Gauge Invariance on Interaction $U(1)$ -Bundles ^{*}

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Abstract

The structure of the algebra of gauge-invariant differential forms on the bundle $C \times_M E$ is determined, where $p: C \rightarrow M$ is the bundle of connections of a $U(1)$ -principal bundle $\pi: P \rightarrow M$, and $E \rightarrow M$ is the associated bundle to P by the representation λ_r , $r \in \mathbb{N}$, of $U(1)$ on $\mathbb{C}$ given by $\lambda_r(z)(w) = z^r w$, $z \in U(1)$, $w \in \mathbb{C}$.

1 Introduction

The aim of this paper is to describe the geometric structure underlying the interaction bundle (*i.e.*, the bundle for interacting particle and gauge fields) in the particular case of $U(1)$ -principal bundles. As is well known, the bundle of connections of an arbitrary principal G -bundle $\pi: P \rightarrow M$ is an affine bundle $p: C = C(P) \rightarrow M$ modelled over the vector bundle $T^*M \otimes \text{ad}P \rightarrow M$ (*cf.* [2, 5, 6]), where $\text{ad}P \rightarrow M$ is the adjoint bundle; *i.e.*, the bundle associated to P by the adjoint representation of G on its Lie algebra $\mathfrak{g}$. In the particular case $G = U(1)$, which corresponds to the classical electromagnetism, the adjoint bundle is canonically isomorphic to the trivial line bundle so that C is an affine bundle modelled over the cotangent bundle T^*M . In two previous papers ([7, 8]) we proved that, in this case, C is endowed with a canonical symplectic form ω_2 that generates over $\Omega^\bullet(M)$ the algebra of differential forms on C which are invariant under the natural representation of the gauge algebra of P (that is, $\text{gau}P = \Gamma(M, \text{ad}P)$) on C . The initial motivation for that result was the geometric formulation of Utiyama's theorem (*cf.* [3, 4, 6, 14]). If $E \rightarrow M$ is the vector bundle associated to P by a linear representation of G on a finite-dimensional real vector space V , the fibred product $C \times_M E$ is usually called

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the interaction bundle since on it the Lagrangian for a particle field interacting with a gauge field, is defined. In fact, this is the bundle where Utiyama's foundational paper [14] takes place (also see [13]), so that it is natural to extend the results of [8] to the interaction bundle in analyzing the geometric structure of gauge forms. Moreover, in dealing with the abelian case we confine ourselves to the linear representations λ_r , $r \in \mathbb{N}$, of $U(1)$ on $\mathbb{C}$ given by $\lambda_r(z)(w) = z^r w$, $z \in U(1)$, $w \in \mathbb{C}$, as they are the inequivalent irreducible real representations of $U(1)$ (e.g., see [1, 3.78]).

$\mathcal{I}_{\text{gau}}(C)$ (resp. $\mathcal{I}_{\text{gau}}(E)$, resp. $\mathcal{I}_{\text{gau}}(C \times_M E)$) denotes the algebra of gauge-invariant differential forms on C (resp. E , resp. $C \times_M E$). We have two homomorphisms of $C^\infty(M)$ -algebras, $\mathcal{I}_{\text{gau}}(C) \rightarrow \mathcal{I}_{\text{gau}}(C \times_M E)$, $\mathcal{I}_{\text{gau}}(E) \rightarrow \mathcal{I}_{\text{gau}}(C \times_M E)$. The most outstanding novelty is that $\mathcal{I}_{\text{gau}}(C \times_M E)$ is not generated by $\mathcal{I}_{\text{gau}}(C)$ and $\mathcal{I}_{\text{gau}}(E)$: roughly speaking, in order to generate all gauge-invariant differential forms on the interaction bundle it is necessary to add to the forms above an specific 1-form $\alpha \in \Omega^1(C \times_M E)$, called the interaction 1-form, which depends on the integer r (i.e., on the charge of the particle), and its exterior differential. Accordingly this form allows one to distinguish different representations by means of the algebra of gauge-invariant forms.

