover of X with card (I) $\leqslant c$. Then $\mathcal{U}$ has a locally finite open refinement of cardinal less or equal than c.

Proof.—By induction we will construct a sequence

$$\{\mathcal{W}_n \mid n \in N \cup \{0\}\}$$

of families of open subsets of X, verifying:

a) For each $n \in N \cup \{0\}$ and each $W \in \mathcal{W}_n$, there exists $U \in \mathcal{U}$ with $W \subset U$.

b) For each $n \in N \cup \{0\}$, $\mathcal{W}_n$ is locally finite and of cardinal $\leqslant c$.

c) If x belongs at most to n members of $\mathcal{U}$,

$$x \in \bigcup \{W \mid W \in \mathcal{W}_k, \quad 0 \leq k \leq n\}.$$

d) For each $n \in N \cup \{0\}$, every

$$x \in \bigcup \{W \mid W \in \mathcal{W}_n\}$$

belongs at least to n elements of $\mathcal{U}$. Let $\mathcal{W}_0 = \{\emptyset\}$.

Suppose that we have obtained $\mathcal{W}_1, \ldots, \mathcal{W}_m$ with the conditions a), b), c) and d). We will construct $\mathcal{W}_{m+1}$. Let

$$\mathcal{H} = \{\mathcal{U}' \subset \mathcal{U} \mid \quad \text{card} \quad (\mathcal{U}') = m + 1\}.$$

It is clear than card $(\mathcal{H}) \leqslant c$. For each $\mathcal{U}' \in \mathcal{H}$, let

$$A(\mathcal{U}') = (X - \bigcup\{W \mid W \in \mathcal{W}_k, 0 \leq k \leq m\}) \cap (X - \bigcup\{U \mid U \in \mathcal{U} - \mathcal{U}'\}).$$

$A(\mathcal{U}')$ is closed and the family

$$\{A(\mathcal{U}') \mid \mathcal{U}' \in \mathcal{H}\}$$

is discrete.

Applying the lemma 8 to the above family we see that there is a discrete family of open subsets of X,

$$\{V(\mathcal{U}') \mid \mathcal{U}' \in \mathcal{H}\}$$

such that

$$A(\mathcal{U}') \subset V(\mathcal{U}')$$

for all $\mathcal{U}' \in \mathcal{H}$. Also $A(\mathcal{U}') \subset U$ for all $U \in \mathcal{U}'$.

For each $\mathcal{U}' \in \mathcal{H}$, let

$$P(\mathcal{U}') = V(\mathcal{U}') \cap \{\cap \{V \mid V \in \mathcal{U}'\}\}.$$

It is clear that $P(\mathcal{U}')$ is open and $A(\mathcal{U}') \subset P(\mathcal{U}')$ for all $\mathcal{U}' \in \mathcal{H}$.

Consider

$$\mathcal{W}_{m+1} = \{P(\mathcal{U}') \mid \mathcal{U}' \in \mathcal{H}\}.$$

One can check that $\mathcal{W}_0, \ldots, \mathcal{W}_{m+1}$ verify a), b), c) and d) and the construction of the sequence $\{\mathcal{W}_n\}_{n \in N \cup \{0\}}$ is complete.

For each $n \in N \cup \{0\}$, let

$$W_n = \bigcup_{W \in \mathcal{W}_n} W.$$

$\{\mathcal{W}_n \mid n \in N \cup \{0\}\}$ is a pointwise finite open cover of the normal space X. So by the above lemma, there exists $\mathcal{V} = \{V_n\}_{n \in N \cup \{0\}}$ locally finite open refinement of $\{W_n \mid n \in N \cup \{0\}\}$ such that $V_n \subset W_n$ for all $n \in N \cup \{0\}$. Then, by a) and b)

$$\{V_n \cap W \mid W \in \mathcal{W}_n, \quad n \in N \cup \{0\}\}$$

is a locally finite open refinement of $\mathcal{U}$ of cardinal $\leqslant c$.

