SOME CHARACTERIZATIONS OF $c$-PARACOMPACT AND $c$-COLLECTIONWISE NORMAL SPACES BY CONTINUOUS SELECTIONS

by

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(Conclusion)

3. A CHARACTERIZATION OF THE $c$-COLLECTIONWISE NORMAL SPACES BY MEANS OF SELECTIONS

A topological space $X$ is said to be $c$-collectionwise normal, where $c$ is an infinite cardinal, if for every discret family $\{C_i\}_{i \in I}$ of closed subsets of $X$ with $\text{card} (I) \leq c$, there exists a family $\{G_i\}_{i \in I}$ of pairwise disjoint open subsets of $X$ such that $C_i \subseteq G_i$ for all $i \in I$.

It is well-known that the $S_c$-collectionwise normal spaces are the normal spaces ([0], Thm. 1.12). Hence, every $c$-collectionwise normal space is normal.

Lemma 8.—Let $X$ be a topological space, $c$ an infinite cardinal. The following are equivalent:

a) $X$ is $c$-collectionwise normal.

b) For every discrete family $\{C_i\}_{i \in I}$ of closed subsets of $X$ with $\text{card} (I) \leq c$, there exists a family $\{G_i\}_{i \in I}$ of open subsets of $X$ such that $C_i \subseteq G_i$ for all $i \in I$.

Proof.—a) $\implies$ b).

By the hypothesis, there is a family $\{G'_i\}_{i \in I}$ of open subsets of $X$ such that $C_i \subseteq G'_i$ for all $i \in I$ and $G'_i \cap G'_j = \emptyset$ for all $i, j \in I$ with $i \neq j$. 
Now $C = \bigcup_{i \in I} C_i$ is a closed subset of $X$ since $\{C_i\}_{i \in I}$ is a locally finite family of closed subsets of $X$. So $C \subset \bigcup_{i \in I} G'_i$ and as $X$ is normal, there exists an open $A \subset X$ such that

$$C \subset A \subset \overline{A} \subset \bigcup_{i \in I} G'_i$$

It is enough to take $G_i = G'_i \cap \Lambda$ for all $i \in I$.

b) $\implies$ a).

Trivial.

**Lemma 9.**—Let $F$ be a closed subset of a $c$-collectionwise normal space $X$ and $\mathcal{U} = \{U_i \mid i \in I\}$ a locally finite open cover of $F$ such that $\text{card}(I) \leq c$. Then there exists a locally finite open cover

$$\mathcal{V} = \{V_i \mid i \in I\}$$

of $X$ such that $F \cap V_i \subset U_i$ for all $i \in I$.

**Proof.**—As $F$ is closed in $X$, $F$ is normal. Hence there exists $\phi = \{\phi_i\}_{i \in I}$ continuous partition of the unity subordinated to $\mathcal{U}$. Then $\mathcal{W} = \{W_i\}_{i \in I}$, where $W_i = \sup_i \phi_i$ for all $i \in I$, is an open cover of $F$. For every $i \in I$, $W_i$ is an $F$-space in $F$ and then in $X$. Also $W_i \subset U_i$.

Let $\leq$ be a well-ordering of $I$. For every $i \in I$, let

$$C_i = W_i - \bigcup_{j < i} W_j.$$ 

We have:

a) For all $i \in I$, $C_i$ is an $F$-space in $F$, and therefore in $X$.

b) $\mathcal{C} = \{C_i\}_{i \in I}$ is a cover of $F$.

c) For all $i, j \in I$, $i \neq j$, $C_i \cap C_j = \emptyset$.

For each $i \in I$, by a),

$$C_i = \bigcup_{n \in N} C_i^n$$

where for every $n \in N$, $C_i^n$ is closed in $X$.

It follows from c) that $\{C_i^n \mid i \in I\}$ is a discrete family of closed
subsets of $X$, for all $n \in \mathbb{N}$, since $C_i^n \subset C_i \subset U_i$ and $\mathcal{U}$ is locally finite in $F$. By the above lemma and $X$ being $\epsilon$-collectionwise normal, for every $n \in \mathbb{N}$ there is a discrete family of open subsets of $X$ \{ $G_i^n \mid i \in I$ \} such that $C_i^n \subseteq G_i^n$. Then

$$C_i^n \subseteq G_i^n \cap (X - (F - U_i)).$$

So, by the Urysohn lemma, there exists an open and $F_\sigma$ set in $X$, $M_i^n$, such that

$$C_i^n \subseteq M_i^n \subseteq G_i^n \cap (X - (F - U_i))$$

for all $i \in I$ and $n \in \mathbb{N}$.