In §2 we recall some properties of the bundle of connections of a principal bundle and we introduce the standard coordinate systems on the bundles C , E , which we use throughout. In §3 we define the action of the group of automorphisms of P on C , on E and on the interaction bundle, and we obtain the corresponding infinitesimal versions of these actions. This leads one to introduce the notions of $\text{gau}P$ -invariant and $\text{aut}P$ -invariant differential forms on the interaction bundle in §4. In §5 it is proved that $J^1P \times \mathbb{C}$ is a $U(1)$ -principal bundle over $C \times_M E$. In §6 this bundle structure is used in order to give a direct definition of the interaction 1-form α as being the projection onto $C \times_M E$ of a certain differential form $J^1P \times_M \mathbb{C}$, explicitly defined in terms of the structure form on the 1-jet bundle. The geometric interpretation of the form α is closely related to the classification of the Lagrangians on $J^1(C \times_M E)$ which are gauge-invariant in the Utiyama sense (cf. [3]). More precisely, for every connection Γ on P and every section $\xi \in \Gamma(M, E)$, $(\sigma_\Gamma, \xi)^* \alpha$ coincides with the imaginary part of $\langle \xi, \nabla \xi \rangle$, where $\sigma_\Gamma: M \rightarrow C$ is the section induced by Γ (see notations below), ∇ is the covariant derivative induced by Γ on E , and $\langle, \rangle$ stands for the standard Hermitian structure on E . Also the physical meaning of α is relevant: it is shown to be the “universal” current of the Yang–Mills–Higgs classical action (see [4, Ch. 5] and §6.6 below).

Let A be the standard basis of $\mathfrak{g} = \mathfrak{u}(1)$ (see §2.2 below for the notations), and let $A^* \in \mathfrak{X}(V)$ be the fundamental vector field associated to A under a linear representation. We set $\mathcal{A}(V) = \{\Omega \in \Omega^\bullet(V); i_{A^*}\Omega = 0, i_{A^*}d\Omega = 0\}$. In §7 we prove that every differential form $\Omega \in \mathcal{A}(V)$, induces a differential form $\Omega_E \in \Omega^\bullet(E)$, which is not only gauge invariant but also invariant under the Lie algebra of all infinitesimal automorphisms of P . In this way, we obtain all $\text{aut}P$ -invariant differential forms on E and furthermore the structure of $\mathcal{I}_{\text{gau}}(E)$ is determined.

Section 8 is devoted to the statement and the proof of the characterization of

$\mathcal{I}_{\text{gau}}(C \times_M E)$; as a consequence we also determine the algebra of $\text{aut} P$ -invariant forms on $C \times_M E$. Finally, in §9 we obtain the basic relations among the forms which generate the algebra of gauge-invariant forms.

2 Preliminaries and notations

2.1 The bundle of connections of a principal G -bundle. Let $\pi: P \rightarrow M$ be a principal G -bundle over a m -dimensional, connected C^∞ manifold M . We set $T_G P = (TP)/G$, and we denote by $[X]$ the orbit of $X \in TP$ in $T_G P$. The sections of $T_G P$ correspond with the G -invariant vector fields on P , and π -vertical G -invariant vector fields on P can be identified to the sections of the adjoint bundle. We have an exact sequence of vector bundles over M ([2]), $0 \rightarrow \text{ad} P \rightarrow T_G P \rightarrow TM \rightarrow 0$. Connections on P correspond with the splittings of this sequence. Hence connections on P are the sections of an affine bundle $p: C = C(P) \rightarrow M$ modelled over $T^*M \otimes \text{ad} P$. We also denote by $\sigma_\Gamma: M \rightarrow C$ the section of the bundle of connections induced from Γ .

2.2 Coordinates in C . Let $\pi: P \rightarrow M$ be a $U(1)$ -principal bundle and let A be the standard basis of $\mathfrak{u}(1)$; that is, the vector field corresponding to the 1-parameter subgroup $\mathbb{R} \rightarrow U(1), t \mapsto \exp(it)$. As $U(1)$ is abelian the fundamental vector field $A^* \in \mathfrak{X}(P)$ is $U(1)$ -invariant. Set $\tilde{A} = [A^*]$. Let $(W; q^1, \dots, q^m)$, $m = \dim M$, be an open coordinate domain of M on which $\pi^{-1}(W) \simeq W \times U(1)$. We parametrize the points in $U(1)$ as $\exp(it)$, $0 \leq t \leq 2\pi$. Thus, the functions $(q^1 \circ \pi, \dots, q^m \circ \pi, t)$ are coordinates on P . We usually identify $q^i \circ \pi$ to q^i . A linear map $\ell: T_q M \rightarrow (T_G P)_q$ is a section of π_* if and only if scalars $\lambda_1, \dots, \lambda_m$ exist such that: $\ell(\partial/\partial q^i)_q = (\partial/\partial q^i)_q + \lambda_i \tilde{A}_q$, $1 \leq i \leq m$. We define $p_1, \dots, p_m$ on $p^{-1}(W)$ by $p_i(\ell) = -\lambda_i$, or equivalently, for every connection Γ , $\sigma_\Gamma(\partial/\partial q^i) = \partial/\partial q^i - (p_i \circ \sigma_\Gamma) \tilde{A}$. Hence the functions q^i (or more properly $q^i \circ p$) and p_i , $1 \leq i \leq m$, are coordinates on $p^{-1}(W)$. The bundle of connections of $M \times U(1)$ can be identified with T^*M , and then the functions (q^i, p_i) have their usual meaning (cf. [7, 8]).