LEMMA 12.—Let X be a c-collectionwise normal topological space, Y a metrizable real or complex topological vector space with separability degree $\leqslant c$, M a non-empty convex subset of Y, and $\psi\colon X \longrightarrow 2^Y$ lower-semicontinuous such that $\psi(x) \subset M$ for all $x \in X$.

Suppose that either $\psi(x)$ is convex relatively compact in Y or $\psi(x) = M$ for all $x \in X$. Let V be a convex open neighborhood of O in Y and f a continuous map of X into Y with $f(X) \subset M$. Suppose that

$$\theta(x) = \psi(x) \cap (f(x) + V) \neq \emptyset$$

for all $x \in X$.

Then, given any convex open neighborhood W of O in Y, there exists a continuous map g of X into Y such that $g(x) \in \theta(x) + W$ for all $x \in X$, and $g(X) \subset \bar{M}$.

PROOF.—Let U be an convex symmetric open neighborhood of O in Y such that $U + U \subset W$. Consider

$$G = \{x \in X \mid f(x) \in \theta(x) + U\} = \{x \in X \mid \theta(x) \cap (f(x) - U) \neq \emptyset\}.$$

By lemma 1 applied to

$$\{(x, y) \mid x - y \in V\}$$

ψ and f, we have that θ is lower-semicontinuous. Hence G is open. In fact: let $x_0 \in G$. Then there exists

$$z_0 \in \theta(x_0) \cap (f(x_0) - U)$$

and

$$z_1 = z_0 - f(x_0) \in U$$

So there are an symmetric open neighborhood U_1 of O and an open neighborhood V^{z_1} of z_1 such that

$$U_1 + V^{z_1} \subset U.$$

By the continuity of f and the lower-semicontinuity of θ there is an neighborhood V^{x_0} of x_0 such that

$$f(V^{x_0}) \subset f(x_0) - U_1$$

and

$$\theta(x') \cap (V^{z_1} + f(x_0)) \neq \varnothing$$

for all $x' \in V^{x_0}$. So, for all $x' \in V^{x_0}$,

$$\theta(x') \cap (f(x') - U) \neq \varnothing$$

and hence $V^{x_0} \subset G$.

It is clear that $\mathcal{V} = \{y - U\}_{y \in \overline{M}}$ is an open cover of $\overline{M}$. But Y is paracompact and hence there exists a locally finite family $\mathcal{W} = \{W_j\}_{j \in J}$ in Y of open subsets of Y such that

$$\overline{M} \subset \bigcup_{j \in J} W_j,$$

and for every $j \in J$, $W_j \cap \overline{M} \neq \varnothing$ and there is $y_j \in \overline{M}$ with $W_j \subset y_j - U$. Y is metrizable and its separability degree is $\leqslant c$, then Y is c-Lindelöf and there is an $I \subset J$, card (I) $\leqslant c$ and

$$\overline{M} \subset \bigcup_{i \in I} W_i.$$

For every $i \in I$, consider

$$U_i = \{x \in X \mid \theta(x) \cap W_i \neq \varnothing\}.$$

As θ is lower-semicontinuous, U_i is an open subset of X for all $i \in I$.

Let $\mathcal{U} = \{U_i \mid i \in I\}$. It holds that $\mathcal{U}$ is an open cover of X. In fact, let $x \in X$. From $\theta(x) \neq \varnothing$ there exists $y \in \theta(x) \subset M$, and therefore there exists $i \in I$ such that $y \in W_i$. So, $x \in U_i$.

Consider the closed subset of X, $A = X - G$. Now then open cover

$$\mathcal{H} = \{A \cap U_i \mid i \in I\}$$

of A of cardinality less or equal than c, is pointwise finite: If $x \in A$ then

$$f(x) \notin \theta(x) + U = [\psi(x) \cap (f(x) + V] + U$$

wich implies

$$\psi(x) \neq M \; (f(x) \in M \quad \text{for all} \quad x \in X).$$

So $\psi(x)$ is relatively compact in Y and hence $\overline{\theta(x)}$ is compact for all $x \in A$. On the other hand $x \in U_i$ if and only if

$$\theta(x) \cap W_i \neq \emptyset.$$

So $\mathcal{H}$ is pointwise finite since $\{W_i \mid i \in I\}$ is locally finite.