Let

$$M = \bigcup_{i \in I} M_i^n.$$

It is clear that $M$ is an open subset of $X$ which includes $F$. So, by the Urysohn lemma again, there is an open and $F_\sigma$ set $M_0$ in $X$ such that

$$X - M \subseteq M_0 \subseteq X - F.$$

Let $i_0$ be the first member of $(I, \ll)$. We consider

$$P_i^i = M_0 \cup M_i^n \quad \text{and} \quad P_i^i = M_i^n \quad \text{if} \quad (i, n) \not\ll (i_0, 1).$$

Then \{ $P_i^n \mid i \in I, n \in \mathbb{N}$ \} is an open cover of $X$, its elements are $F_\sigma$ and such that

$$C_i^n \subseteq P_i^n \subseteq X - (F - U_i).$$

Also \{ $P_i^n \mid i \in I$ \} is locally finite in $X$ for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, consider

$$P_n = \bigcup_{i \in I} P_i^n.$$

Then \{ $P_n \mid n \in \mathbb{N}$ \} is an open cover of $X$ and $P_n$ is $F_\sigma$ for all $n \in \mathbb{N}$. So \{ $P_n \mid n \in \mathbb{N}$ \} has a locally finite open refinement \{ $Q_n \mid n \in \mathbb{N}$ \}. 

Let $V_i^n = P_i^n \cap Q_i$ and 

$$V_i = \bigcup_{n \in \mathbb{N}} V_i^n.$$ 

Then:

1) $\mathcal{V} = \{V_i \mid i \in I\}$ is an open cover of $X$.

2) 

$$F \cap V_i \subset F \cap \left( \bigcup_{n \in \mathbb{N}} P_i^n \right) \subset F \cap (X - (F - U_i)) \subset U_i.$$ 

3) $\mathcal{V}$ is locally finite.

**Lemma 10 ([12]).—** Let $X$ be a normal space and $\mathcal{U} = \{U_n \mid n \in \mathbb{N}\}$ a pointwise finite open cover of $X$. Then $\mathcal{U}$ admits a locally finite open refinement

$$\mathcal{V} = \{V_n \mid n \in \mathbb{N}\}$$ 

such that $V_n \subset U_n$ for all $n \in \mathbb{N}$.

**Lemma 11.**—Let $X$ be a $\mathcal{C}$-collectionwise normal space and $\mathcal{U} = \{U_i\}_{i \in I}$ a pointwise finite open cover of $X$ with $\text{card}(I) \leq \mathcal{C}$. Then $\mathcal{U}$ has a locally finite open refinement of cardinal less or equal than $\mathcal{C}$.

**Proof.**—By induction we will construct a sequence

$$\{\mathcal{W}_n \mid n \in \mathbb{N} \cup \{0\}\}$$

of families of open subsets of $X$, verifying:

a) For each $n \in \mathbb{N} \cup \{0\}$ and each $W \in \mathcal{W}_n$, there exists $U \in \mathcal{U}$ with $W \subset U$.

b) For each $n \in \mathbb{N} \cup \{0\}$, $\mathcal{W}_n$ is locally finite and of cardinal $\leq \mathcal{C}$.

c) If $x$ belongs at most to $n$ members of $\mathcal{U}$,

$$x \in \bigcup \{W \mid W \in \mathcal{W}_k, \quad 0 \leq k \leq n\}.$$ 

d) For each $n \in \mathbb{N} \cup \{0\}$, every

$$x \in \bigcup \{W \mid W \in \mathcal{W}_n\}$$ 

belongs at least to $n$ elements of $\mathcal{U}$. Let $\mathcal{W}_0 = \{\emptyset\}.$
Suppose that we have obtained \( \mathcal{W}_1, ..., \mathcal{W}_m \) with the conditions a), b), c) and d). We will construct \( \mathcal{W}_{m+1} \). Let

\[
\mathfrak{A} = \{ \mathcal{U} \subset \mathcal{U} \mid \text{card} (\mathcal{U}) = m + 1 \}.
\]

It is clear that \( \text{card}(\mathfrak{A}) \leq c \). For each \( \mathcal{U} \in \mathfrak{A} \), let

\[
\Lambda (\mathcal{U}) = (X \setminus \bigcup \{ W | W \in \mathcal{W}_k, 0 \leq k \leq m \}) \cap (X \setminus \bigcup \{ U | U \in \mathcal{U} \setminus \mathcal{U} \}).
\]

\( \Lambda (\mathcal{U}) \) is closed and the family

\[
\{ \Lambda (\mathcal{U}) \mid \mathcal{U} \in \mathfrak{A} \}
\]

is discrete.