2.3 Coordinates in E . Let $\pi_E: E \rightarrow M$ be the vector bundle associated to the linear representation λ_r defined in the Introduction and let $(W; q^i)$ be as in §2.2. We denote by $[u, w]$ the orbit of the pair $(u, w) \in P \times \mathbb{C}$ in $E = (P \times \mathbb{C})/U(1)$, and let $s_0: W \rightarrow P$ be the section corresponding to the trivialization $\pi^{-1}(W) \simeq W \times U(1)$. We define functions $\mathbf{x}, \mathbf{y}$ on $\pi_E^{-1}(W)$, by setting $e = [s_0(\pi_E e), \mathbf{x}(e) + i\mathbf{y}(e)]$, $e \in \pi_E^{-1}(W)$, and $(q^1, \dots, q^m, \mathbf{x}, \mathbf{y})$ are coordinates on $\pi_E^{-1}(W)$.

2.4 Infinitesimal contact transformations. Let $p: N \rightarrow M$ be an arbitrary fibred manifold; i.e., p is a surjective submersion. We denote by $p_1: J^1 N \rightarrow M$ the 1-jet bundle of local sections of p . For every section $s: W \rightarrow N$ of p defined on an open subset $W \subseteq M$, we denote by $j^1 s: W \rightarrow J^1 N$ its jet prolongation. Set $\dim N = m + n$. Every fibred coordinate system (q^i, y^j) , $1 \leq i \leq m$, $1 \leq j \leq n$, for the projection p induces a coordinate system (q^i, y^j, y_i^j) on $J^1 N$ by $y_i^j(j_x^1 s) = (\partial(y^j \circ s)/\partial q^i)(x)$. A differential 1-form θ on $J^1 N$ is a contact

form if $(j^1 s)^* \theta = 0$ for every local section s of p . The set of all contact forms is a differential system $\mathcal{C}$ of rank n locally generated by $\theta^j = dy^j - y_i^j dq^i$, $1 \leq j \leq n$. A vector field $X \in \mathfrak{X}(J^1 N)$ is said to be an infinitesimal contact transformation if $L_X \mathcal{C} \subseteq \mathcal{C}$. For every vector field $X \in \mathfrak{X}(N)$ there exists a unique infinitesimal contact transformation $X^{(1)} \in \mathfrak{X}(J^1 N)$ projecting onto X via the projection $p_{10}: J^1 N \rightarrow J^0 N = N$, and the mapping $\mathfrak{X}(N) \rightarrow \mathfrak{X}(J^1 N)$, $X \mapsto X^{(1)}$, is a Lie algebra monomorphism (e.g., see [12]).

3 The basic liftings

3.1 The action of the group of automorphism on $C \times_M E$. Let us denote by $\text{Aut}P$ the group of automorphisms of P . Every $\Phi \in \text{Aut}P$ induces a unique diffeomorphism $\phi: M \rightarrow M$, such that $\pi \circ \Phi = \phi \circ \pi$. The mapping $\Phi \mapsto \phi$ is a group homomorphism whose kernel is the gauge group, $\text{Gau}P$.

Let ω_Γ be the connection form of a connection Γ on P . Given $\Phi \in \text{Aut}P$ we set $\Phi \cdot \Gamma = \Gamma'$, where $(\Phi^{-1})^* \omega_\Gamma = \omega_{\Gamma'}$ (see [10, II.6.2]). As $(\omega_{\Gamma'})|_{\pi^{-1}(\phi x)}$ depends only on $(\omega_\Gamma)|_{\pi^{-1}(x)}$, we can define a unique diffeomorphism $\Phi_C: C \rightarrow C$ such that for every connection Γ and every $x \in M$, $\Phi_C(\Gamma(x)) = (\Phi \cdot \Gamma)(x)$. We have: 1) $p \circ \Phi_C = \phi \circ p$, 2) $(\Phi \circ \Psi)_C = \Phi_C \circ \Psi_C$, $\forall \Phi, \Psi \in \text{Aut}P$.

Similarly, $\text{Aut}P$ acts on E and on $C \times_M E$ (notations of §2.3) by setting $\Phi_E([u, w]) = [\Phi(u), w]$, $\bar{\Phi}(\Gamma_x, [u, w]) = (\Phi_C(\Gamma_x), \Phi_E([u, w]))$, respectively, for every $\Phi \in \text{Aut}P$, with $p(\Gamma_x) = \pi(u) = q$. The definition of $\bar{\Phi}$ makes sense since $p(\Phi_C(\Gamma_x)) = \pi_E(\Phi_E([u, w])) = q$.