A is a closed in the c-collectionwise normal space X, so A is c-collectionwise normal. By lemma 11, there is a locally finite open refinement $\{H_\lambda\}_{\lambda \in \Lambda}$ of $\mathcal{H}$ in A with card $(\Lambda) \leqslant c$. By lemma 9 there is a locally finite open cover $\{A_\lambda \mid \lambda \in \Lambda\}$ of X such that $A_\lambda \cap A \subset H_\lambda$ for all $\lambda \in \Lambda$.

For every $\lambda \in \Lambda$, let $i_\lambda \in I$ such that $H_\lambda \subset U_{i_\lambda}$ and

$$B_\lambda = A_\lambda \cap U_{i_\lambda}.$$

Then

$$\mathcal{M}_1 = \{B_\lambda \mid \lambda \in \Lambda\}$$

is a locally finite family of open subsets of X which cover $X - G = A$.

On the other hand,

$$\mathcal{M}_2 = \{f^{-1}(W_i) \cap G \mid i \in I\}$$

is a locally finite family of open sets in X which cover G ($f(X) \subset M$) since $\{W_i \mid i \in I\}$ is a cover of M locally finite in Y. So

$$\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 = \{M_\gamma \mid \gamma \in \Gamma\}$$

where $\Gamma = \Lambda + I$ is a locally finite open cover of X. X is normal and so there is a locally finite continuous partition of the unity $\{h_\gamma \mid \gamma \in \Gamma\}$ subordinated to $\mathcal{M}$. If $\gamma = \lambda \in \Lambda$, let $y_\gamma \in \overline{M}$ such that

$$M_\gamma = B_\lambda \subset U_{i_\lambda} \quad \text{and} \quad W_{i_\lambda} \subset y_\gamma - U.$$

If $\gamma = i \in I$, let $y_\gamma \in \overline{M}$ such that $W_i \subset y_\gamma - U$.

Consider $g: X \longrightarrow Y$ defined by

$$g(x) = \sum_{\gamma \in \Gamma} h_\gamma(x) y_\gamma.$$

We have:

a) g is continuous, since $\{\sup(h_\gamma) \mid \gamma \in \Gamma\}$ is locally finite.

b) For every

$$x \in X, \quad g(x) \in \theta(x) + W.$$

In fact, let $x \in X$.

CASE 1.—$x \in G$.

The $h_\gamma(x) \neq \varnothing$ implies either

$$x \in M_\gamma = B_\lambda \subset U_{i_\lambda}$$

if

$$\gamma = \lambda \in \Lambda \quad \text{or} \quad x \in M_\gamma = f^{-1}(W_i) \cap G$$

if $\gamma = i \in I$.

In the case of

$$x \in B_\lambda \subset U_{i_\lambda} \quad \theta(x) \cap W_{i_\lambda} \neq \varnothing$$

and hence

$$\theta(x) \cap (y_\gamma - U) \neq \varnothing.$$

So

$$y_\gamma \in \theta(x) + U \subset \theta(x) + W.$$

If

$$x \in f^{-1}(W_i) \cap G, \quad f(x) \in W_i \subset y_\gamma - U$$

and

$$f(x) \in \theta(x) + \mathrm{U}.$$

So

$$y_\gamma = f(x) + u_1 = v + u_2 + u_1 \in \theta(x) + \mathrm{W} \quad (v \in \theta(x);\ u_1, u_2 \in \mathrm{U}).$$

In this way we conclude

$$g(x) \in \theta(x) + \mathrm{W}$$

since $\theta(x) + \mathrm{W}$ is convex and

$$\sum_{\gamma \in \Gamma} h_\gamma(x) = 1.$$

CASE 2.—$x \notin \mathrm{G}$.