Applying the lemma 8 to the above family we see that there is a discrete family of open subsets of \( X \),

\[
\{ V (\mathcal{U}) \mid \mathcal{U} \in \mathfrak{A} \}
\]

such that

\[
\Lambda (\mathcal{U}) \subseteq V (\mathcal{U})
\]

for all \( \mathcal{U} \in \mathfrak{A} \). Also \( \Lambda (\mathcal{U}) \subseteq U \) for all \( U \in \mathcal{U} \).

For each \( \mathcal{U} \in \mathfrak{A} \), let

\[
P (\mathcal{U}) = V (\mathcal{U}) \cap \{ \cap \{ V | V \in \mathcal{U} \} \}.
\]

It is clear that \( P (\mathcal{U}) \) is open and \( \Lambda (\mathcal{U}) \subseteq P (\mathcal{U}) \) for all \( \mathcal{U} \in \mathfrak{A} \).

Consider

\[
\mathcal{W}_{m+1} = \{ P (\mathcal{U}) \mid \mathcal{U} \in \mathfrak{A} \}.
\]

One can check that \( \mathcal{W}_0, ..., \mathcal{W}_{m+1} \) verify a), b), c) and d) and the construction of the sequence \( \{ \mathcal{W}_n \}_{n \in \mathbb{N} \cup \{ 0 \}} \) is complete.

For each \( n \in \mathbb{N} \cup \{ 0 \} \), let

\[
\mathcal{W}_n = \bigcup_{W \in \mathcal{W}_n} W.
\]

\( \{ \mathcal{W}_n \mid n \in \mathbb{N} \cup \{ 0 \} \} \) is a pointwise finite open cover of the normal
space $X$. So by the above lemma, there exists $\mathcal{V} = \{V_n\}_{n \in \mathbb{N} \cup \{0\}}$ locally finite open refinement of $\{W_n \mid n \in \mathbb{N} \cup \{0\}\}$ such that $V_n \subseteq W_n$ for all $n \in \mathbb{N} \cup \{0\}$. Then, by a) and b)
\[
|V_* \cap W \mid W \in \mathcal{W}_*, \quad n \in \mathbb{N} \cup \{0\}\]
is a locally finite open refinement of $\mathcal{U}$ of cardinal $\leq c$.

**Lemma 12.**—Let $X$ be a $c$-collectionwise normal topological space, $Y$ a metrizable real or complex topological vector space with separability degree $\leq c$, $M$ a non-empty convex subset of $Y$, and $\hat{\psi} : X \to 2^M$ lower-semicontinuous such that $\hat{\psi}(x) \subseteq M$ for all $x \in X$.

Suppose that either $\hat{\psi}(x)$ is convex relatively compact in $Y$ or $\hat{\psi}(x) = M$ for all $x \in X$. Let $V$ be a convex open neighborhood of $O$ in $Y$ and $f$ a continuous map of $X$ into $Y$ with $f(X) \subseteq M$. Suppose that
\[
\theta(x) = \hat{\psi}(x) \cap (f(x) + V) \neq \emptyset
\]
for all $x \in X$.

Then, given any convex open neighborhood $W$ of $O$ in $Y$, there exists a continuous map $g$ of $X$ into $Y$ such that $g(x) \in \theta(x) + W$ for all $x \in X$, and $g(X) \subseteq M$.

**Proof.**—Let $U$ be an convex symmetric open neighborhood of $O$ in $Y$ such that $U + U \subseteq W$. Consider
\[
G = \{x \in X \mid f(x) \in \theta(x) + U\} = \{x \in X \mid \theta(x) \cap (f(x) - U) \neq \emptyset\}.
\]
By lemma 1 applied to
\[
|\{(x,y) \mid x - y \in V\}|
\]
and $f$, we have that $\theta$ is lower-semicontinuous. Hence $G$ is open. In fact: let $x_0 \in G$. Then there exists
\[
x_0 \in \theta(x_0) \cap (f(x_0) - U)
\]
and
\[
x_1 = x_0 - f(x_0) \in U
\]
So there are an symmetric open neighborhood $U_1$ of $O$ and an open neighborhood $V^x$ of $z$, such that

$$U_1 + V^x \subset U.$$ 

By the continuity of $f$ and the lower-semicontinuity of $\theta$ there is an neighborhood $V_{x_0}$ of $x_0$ such that

$$f(V_{x_0}) \subset f(x_0) - U_1$$

and

$$\theta(x') \cap (V_{x_0} + f(x_0)) \neq \emptyset$$

for all $x' \in V_{x_0}$. So, for all $x' \in V_{x_0}$,

$$\theta(x') \cap (f(x') - U) \neq \emptyset$$

and hence $V_{x_0} \subset G$.