3.2 The homomorphism $\text{aut}P \rightarrow \mathfrak{X}(C \times_M E)$. Let Φ_t be the local flow of a vector field X on P . Then, X is a $U(1)$ -invariant vector field if and only if $\Phi_t \in \text{Aut}P$, $\forall t$. Because of this we denote by $\text{aut}P$ the Lie algebra of all $U(1)$ -invariant vector fields on P and we think of the elements of $\text{aut}P$ as being the infinitesimal automorphisms of P . We have an identification $\text{aut}P = \Gamma(M, T_G P)$.

Let Φ_t be the local flow of $X \in \text{aut}P$. We denote by $X_C \in \mathfrak{X}(C)$, $X_E \in \mathfrak{X}(E)$, $\bar{X} \in \mathfrak{X}(C \times_M E)$, the infinitesimal generators of the flows $(\Phi_t)_C$, $(\Phi_t)_E$, $\bar{\Phi}_t$ on C , E , $C \times_M E$, respectively, defined in §3.1. We have Lie algebra homomorphisms

$$\begin{aligned} \text{aut}P &\rightarrow \mathfrak{X}(C), & X &\mapsto X_C, \\ \text{aut}P &\rightarrow \mathfrak{X}(E), & X &\mapsto X_E, \\ \text{aut}P &\rightarrow \mathfrak{X}(C \times_M E), & X &\mapsto \bar{X}. \end{aligned}$$

If $(W; q^i)$ is as in §2.2, then it is not difficult to see that a vector field $X \in \mathfrak{X}(\pi^{-1}W)$ is $U(1)$ -invariant if and only if there exist functions $f_i, g \in C^\infty(W)$, such that

$$(1) \quad X = f^i(q^1, \dots, q^m) \frac{\partial}{\partial q^i} + g(q^1, \dots, q^m) A^*,$$

and we have (see [7, 9]):

$$(2) \quad X_C = f^i \frac{\partial}{\partial q^i} - \left(\frac{\partial g}{\partial q^i} + \frac{\partial f^h}{\partial q^i} p_h \right) \frac{\partial}{\partial p_i}.$$

$$(3) \quad X_E = f^i \frac{\partial}{\partial q^i} - rg \left(\mathbf{y} \frac{\partial}{\partial \mathbf{x}} - \mathbf{x} \frac{\partial}{\partial \mathbf{y}} \right)$$

$$(4) \quad \bar{X} = f^i \frac{\partial}{\partial q^i} - \left(\frac{\partial g}{\partial q^i} + \frac{\partial f^h}{\partial q^i} p_h \right) \frac{\partial}{\partial p_i} - rg \left(\mathbf{y} \frac{\partial}{\partial \mathbf{x}} - \mathbf{x} \frac{\partial}{\partial \mathbf{y}} \right).$$

Note that every $X \in \text{aut}P$ is π -projectable and that the projections of X_C , X_E and $\bar{X}$ onto M coincide with that of X . Moreover, the vector field $X_C + X_E \in \mathfrak{X}(C \times E)$ is tangent to the submanifold $C \times_M E$ and we have $\bar{X} = X_C + X_E$.

4 Invariance

4.1 Definitions. A differential form Ω on C (resp. E , resp. $C \times_M E$) is said to be *autP-invariant* if for every $X \in \text{aut}P$, we have

$$(5) \quad L_{X_C} \Omega = 0 \text{ (resp. } L_{X_E} \Omega = 0, \text{ resp. } L_{\bar{X}} \Omega = 0).$$

We denote by $\mathcal{I}_{\text{aut}}(C)$ (resp. $\mathcal{I}_{\text{aut}}(E)$, resp. $\mathcal{I}_{\text{aut}}(C \times_M E)$) the algebra of *autP*-invariant differential forms on C (resp. E , resp. $C \times_M E$). From the definitions it follows that there are natural inclusions

$$\mathcal{I}_{\text{aut}}(C) \subset \mathcal{I}_{\text{aut}}(C \times_M E), \quad \mathcal{I}_{\text{aut}}(E) \subset \mathcal{I}_{\text{aut}}(C \times_M E),$$

induced by the canonical projections $pr_1: C \times_M E \rightarrow C$, $pr_2: C \times_M E \rightarrow E$, respectively.