Then $h_\gamma(x) \neq 0$ implies $x \in \mathrm{M}_\gamma$ where $\gamma = \lambda \in \Lambda$. Then

$$\theta(x) \cap \mathrm{W}_{i_j} \neq \varnothing$$

and hence

$$\theta(x) \cap (y_\gamma - \mathrm{U}) \neq \varnothing.$$

So

$$y_\gamma \in \theta(x) + \mathrm{U} \subset \theta(x) + \mathrm{W}.$$

We have

$$g(x) \in \theta(x) + \mathrm{W}$$

since $\theta(x) + \mathrm{W}$ is convex and

$$\sum_{\gamma \in \Gamma} h_\gamma(x) = 1.$$

c) $g(\mathrm{X}) \subset \overline{\mathrm{M}}$, since $y_\gamma \in \overline{\mathrm{M}}$ for all $\gamma \in \Gamma$ and

$$\sum_{\gamma \in \Gamma} h_\gamma = 1$$

($\overline{\mathrm{M}}$ is convex).

COROLLARY 13.—Let X be a c-collectionwise normal space, Y a metrizable real or complex topological vector space with separability degree $\leqslant c$, M a non-empty convex subset of Y, and $\psi: X \longrightarrow 2^Y$ lower-semicontinuous such that $\psi(x) \subset M$ for all $x \in X$. Suppose that either $\psi(x)$ is convex relatively compact in Y or $\psi(x) = M$ for all $x \in X$. Then given any convex open neighborhood W of O in Y, there is a continuous map g of X into Y such that

$$g(x) \in (\psi(x) + W)$$

for all $x \in X$ and $g(X) \subset M$.

PROOF.—It is enough to take in the lemma $V = Y$ and f a constant map of X into M.

THEOREM 14.—Let X be a topological space and c an infinite cardinal. The following are equivalent:

a) X is c-collectionwise normal.

b) For every set A of cardinality less or equal than c, every convex closed non-empty subset M of $l_1(A)$ and every lower-semicontinuous.

$$\phi : X \longrightarrow 2^{l_1(A)}$$

such that either $\phi(x)$ is a convex compact subset of M or $\phi(x) = M$ for all $x \in X$, it holds that ϕ admits a selection.

c) If Y is a real or complex Banach space with separability degree less or equal than c and M is a convex closed non-empty subset of Y, then every $\phi: X \longrightarrow 2^Y$ lower semicontinuous such that either $\phi(x)$ is a convex compact subset of M or $\phi(x) = M$ for all $x \in X$, admits a selection.

d) If Y is a real or complex Frechet space with separability degree less or equal than c, and M is a convex closed non-empty subset of Y, then every lower-semicontinuous $\phi: X \longrightarrow 2^Y$ such that either $\phi(x)$ is a convex compact subset of M or $\phi(x) = M$ for all $x \in M$, admits a selection.

e) If Y is a metrizable locally convex real or complex topological vector space of separability degree less or equal than c and M is a convex complete non-empty subset of Y, then every lower-semicontinuous $\phi: X \longrightarrow 2^Y$ such that either $\phi(x)$ is a convex compact subset of M or $\phi(x) = M$ for all $x \in X$, admits a selection.

PROOF.—a) $\Longrightarrow$ d).

As Y is a Frechet space, there exists a metric d on Y such that d describes the topology of Y and d is translations invariant. Let $\{V_n\}_{n \in N}$ a countable base of convex symmetric open neighborhoods of O in Y such that

$$V_{n+1} \subset \frac{1}{2} V_n$$

and

$$2 V_n \subset \left\{ y \in Y \mid d(y, 0) < \frac{1}{2^n} \right\} = B_{\frac{1}{2^n}}(0)$$

for all $n \in N$.