It is clear that $\mathcal{P} = \{y - U_j\}_{j \in J}$ is an open cover of $\overline{M}$. But $Y$ is paracompact and hence there exists a locally finite family $\mathcal{W} = \{W_j\}_{j \in J}$ in $Y$ of open subsets of $Y$ such that

$$\overline{M} \subset \bigcup_{j \in J} W_j,$$

and for every $j \in J$, $W_j \cap \overline{M} \neq \emptyset$ and there is $y_j \in \overline{M}$ with $W_j \subset y_j - U$. $Y$ is metrizable and its separability degree is $\leq \varsigma$, then $Y$ is $\varsigma$-Lindelöf and there is an $I \subset J$, card $I \leq \varsigma$ and

$$\overline{M} \subset \bigcup_{i \in I} W_i.$$ 

For every $i \in I$, consider

$$U_i = \{x \in X \mid \theta(x) \cap W_i \neq \emptyset\}.$$ 

As $\theta$ is lower-semicontinuous, $U_i$ is an open subset of $X$ for all $i \in I$.

Let $\mathcal{U} = \{U_i \mid i \in I\}$. It holds that $\mathcal{U}$ is an open cover of $X$. In fact, let $x \in X$. From $\theta(x) \neq \emptyset$ there exists $y \in \theta(x) \subset M$, and therefore there exists $i \in I$ such that $y \in W_i$. So, $x \in U_i$. 

Consider the closed subset of $X$, $A = X - G$. Now then open cover

$$\mathcal{H} = \{ A \cap U_i \mid i \in I \}$$

of $A$ of cardinality less or equal than $c$, is pointwise finite: If $x \in A$, then

$$f(x) \notin \emptyset (x) + U = [\emptyset (x) \cap (f(x) + V)] + U$$

which implies

$$\emptyset (x) \neq M (f(x) \in M \text{ for all } x \in X).$$

So $\emptyset (x)$ is relatively compact in $Y$ and hence $\emptyset (x)$ is compact for all $x \in A$. On the other hand $x \in U_i$ if and only if

$$\emptyset (x) \cap U_i \neq \emptyset.$$

So $\mathcal{H}$ is pointwise finite since $\{ W_i \mid i \in I \}$ is locally finite.

$A$ is a closed in the $c$-collectionwise normal space $X$, so $A$ is $c$-collectionwise normal. By lemma 11, there is a locally finite open refinement $\{ H_\lambda \}_{\lambda \in \Lambda}$ of $\mathcal{H}$ in $A$ with $\text{card} (\Lambda) \leq c$. By lemma 9 there is a locally finite open cover $\{ A_\lambda \mid \lambda \in \Lambda \}$ of $X$ such that $A_\lambda \cap \Lambda \subseteq H_\lambda$ for all $\lambda \in \Lambda$.

For every $\lambda \in \Lambda$, let $i_\lambda \in I$ such that $H_\lambda \subseteq U_{i_\lambda}$ and

$$B_\lambda = A_\lambda \cap U_{i_\lambda}.$$

Then

$$\mathcal{M}_1 = \{ B_\lambda \mid \lambda \in \Lambda \}$$

is a locally finite family of open subsets of $X$ which cover $X - G = A$.

On the other hand,

$$\mathcal{M}_2 = \{ f^{-1} (W_i) \cap G \mid i \in I \}$$

is a locally finite family of open sets in $X$ which cover $G (f (X) \subseteq M)$ since $\{ W_i \mid i \in I \}$ is a cover of $M$ locally finite in $Y$. So

$$\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 = \{ M_\tau \mid \tau \in \Gamma \}$$
where $\Gamma = \Lambda + I$ is a locally finite open cover of $X$. $X$ is normal and so there is a locally finite continuous partition of the unity $\{h_{\gamma} \mid \gamma \in \Gamma\}$ subordinated to $\mathcal{M}$. If $\gamma = \lambda \in \Lambda$, let $y_\gamma \in \overline{M}$ such that

$$M_\gamma = B_\lambda \subset U_\lambda \quad \text{and} \quad W_\lambda \subset y_\gamma - U.$$ 

If $\gamma = i \in I$, let $y_\gamma \in \overline{M}$ such that $W_i \subset y_\gamma - U$.

Consider $g : X \to Y$ defined by

$$g(x) = \sum_{\gamma \in \Gamma} h_\gamma(x) y_\gamma.$$

We have:

a) $g$ is continuous, since $\{\sup (h_\gamma) \mid \gamma \in \Gamma\}$ is locally finite.

b) For every $x \in X$, $g(x) \in \theta(x) + W$.