A differential form Ω on C (resp. E , resp. $C \times_M E$) is said to be *gauge invariant* if the corresponding equation in (5) holds true for every $X \in \text{gau}P$. We denote by $\mathcal{I}_{\text{gau}}(C)$ (resp. $\mathcal{I}_{\text{gau}}(E)$, resp. $\mathcal{I}_{\text{gau}}(C \times_M E)$) the algebra of *gauP*-invariant forms on C (resp. E , resp. $C \times_M E$). We remark that $\mathcal{I}_{\text{gau}}(C)$, $\mathcal{I}_{\text{gau}}(E)$, $\mathcal{I}_{\text{gau}}(C \times_M E)$ are endowed with an structure of algebra over $\Omega^\bullet(M)$ via the natural projections. We thus have natural inclusions

$$\mathcal{I}_{\text{gau}}(C) \subset \mathcal{I}_{\text{gau}}(C \times_M E), \quad \mathcal{I}_{\text{aut}}(C) \subset \mathcal{I}_{\text{gau}}(C),$$

$$\mathcal{I}_{\text{gau}}(E) \subset \mathcal{I}_{\text{gau}}(C \times_M E), \quad \mathcal{I}_{\text{aut}}(E) \subset \mathcal{I}_{\text{gau}}(E),$$

$$\mathcal{I}_{\text{aut}}(C \times_M E) \subset \mathcal{I}_{\text{gau}}(C \times_M E).$$

4.2 Structure of $\mathcal{I}_{\text{gau}}(C)$. The algebra of gauge-invariant forms on the bundle of connections of a $U(1)$ -principal bundle $\pi: P \rightarrow M$ has been characterized in [7, 8]. It turns out that

$$\mathcal{I}_{\text{gau}}(C) = p^* \Omega^\bullet(M)[\omega_2],$$

where ω_2 is a symplectic form on C whose local expression in the system of coordinates defined in §2.2 is $\omega_2 = dp_i \wedge dq^i$; *i.e.*, the general expression of a gauge-invariant r -form on C is $\xi_r = \sum p^* \nu_{r-2s} \wedge (\omega_2)^s$, $\nu_{r-2s} \in \Omega^\bullet(M)$, $s = 0, \dots, [r/2]$. In particular, it is not difficult to prove that the subalgebra of $\text{aut}P$ -invariant forms are polynomial expression of the form ω_2 , that is,

$$\mathcal{I}_{\text{aut}}(C) = \mathbb{R}[\omega_2].$$

5 The identification $(J^1P \times \mathbb{C})/U(1) \simeq C \times_M E$

5.1 The connection associated to a 1-jet of section. Each section $s: W \rightarrow P$ of $\pi: P \rightarrow M$, defined on an open neighbourhood of $q \in M$, defines “an element of connection at q ”; *i.e.*, a point $\Gamma_q \in C_q$, which is determined by giving a retract $\Gamma_q: T_uP \rightarrow V_uP$ of the inclusion of the vertical subspace $V_uP \subset T_uP$, $\forall u \in \pi^{-1}(q)$, as follows: $\Gamma_q(X) = X - (R_z)_* s_* \pi_*(X)$, $X \in T_uP$, where $z \in U(1)$ is the unique element such that $u = s(q) \cdot z$. Note that for every $z \in U(1)$, we have $(R_z)_* \circ \Gamma_q = \Gamma_q \circ (R_z)_*$. It is easy to see that Γ_q depends only on $j_q^1 s$, so that we can define a map of fibred manifolds over M , $\gamma: J^1P \rightarrow C$ by setting $\gamma(j_q^1 s) = \Gamma_q$. We say that $\gamma(j_q^1 s)$ is the element of connection at the point q associated to the 1-jet $j_q^1 s$.

5.2 Proposition. *Let us consider the induced action of $U(1)$ on J^1P ; *i.e.*, $j_x^1 s \cdot z = j_x^1 (R_z \circ s)$, for $z \in U(1)$, and the action of $U(1)$ on $\mathbb{C}$ defined by the representation λ_r . With the same notations as in §2.3, §5.1, let $\varphi: J^1P \times \mathbb{C} \rightarrow C \times_M E$ be the map of fibred manifolds over M given by $\varphi(j_q^1 s, w) = (\gamma(j_q^1 s), [s(q), w])$. Then, φ is a surjective submersion whose fibres are the orbits of the action of $U(1)$ on $J^1P \times \mathbb{C}$ given by $(j_q^1 s, w) \cdot z = (j_q^1 s \cdot z, z^{-1} \cdot w)$. Hence we have a natural identification $(J^1P \times \mathbb{C})/U(1) \simeq C \times_M E$.*