We will show by induction that there exists a sequence of continuous maps $\{f_n\}_{n \in N}$ of X into Y such that:

a) $f_n(x) \in f_{n-1}(x) + 2 V_{n-1}$ for all $x \in X$ and $n \in N - \{1\}$.

b) $f_n(x) \in \phi(x) + V_n$ for all $x \in X$ and $n \in N$.

c) $f_n(X) \subset M$ for all $n \in N$.

Let $n = 1$. By the above Corollary applied to M, ϕ and V_1, it holds that there is a continuous map $f_1: X \longrightarrow Y$ such that

$$f_1(x) \in \phi(x) + V_1$$

for all $x \in X$ and $f_1(X) \subset M$.

Suppose that we have $f_2, \ldots, f_n$ continuous maps of X into Y satisfying a), b) and c). By b),

$$f_n(x) \in \phi(x) + V_n$$

for all $x \in X$. Then

$$\phi(x) \cap (f_n(x) + V_n) \neq \emptyset$$

for all $x \in X$ since V_n is symmetric. So, by the Lemma 12 applied to M, ϕ, V_n, f_n and V_{n+1}, there is a continuous map f_{n+1} of X into Y such that

$$f_{n+1}(x) \in [\phi(x) \cap (f_n(x) + V_n)] + V_{n+1} \subset \phi(x) + V_{n+1}$$

for all $x \in X$ and $f_{n+1}(X) \subset M$. On the other hand

$$f_{n+1}(x) \in f_n(x) + V_n + V_{n+1} \subset f_n(x) + V_n + \frac{1}{2} V_n \subset f_n(x) + V_n + V_n \subset$$
$$\subset f_n(x) + 2 V_n.$$

This completes the construction of the sequence $\{f_n\}_{n \in N}$. Let $x \in X$. We show that $\{f_n(x)\}_{n \in N}$ is a Cauchy sequence in (Y, d). Let $\xi > 0$ and $n_0 \in N$ such that

$$\sum_{m=n_0}^{\infty} \frac{1}{2^m} < \xi.$$

The for every $m, n \geqslant n_0$ if $m = n + r$ $(r > 0)$, by a), we have

$$d(f_m(x), f_n(x)) \leq d(f_m(x), f_{m-1}(x)) + \ldots + d(f_{n+1}(x), f_n(x)) <$$
$$< \frac{1}{2^{m-1}} + \ldots + \frac{1}{2^n} < \xi.$$

So for every $x \in X$ there exists

$$\alpha(x) = \lim \{f_n(x)\}_{n \in N}$$

Furthermore the above proof shows that $\{f_n\}_{n \in N}$ converges uniformily to α. So α is a continuous map of X into Y.

On the other hand given $x \in X$, for every $n \in N$ there are $v_n \in V_n$ and $z_n \in \phi(x)$ such that

$$f_n(x) = z_n + v_n.$$

But

$$\lim \{v_n\}_{n \in N} = 0,$$

so

$$\alpha(x) = \lim \{z_n\}_{n \in N} \in \phi(x).$$

d) $\Longrightarrow$ c).

Consequence that every Banach space is a Frechet space.

c) $\Longrightarrow$ b).

This is a consequence of the fact that for every set A such that card (A) $\leqslant c$, l_1 (A) is a real Banach space with separability degree less or equal than c.

b) $\Longrightarrow$ a).

Let A be a set, card(A) $\leqslant c$. Let F be a closed in X and g a continuous map of F into l_1 (A). Consider the map

$$\phi : X \longrightarrow 2^{l_1(A)}$$

defined by

$$\phi(x) = \begin{cases} \{g(x)\} & \text{if} \quad x \in F \\ l_1(A) & \text{if} \quad x \notin F. \end{cases}$$

Then ϕ is lower-semicontinuous. By the hypothesis with M $= l_1$ (A), there exists a selection $\bar{g}$ of ϕ. It is clear that $\bar{g}$ is a continuous map of X into l_1 (A) such that $\bar{g}|_F = g$.