In fact, let $x \in X$.

**CASE 1.** $x \in G$.

The $h_\gamma(x) \neq \emptyset$ implies either

$$x \in M_\gamma = B_\lambda \subset U_\lambda$$

if $\gamma = \lambda \in \Lambda$ or $x \in M_i = f^{-1}(W_i) \cap G$

if $\gamma = i \in I$.

In the case of

$$x \in B_\lambda \subset U_\lambda \quad \theta(x) \cap W_\lambda \neq \emptyset$$

and hence

$$\theta(x) \cap (y_\gamma - U) \neq \emptyset.$$

So

$$y_\gamma \in \theta(x) + U \subset \theta(x) + W.$$

If

$$x \in f^{-1}(W_i) \cap G, \quad f(x) \in W_i \subset y_\gamma - U$$
\[ f(x) \in \theta(x) + U. \]

So
\[ y_\gamma = f(x) + u_k = v + u_\delta + u_\gamma \in \theta(x) + W \quad (v \in \theta(x); \ u_\delta, u_\gamma \in U). \]

In this way we conclude
\[ g(x) \in \theta(x) + W \]

since \( \theta(x) + W \) is convex and
\[ \sum_{\gamma \in I'} h_\gamma(x) = 1. \]

Case 2.—\( x \in G. \)

Then \( h_\gamma(x) \neq 0 \) implies \( x \in M_\gamma \) where \( \gamma = \lambda \in \Lambda \). Then
\[ \theta(x) \cap W_{ij} \neq \emptyset \]

and hence
\[ \theta(x) \cap (y_\gamma - U) \neq \emptyset. \]

So
\[ y_\gamma \in \theta(x) + U \subset \theta(x) + W. \]

We have
\[ g(x) \notin \theta(x) + W \]

since \( \theta(x) + W \) is convex and
\[ \sum_{\gamma \in I'} h_\gamma(x) = 1. \]

c) \( g(X) \subset \overline{M} \), since \( y_\gamma \in \overline{M} \) for all \( \gamma \in I' \) and
\[ \sum_{\gamma \in I'} h_\gamma = 1 \]

(\( \overline{M} \) is convex).
Corollary 13.—Let $X$ be a $c$-collectionwise normal space, $Y$ a metrizable real or complex topological vector space with separability degree $\leq c$, $M$ a non-empty convex subset of $Y$, and $\varphi: X \to 2^Y$ lower-semicontinuous such that $\varphi(x) \subseteq M$ for all $x \in X$. Suppose that either $\varphi(x)$ is convex relatively compact in $Y$ or $\varphi(x) = M$ for all $x \in X$. Then given any convex open neighborhood $W$ of $O$ in $Y$, there is a continuous map $g$ of $X$ into $Y$ such that

$$g(x) \in (\varphi(x) + W)$$

for all $x \in X$ and $g(x) \subseteq M$.

Proof.—It is enough to take in the lemma $V = Y$ and $f$ a constant map of $X$ into $M$.

Theorem 14.—Let $X$ be a topological space and $c$ an infinite cardinal. The following are equivalent:

a) $X$ is $c$-collectionwise normal.

b) For every set $A$ of cardinality less or equal than $c$, every convex closed non-empty subset $M$ of $l_1(A)$ and every lower-semicontinuous

$$\varphi : X \to 2^{l_1(A)}$$

such that either $\varphi(x)$ is a convex compact subset of $M$ or $\varphi(x) = M$ for all $x \in X$, it holds that $\varphi$ admits a selection.

c) If $Y$ is a real or complex Banach space with separability degree less or equal than $c$ and $M$ is a convex closed non-empty subset of $Y$, then every $\varphi : X \to 2^Y$ lower semicontinuous such that either $\varphi(x)$ is a convex compact subset of $M$ or $\varphi(x) = M$ for all $x \in X$, admits a selection.

d) If $Y$ is a real or complex Frechet space with separability degree less or equal than $c$, and $M$ is a convex closed non-empty subset of $Y$, then every lower-semicontinuous $\varphi : X \to 2^Y$ such that either $\varphi(x)$ is a convex compact subset of $M$ or $\varphi(x) = M$ for all $x \in M$, admits a selection.

e) If $Y$ is a metrizable locally convex real or complex topological vector space of separability degree less or equal than $c$ and $M$ is a convex complete non-empty subset of $Y$, then every lower-semicon-
tinuous $\phi : X \to 2^\mathcal{Y}$ such that either $\phi (x)$ is a convex compact subset of $\mathcal{M}$ or $\phi (x) = \mathcal{M}$ for all $x \in X$, admits a selection.