Proof. Let $\pi_{10}: J^1P \rightarrow P$ be the canonical projection, $\pi_{10}(j_q^1 s) = s(q)$. With the same notations as in §2.2, §2.4, let (q_i, t, t_i) , $1 \leq i \leq m$, be the coordinate system induced on $\pi_{10}^{-1}(\pi^{-1}(W))$ by $(\pi^{-1}(W); q_i, t)$; *i.e.*, $t_i(j_q^1 s) = (\partial(t \circ s)/\partial q^i)(q)$. On $p^{-1}(W) \times_W \pi_E^{-1}(W) \subset C \times_M E$, we consider the coordinate system $(q^i, p_i, \mathbf{x}, \mathbf{y})$ defined in §2.2, §2.3. In these systems, the equations of φ are

$$(6) \quad q^i \circ \varphi = q^i, \quad p_i \circ \varphi = -t_i \quad (1 \leq i \leq m), \quad (\mathbf{x} + i\mathbf{y}) \circ \varphi = \exp(irt)(x + iy),$$

thus proving that φ is a submersion. In fact, $(p_i \circ \varphi)(j_q^1 s, w) = p_i(\gamma(j_q^1 s))$, and from the very definition of the coordinates p_i in §2.2 we have

$$\sigma_{\gamma(j_q^1 s)}(\partial/\partial q^i)_q = [\partial/\partial q^i]_q - p_i(\gamma(j_q^1 s)) \tilde{A}_q.$$

Hence

$$(\partial/\partial q^i)_{s(q)}^* = (\partial/\partial q^i)_{s(q)} - p_i(\gamma(j_q^1 s)) A_{s(q)}^*.$$

Moreover, according to the definition of the connection associated to $j_q^1 s$ in §5.1, $\gamma(j_q^1 s)$ is obtained by imposing

$$\begin{aligned} 0 &= \gamma(j_q^1 s) (\partial/\partial q^i)_{s(q)}^* = (\partial/\partial q^i)_{s(q)}^* - s_*(\partial/\partial q^i)_q \\ &= \left((\partial/\partial q^i)_{s(q)} - p_i(\gamma(j_q^1 s)) A_{s(q)}^* \right) \\ &\quad - \left((\partial/\partial q_i)_{s(q)} + (\partial(t \circ s)/\partial q^i)(q) (\partial/\partial t)_{s(q)} \right), \end{aligned}$$

and thus,

$$p_i(\gamma(j_q^1 s)) = -(\partial(t \circ s)/\partial q^i)(q) = -t_i(j_q^1 s).$$

Similarly, we have

$$\begin{aligned} ((\mathbf{x} + i\mathbf{y}) \circ \varphi)(j_q^1 s, w) &= (\mathbf{x} + i\mathbf{y})([s(q), w]) \\ &= (\mathbf{x} + i\mathbf{y})(s_0(q) \cdot \exp(it \cdot s(q)), w) \\ &= (\mathbf{x} + i\mathbf{y})(s_0(q), \exp(irt \cdot s(q))w) \\ &= \exp(irt \cdot s(q))w, \end{aligned}$$

as follows from the very definition of $\mathbf{x}, \mathbf{y}$ in §2.3. Given a point $(\Gamma_q, [u, w]) \in C \times_M E$, $q = \pi(u)$, since Γ_q is a retract of $V_u P \subset T_u P$, we have

$$(\Gamma_q)|_{T_u P} = ((dt)_u - \lambda_i(dq^i)_u) \otimes (\partial/\partial t)_u.$$

Hence we can define a point $j_q^1 s \in J^1 P$, by imposing $s(q) = u$, $(\partial(t \circ s)/\partial q^i)(q) = \lambda_i$. Accordingly, Γ_q and $\gamma(j_q^1 s)$ coincide over $T_u P$, and since Γ_q and $\gamma(j_x^1 s)$ commute with the action of G , we can conclude that Γ_q and $\gamma(j_q^1 s)$ coincide at each point of the fibre $\pi^{-1}(q)$. Therefore, φ is surjective. Moreover, since $u = s(q) \cdot z = (s(q) \cdot \zeta) \cdot (\zeta^{-1}z)$, for every $\zeta \in U(1)$, from the definition of γ for every $X \in T_u P$, $u \in \pi^{-1}(q)$, we obtain

$$\begin{aligned} \gamma(j_q^1 (R_\zeta \circ s))(X) &= X - (R_{\zeta^{-1}z})_*(R_\zeta \circ s)_*(\pi_* X) \\ &= X - s_* \pi_* X = \gamma(j_q^1 s)(X). \end{aligned}$$

Hence

$$\begin{aligned} \varphi(j_q^1 s \cdot \zeta, \zeta^{-1} \cdot w) &= (\gamma(j_q^1 s \cdot \zeta), [s(q) \cdot \zeta, \zeta^{-1} \cdot w]) \\ &= (\gamma(j_q^1 s), [s(q), w]) = \varphi(j_q^1 s, w). \end{aligned}$$