Let $\{C_i\}_{i \in I}$ be a discrete family of closed subsets of X such that card (I) $\leqslant c$. Consider the real Banach space l_1 (A), and the map:

$$g : \bigcup_{i \in I} C_i \longrightarrow l_1(I)$$

defined by

$$g(C_i) = \{x_i\}$$

where

$$x_i(j) = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

g is a continuous map of $\bigcup_{i \in I} C_i$ into l_1 (I) since $\{C_i\}_{i \in I}$ is locally finite. So, there exists a continuous map $\bar{g}$ of X into l_1 (I) such that

$$\bar{g} \,\Big|\, \bigcup_{i \in I} C_i = g.$$

For every $i \in I$, let

$$G_i = \bar{g}^{-1}(B_{\frac{1}{2}}(x_i))$$

The G_i is open and $C_i \subset G_i$.

Finally if $i, j \in I$ are such that $i \neq j$, then

$$G_i \cap G_j = \bar{g}^{-1} (B_{\frac{1}{2}} (x_i) \cap B_{\frac{1}{2}} (x_j)) = \emptyset .$$

This shows that X is c-collectionwise normal.

e) $\Longrightarrow$ d).

This is a consequence of the facts that every Frechet space is metrizable, complete and locally convex and that every closed subset of a complete space is complete.

d) $\Longrightarrow$ e).

Let $\mathcal{U}$ be the uniformity associated to Y and let $(\tilde{Y}, \tilde{\mathcal{U}})$ the T_2 complection of $(Y, \mathcal{U})$. The inmersion i of Y into $\tilde{Y}$ is a linear homeophorphism onto the image. Furthermore, as Y is an metrizable locally convex topological vector space with separability degree less or equal than c, $\tilde{Y}$ is a Frechet topological vector space ([10]), with separability degree less or equal than c. On the other hand, $\tilde{M} = i(M)$ is convex and complete. Hence $\tilde{M}$ is convex and closed in $\tilde{Y}$.

Finally $\tilde{\phi}(x) = i(\phi(x))$ is lower-semicontinuous and either $\tilde{\phi}(x)$ is a convex compact subset of $\tilde{M}$ or $\tilde{\phi}(x) = \tilde{M}$ for all $x \in X$. By hypothesis there exists a selection $\tilde{f}$ of $\tilde{\phi}$. It is clear thar $f = i^{-1} \circ \tilde{f}$ is a selection of ϕ.

Corollary 15.—Let X bet a topological space. The following are equivalent:

a) X is collectionwise normal.

b) For every set A, every closed convex non-empty subset M of $l_1(A)$ and every

$$\phi : X \longrightarrow 2^{l_1(A)}$$

lower-semicontinuous such that either $\phi(x)$ is a convex compact subset of M or $\phi(x) = M$ for all $x \in X$, it holds that ϕ has a selection ([14]).

c) If Y is a real or complex Banach space and M is a closed convex non-empty subset of Y, every ϕ: $X \longrightarrow 2^Y$ lower-semicontinuous such that either $\phi(x)$ is a convex compact subset of M or $\phi(x) = M$ for all $x \in X$, admits a selection ([14]).

d) If Y is a real or complex Frechet space and M is a closed convex and non-empty subset of Y, every ϕ: X $\longrightarrow 2^{Y}$ lower-semicontinuous such that either $\phi(x)$ is a convex compact subset of M or $\phi(x) = M$ for all $x \in X$, admits a selection.

e) If Y is a metrizable locally convex topological vector space and M is a convex complete non-empty subset of Y, every ϕ: X $\longrightarrow 2^{Y}$ lower-semicontinuous such that either $\phi(x)$ is a convex compact subset of M or $\phi(x) = M$ for all $x \in X$, admits a selection.

Corollary 16.—Every c-paracompact normal space is c-collectionwise normal.

Proof.—Consequence of Thm. 5 and 14.

The converse of Corollary 15 is not true: [23] provides an example of a collectionwise normal space non $\aleph_0$-paracompact.