**Proof.** a) $\Rightarrow$ d).

As $Y$ is a Frechet space, there exists a metric $d$ on $Y$ such that $d$ describes the topology of $Y$ and $d$ is translations invariant. Let \{\{V_n\}_{n \in \mathbb{N}}\} a countable base of convex symmetric open neighborhoods of $0$ in $Y$ such that

$$V_{n+1} \subseteq \frac{1}{2} V_n$$

and

$$2V_n \subseteq \left\{ y \in Y \mid d(y, 0) < \frac{1}{2^n} \right\} = B_{\frac{1}{2^n}}(0)$$

for all $n \in \mathbb{N}$.

We will show by induction that there exists a sequence of continuous maps \{\{f_n\}_{n \in \mathbb{N}}\} of $X$ into $Y$ such that:

a) $f_n (x) \in f_{n-1} (x) + 2V_{n-1}$ for all $x \in X$ and $n \in \mathbb{N} - \{1\}$.

b) $f_n (x) \in \phi (x) + V_n$ for all $x \in X$ and $n \in \mathbb{N}$.

c) $f_n (X) \subseteq \mathcal{M}$ for all $n \in \mathbb{N}$.

Let $n = 1$. By the above Corollary applied to $\mathcal{M}$, $\phi$ and $V_1$, it holds that there is a continuous map $f_1 : X \to Y$ such that

$$f_1 (x) \in \phi (x) + V_1$$

for all $x \in X$ and $f_1 (X) \subseteq \mathcal{M}$.

Suppose that we have $f_2, ..., f_n$ continuous maps of $X$ into $Y$ satisfying a), b) and c). By b),

$$f_n (x) \in \phi (x) + V_n$$

for all $x \in X$. Then

$$\phi (x) \cap (f_n (x) + V_n) \neq \emptyset$$

for all $x \in X$ since $V_n$ is symmetric. So, by the Lemma 12 applied to $\mathcal{M}$, $\phi$, $V_n$, $f_n$ and $V_{n+1}$, there is a continuous map $f_{n+1}$ of $X$ into $Y$ such that

$$f_{n+1} (x) \in [\phi (x) \cap (f_n (x) + V_n)] + V_{n+1} \subseteq \phi (x) + V_{n+1}$$
for all \( x \in X \) and \( f_{n+1}(X) \subseteq M \). On the other hand

\[
f_{n+1}(x) \in f_n(x) + V_n + V_{n+1} \subseteq f_n(x) + V_n + \frac{1}{2} V_n \subseteq f_n(x) + V_n + V_n + V_n \subseteq f_n(x) + 2V_n.
\]

This completes the construction of the sequence \( \{f_n\} \subseteq M \). Let \( x \in X \).

We show that \( \{f_n(x)\} \subseteq M \) is a Cauchy sequence in \( (Y, d) \). Let \( \varepsilon > 0 \) and \( n_0 \in \mathbb{N} \) such that

\[
\sum_{m=n_0}^{n} \frac{1}{2^m} < \varepsilon.
\]

The for every \( m, n \geq n_0 \) if \( m = n + r \ (r > 0) \), by a), we have

\[
d(f_m(x), f_n(x)) \leq d(f_m(x), f_{m-1}(x)) + \ldots + d(f_{n+1}(x), f_n(x)) < \frac{1}{2^{m-1}} + \ldots + \frac{1}{2^n} < \varepsilon,
\]

So for every \( x \in X \) there exists

\[
a(x) = \lim_{n \to \infty} f_n(x) \in N
\]

Furthermore the above proof shows that \( \{f_n\} \subseteq M \) converges uniformly to \( a \). So \( a \) is a continuous map of \( X \) into \( Y \).

On the other hand given \( x \in X \), for every \( n \in \mathbb{N} \) there are \( v_n \in V_n \) and \( z_n \in \phi(x) \) such that

\[
f_n(x) = z_n + v_n.
\]

But

\[
\lim_{n \to \infty} |v_n| = 0,
\]

so

\[
a(x) = \lim_{n \to \infty} |z_n| \in \phi(x).
\]

d) \( \implies \) c).

Consequence that every Banach space is a Frechet space.

c) \( \implies \) b).
This is a consequence of the fact that for every set \( A \) such that \( \text{card}(A) \leq c \), \( l_1(A) \) is a real Banach space with separability degree less or equal than \( c \).

b) \( \iff \) a).