Conversely, assume $\gamma(j_q^1 s) = \gamma(j_q^1 s')$, $[s(q), w] = [s'(q), w']$. Then there exists $\zeta \in U(1)$ such that $s'(q) = s(q) \cdot \zeta$, $w' = \zeta^{-1} \cdot w$. Hence $\gamma(j_q^1 (R_\zeta \circ s)) = \gamma(j_q^1 s')$, and since $(R_\zeta \circ s)(q) = s'(q) = u$, from the definition of γ , we obtain

$$(\partial(t \circ s')/\partial q^i)(q) = (\partial(t \circ R_\zeta \circ s)/\partial q^i)(q).$$

Thus, $j_q^1 (R_\zeta \circ s) = j_q^1 s \cdot \zeta = j_q^1 s'$. □

6 The interaction 1-form

6.1 The structure form. As is well-known (*e.g.*, see [12]), J^1P is endowed with a $V(P)$ -valued 1-form θ , called the structure form on the 1-jet bundle. For a $U(1)$ -bundle $\pi: P \rightarrow M$, the vertical bundle $V(P)$ is a trivial line bundle, so that we can think of the structure form as an ordinary (*i.e.*, real-valued) 1-form on J^1P . With the same notations as in §2.2, let (q^i, t, t_i) , $1 \leq i \leq m$, be the coordinate system induced on $\pi_{10}^{-1}(\pi^{-1}(W))$ by $(\pi^{-1}(W); q^i, t)$. Then, the local expression of the structure form is $\theta = dt - t_i dq^i$.

6.2 Proposition. *Let $z = x + iy$ be the complex coordinate on $\mathbb{C}$, and let $\varphi: J^1P \times \mathbb{C} \rightarrow C \times_M E$ be the submersion defined in §5.2. We have*

- (i) *The 1-form $\Im(\bar{z}dz) + r z \bar{z} \theta$ on $J^1P \times \mathbb{C}$, where θ denotes the structure form and $\Im$ the imaginary part, is φ -projectable onto $C \times_M E$; that is, there exists a unique 1-form α on $C \times_M E$ such that*

$$\varphi^*(\alpha) = \Im(\bar{z}dz) + r z \bar{z} \theta.$$

- (ii) *Furthermore, α is $\text{aut}P$ -invariant. It is called the interaction 1-form on the bundle $C \times_M E$, and its local expression on the coordinate system $(q^i, p_i, \mathbf{x}, \mathbf{y})$ (cf. §2.2, §2.3) is*

$$(7) \quad \alpha = \mathbf{x}d\mathbf{y} - \mathbf{y}d\mathbf{x} + r(\mathbf{x}^2 + \mathbf{y}^2) p_i dq^i.$$

Proof. (i) With the notations above the local expression of $\Im(\bar{z}dz) + r z \bar{z} \theta$ is $x dy - y dx + r(x^2 + y^2)(dt - t_i dq^i)$. Let $A^\bullet \in \mathfrak{X}(J^1P \times \mathbb{C})$ be the fundamental vector field associated to the standard basis $A \in \mathfrak{u}(1)$ under the action of $U(1)$ on $J^1P \times \mathbb{C}$ defined in §5.2. We have $A^\bullet = \partial/\partial t + r(y \partial/\partial x - x \partial/\partial y)$. Hence

- (a) $i_{A^\bullet}(\Im(\bar{z}dz) + r z \bar{z} \theta) = 0$,
(b) $i_{A^\bullet}d(\Im(\bar{z}dz) + r z \bar{z} \theta) = 0$.

From (a), (b) we obtain $L_{A^\bullet}(\Im(\bar{z}dz) + r z \bar{z} \theta) = 0$, or equivalently,

$$(c) \quad (R_{\exp(it)})^*(\Im(\bar{z}dz) + r z \bar{z} \theta) = \Im(\bar{z}dz) + r z \bar{z} \theta, \forall t \in \mathbb{R}.$$

Taking into account that $\ker \varphi_* = \langle A^\bullet \rangle$, by virtue of §5.2, from the equation (a) it follows $(\Im(\bar{z}dz) + r z \bar{z} \theta)(X) = 0$, for every φ -vertical tangent vector $X \in T_{(j_q^1 s, w)}(J^1P \times \mathbb{C})$. Moreover, from §6.2 and the equation (c) we obtain

$$(\Im(\bar{z}dz) + r z \bar{z} \theta)((R_z)_*X) = (\Im(\bar{z}dz) + r z \bar{z} \theta)(X).$$

This proves that there exists a unique 1-form α on $C \times_M E$, such that for every $X \in T(J^1P \times \mathbb{C})$, $\alpha(\varphi_*X) = (\Im(\bar{z}dz) + r z \bar{z} \theta)(X)$.