Proposition 17.—Let X be a topological space and c an infinite cardinal. The following are equivalent:

a) X is c-collectionwise normal.

b) For every set A with card (A) $\leqslant c$, every closed C in X and every continuous map g of C in l_1 (A), there exists a continuous map $\bar{g}$ of X in l_1 (A) such that

$$\bar{g}|_C = g \quad \text{and} \quad \bar{g}(X) \subset \overline{\text{Conv}(g(C))}.$$

c) For every real or complex Banach space Y with separability degree less or equal than c, every closed subset C of X and every continuous map g of C into Y, there exists a continuous map $\bar{g}$ of X into Y such that

$$\bar{g} \mid C = g \quad \text{and} \quad \bar{g}(X) \subset \overline{\text{Conv}(g(C))}.$$

d) For every real or complex Frechet topological vector space Y with separability degree $\leqslant c$, every closed C in X and every continuous map g of C into Y, there exists a continuous map $\bar{g}$ of X into Y such that

$$\bar{g} \mid C = g \quad \text{and} \quad \bar{g}(X) \subset \overline{\text{Conv}(g(C))}.$$

e) For every metrizable locally convex real or complex topological vector space Y with separability degree $\leqslant c$ every closed C in X and every continuous map g of C into Y such that $\overline{\text{Conv } g \text{ (C)}}$ is complete, there exists an extension such that its image is contained in $\overline{\text{Conv } (g \text{ (C)})}$.

Proof.—a) $\Longrightarrow$ e).

Consider ϕ: $X \longrightarrow 2^Y$ defined by

$$\phi(x) = \begin{cases} \{g(x)\} & \text{if } x \in C \\ \overline{\text{Conv}(g(C)} & \text{if } x \notin C \end{cases}$$

Then ϕ is lower-semicontinuous. By

Theorem 14 with $M = \overline{\text{Conv } (g \text{ (C)})}$ there exists a selection $\bar{g}$ of ϕ. It is clear that $\bar{g}$ is a continuous map of X into Y,

$$\bar{g} \mid C = g \quad \text{and} \quad \bar{g}(X) \subset \overline{\text{Conv}(g(C))}.$$

e) $\Longrightarrow$ d).

This is a consequence of the facts that every Frechet space is metrizable locally convex and complete, and that every closed in a complete space is complete.

d) $\Longrightarrow$ c).

This is a consequence of the fact that every Banach space is a Frechet space.

c) $\Longrightarrow$ b).

This is consequence of that for every set A with card (A) $\leqslant c$, l_1 (A) is a real Banach space with separability degree less or equal than c.

b) $\Longrightarrow$ a).

Let $\{C_i\}_{i \in I}$ be a discrete family of closed subsets of X with card (I) $\leqslant c$. Consider the real Banach space l_1 (I) and the map

$$g : \bigcup_{i \in I} C_i \longrightarrow l_1(I)$$

defined by $g(C_i) = \{x_i\}$, where

$$x_i(j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Since $\{C_i\}_{i \in I}$ is locally finite, g is continuous and, since $\bigcup_{i \in I} C_i$ is closed in X, by hypothesis there exists a continuous map $\bar{g}$ of X into l_1 (I) such that

$$\bar{g} \mid \bigcup_{i \in I} C_i = g.$$

For every $i \in I$, let

$$G_i = \bar{g}^{-1} (B_{\frac{1}{2}} (x_i))$$

Then G_i is open and $C_i \subset G_i$. Moreover for every $i, j \in I$ with $i \neq j$, $G_i \cap G_j = \emptyset$ since

$$B_{\frac{1}{2}} (x_i) \cap B_{\frac{1}{2}} (x_j) = \emptyset .$$

So, X is c-collectionwise normal.

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Instituto «Jorge Juan»
del C. S. I. C.
Serrano, 123, Madrid

Facultad de Ciencias Matemáticas
Universidad Complutense
Madrid-3