Let \( A \) be a set, \( \text{card}(A) \leq c \). Let \( F \) be a closed in \( X \) and \( g \) a continuous map of \( F \) into \( l_1(A) \). Consider the map

\[ \phi : X \to 2^\lambda(A) \]

defined by

\[ \phi(x) = \begin{cases} 1 & \text{if } x \in F \\ l_1(A) & \text{if } x \notin F. \end{cases} \]

Then \( \phi \) is lower-semicontinuous. By the hypothesis with \( M = l_1(A) \), there exists a selection \( \bar{g} \) of \( \phi \). It is clear that \( \bar{g} \) is a continuous map of \( X \) into \( l_1(A) \) such that \( \bar{g} |_F = g \).

Let \( \{C_i\}_{i \in I} \) be a discrete family of closed subsets of \( X \) such that \( \text{card}(I) \leq c \). Consider the real Banach space \( l_1(A) \), and the map:

\[ g : \bigcup_{i \in I} C_i \to l_1(I) \]

defined by

\[ g(C_i) = |x_i| \]

where

\[ x_i(j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \]

\( g \) is a continuous map of \( \bigcup_{i \in I} C_i \) into \( l_1(I) \) since \( \{C_i\}_{i \in I} \) is locally finite. So, there exists a continuous map \( \tilde{g} \) of \( X \) into \( l_1(I) \) such that

\[ \tilde{g} |_{\bigcup_{i \in I} C_i} = g. \]

For every \( i \in I \), let

\[ G_i = \tilde{g}^{-1}(B_{\frac{1}{2}}(x_i)) \]

The \( G_i \) is open and \( C_i \subset G_i \).
Finally if \( i, j \in I \) are such that \( i \neq j \), then

\[
G_i \cap G_j = \widetilde{g}^{-1}\left( B_{\frac{1}{2}}(x_i) \cap B_{\frac{1}{2}}(x_j) \right) = \emptyset.
\]

This shows that \( X \) is \( c \)-collectionwise normal.

e) \iff d).

This is a consequence of the facts that every Frechet space is metrizable, complete and locally convex and that every closed subset of a complete space is complete.

d) \iff e).

Let \( \mathcal{U} \) be the uniformity associated to \( Y \) and let \((\tilde{Y}, \tilde{\mathcal{U}})\) the \( T_2 \) completion of \((Y, \mathcal{U})\). The immersion \( i \) of \( Y \) into \( \tilde{Y} \) is a linear homeomorphism onto the image. Furthermore, as \( Y \) is an metrizable locally convex topological vector space with separability degree less or equal than \( c \), \( \tilde{Y} \) is a Frechet topological vector space \((\| \|)\), with separability degree less or equal than \( c \). On the other hand, \( \tilde{M} = i(M) \) is convex and complete. Hence \( \tilde{M} \) is convex and closed in \( \tilde{Y} \).

Finally \( \tilde{\phi}(x) = i(\phi(x)) \) is lower-semicontinuous and either \( \tilde{\phi}(x) \) is a convex compact subset of \( \tilde{M} \) or \( \tilde{\phi}(x) = \tilde{M} \) for all \( x \in X \). By hypothesis there exists a selection \( \tilde{f} \) of \( \tilde{\phi} \). It is clear that \( f = i^{-1} \circ \tilde{f} \) is a selection of \( \phi \).

**Corollary 15.**—Let \( X \) be a topological space. The following are equivalent:

a) \( X \) is collectionwise normal.

b) For every set \( A \), every closed convex non-empty subset \( M \) of \( l_1(A) \) and every

\[
\phi : X \to 2^{l_1(A)}
\]

lower-semicontinuous such that either \( \phi(x) \) is a convex compact subset of \( M \) or \( \phi(x) = M \) for all \( x \in X \), it holds that \( \phi \) has a selection \((\| \|)\).

c) If \( Y \) is a real or complex Banach space and \( M \) is a closed convex non-empty subset of \( Y \), every \( \phi : X \to 2^Y \) lower-semicontinuous such that either \( \phi(x) \) is a convex compact subset of \( M \) or \( \phi(x) = M \) for all \( x \in X \), admits a selection \((\| \|)\).
d) If $Y$ is a real or complex Fréchet space and $M$ is a closed convex and non-empty subset of $Y$, every $\phi : X \rightarrow 2^Y$ lower-semicontinuous such that either $\phi(x)$ is a convex compact subset of $M$ or $\phi(x) = M$ for all $x \in X$, admits a selection.

e) If $Y$ is a metrizable locally convex topological vector space and $M$ is a convex complete non-empty subset of $Y$, every $\phi : X \rightarrow 2^Y$ lower-semicontinuous such that either $\phi(x)$ is a convex compact subset of $M$ or $\phi(x) = M$ for all $x \in X$, admits a selection.

**Corollary 16.**—Every $c$-paracompact normal space is $c$-collectionwise normal.

**Proof.**—Consequence of Thm. 5 and 14.

The converse of Corollary 15 is not true: [23] provides an example of a collectionwise normal space non $S_c$-paracompact.

**Proposition 17.**—Let $X$ be a topological space and $c$ an infinite cardinal. The following are equivalent:

a) $X$ is $c$-collectionwise normal.

b) For every set $A$ with $\text{card}(A) \leq c$, every closed $C$ in $X$ and every continuous map $g$ of $C$ in $l_1(A)$, there exists a continuous map $\tilde{g}$ of $X$ in $l_1(A)$ such that

\[ \tilde{g} |_C = g \quad \text{and} \quad \tilde{g}(X) \subset \overline{\text{Conv}}(g(C)). \]

c) For every real or complex Banach space $Y$ with separability degree less or equal than $c$, every closed subset $C$ of $X$ and every continuous map $g$ of $C$ into $Y$, there exists a continuous map $\tilde{g}$ of $X$ into $Y$ such that

\[ \tilde{g} |_C = g \quad \text{and} \quad \tilde{g}(X) \subset \overline{\text{Conv}}(g(C)). \]

d) For every real or complex Frechet topological vector space $Y$ with separability degree $\leq c$, every closed $C$ in $X$ and every continuous map $g$ of $C$ into $Y$, there exists a continuous map $\tilde{g}$ of $X$ into $Y$ such that

\[ \tilde{g} |_C = g \quad \text{and} \quad \tilde{g}(X) \subset \overline{\text{Conv}}(g(C)). \]

e) For every metrizable locally convex real or complex topological vector space $Y$ with separability degree $\leq c$ every closed $C$ in $X$ and
every continuous map $g$ of $C$ into $Y$ such that $\overline{\text{Conv} \ g \ (C)}$ is complete, there exists an extension such that its image is contained in $\overline{\text{Conv} \ g \ (C)}$.

**Proof.** a) $\implies$ e).

Consider $\phi : X \to 2^Y$ defined by

$$\phi (x) = \begin{cases} \frac{\|g(x)\|}{\overline{\text{Conv}} \ g \ (C)} & \text{if } x \notin C \\ g(x) & \text{if } x \in C \end{cases}$$

Then $\phi$ is lower-semicontinuous. By

Theorem 14 with $M = \overline{\text{Conv} \ g \ (C)}$ there exists a selection $\bar{g}$ of $o$.

It is clear that $\bar{g}$ is a continuous map of $X$ into $Y$,

$$\overline{\bar{g} \mid C} = g \quad \text{and} \quad \overline{\bar{g} \ (X)} \subseteq \overline{\text{Conv} \ g \ (C)}.$$  

e) $\implies$ d).

This is a consequence of the facts that every Frechet space is metrizable locally convex and complete, and that every closed in a complete space is complete.

d) $\implies$ c).

This is a consequence of the fact that every Banach space is a Frechet space.

c) $\implies$ b).

This is consequence of that for every set $A$ with $\text{card} \ (A) \leq c$, $l_1 \ (A)$ is a real Banach space with separability degree less or equal than $c$.

b) $\implies$ a).

Let $\{C_i\}_{i \in I}$ be a discrete family of closed subsets of $X$ with $\text{card} \ (I) \leq c$. Consider the real Banach space $l_1 \ (I)$ and the map

$$g : \bigcup_{i \in I} C_i \to l_1 \ (I)$$

defined by $g \ (C_i) = \{x_i\}$, where

$$x_i \ (f) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$
Since \( \{ C_i \}_{i \in I} \) is locally finite, \( g \) is continuous and, since \( \bigcup_{i \in I} C_i \) is closed in \( X \), by hypothesis there exists a continuous map \( \tilde{g} \) of \( X \) into \( l_1 \) (I) such that

\[
\tilde{g} \mid \bigcup_{i \in I} C_i = g.
\]

For every \( i \in I \), let

\[
G_i = \tilde{g}^{-1}(B_{1/2}(x_i))
\]

Then \( G_i \) is open and \( C_i \subset G_i \). Moreover for every \( i, j \in I \) with \( i \neq j \),
\( G_i \cap G_j = \emptyset \) since

\[
B_{1/2}(x_i) \cap B_{1/2}(x_j) = \emptyset.
\]

So, \( X \) is \( c \)-collectionwise normal.

References


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