- (ii) By using the equations of φ in the formula (6), the local expression for α in the statement is easily deduced and, as a simple calculation shows, for every $X \in \text{aut}P$, from the formula (4) we obtain $L_{\bar{X}}\alpha = 0$. $\square$

6.3 Hermitian structure on E . As λ_r is a unitary representation, E is endowed with a canonical Hermitian structure $\langle \cdot, \cdot \rangle: E \times_M E \rightarrow \mathbb{C}$, which is uniquely determined by imposing $\langle [u, w_1], [u, w_2] \rangle = \bar{w}_1 w_2$, for all $u \in P$, $w_1, w_2 \in \mathbb{C}$, where we have used the notation introduced in §2.3 and $\bar{w}$ stands for the complex conjugate of $w \in \mathbb{C}$.

The geometric interpretation of the interaction 1-form is as follows:

6.4 Proposition. *With the hypotheses and notations as in §2.1, §2.3 and §6.3, for every connection Γ on $\pi: P \rightarrow M$, and every section $\xi \in \Gamma(M, E)$, we have*

$$(8) \quad (\sigma_\Gamma, \xi)^* \alpha = \Im \langle \xi, \nabla \xi \rangle,$$

where ∇ stands for the covariant derivative induced by Γ on E . Conversely, if β is a pr_2 -horizontal 1-form on $C \times_M E$, $pr_2: C \times_M E \rightarrow E$ being the projection onto the second factor, which satisfies the same property stated above then $\beta = \alpha$.

Proof. As is well-known (e.g., see [11, 3.5.2]) to each section $\xi \in \Gamma(M, E)$ we can associate a function $F_\xi: P \rightarrow \mathbb{C}$, by imposing for every $u \in P$, $\xi(\pi(u)) = [u, F_\xi(u)]$ (notations as in §2.3). If $\chi = (\mathbf{x} + i\mathbf{y}) \circ \xi$, locally, then on a trivializing open subset $\pi^{-1}(W) \simeq W \times U(1)$, we have

$$F_\xi(q^1, \dots, q^m; t) = \exp(-irt) \chi(q^1, \dots, q^m),$$

since $F_\xi(u \cdot z) = z^{-1} F_\xi(u)$, for every $u \in P$, $z \in U(1)$. Given a connection Γ on P , and a vector field X of M , then $X^* F_\xi$ is the function corresponding to the section $\nabla_X \xi$ (cf. [10, III.1.3]). We have $(\partial/\partial q^i)^* = \partial/\partial q^i - (p_i \circ \sigma_\Gamma)(\partial/\partial t)$. Hence

$$\begin{aligned} (\partial/\partial q^i)^* F_\xi &= \exp(-irt) \frac{\partial \chi}{\partial q^i} + ir \exp(-irt) (p_i \circ \sigma_\Gamma) \chi \\ &= \exp(-irt) \left(\frac{\partial \chi}{\partial q^i} + ir (p_i \circ \sigma_\Gamma) \chi \right), \end{aligned}$$

and accordingly,

$$(\mathbf{x} + i\mathbf{y}) \circ (\nabla_{\partial/\partial q^i} \xi) = \partial \chi / \partial q^i + ir (p_i \circ \sigma_\Gamma) \chi.$$

Therefore,

$$\langle \xi, \nabla_{\partial/\partial q^i} \xi \rangle = \bar{\chi} (\partial \chi / \partial q^i) + ir \chi \bar{\chi} (p_i \circ \sigma_\Gamma),$$

and the result follows from the local expression of α (see the formula (7) in §6.2). Moreover assume that a pr_2 -horizontal 1-form β satisfies the same property as the interaction 1-form. Locally we have $\beta = A d\mathbf{x} + B d\mathbf{y} + C_j dq^j$. Hence

$$\begin{aligned} (A \circ (\sigma_\Gamma, \xi)) \frac{\partial (\mathbf{x} \circ \xi)}{\partial q^j} + (B \circ (\sigma_\Gamma, \xi)) \frac{\partial (\mathbf{y} \circ \xi)}{\partial q^j} + C_j \circ (\sigma_\Gamma, \xi) \\ = (\mathbf{x} \circ \xi) \frac{\partial (\mathbf{y} \circ \xi)}{\partial q^j} - (\mathbf{y} \circ \xi) \frac{\partial (\mathbf{x} \circ \xi)}{\partial q^j} + r \chi \bar{\chi} p_j \circ \sigma_\Gamma. \end{aligned}$$

Since $\mathbf{x} \circ \xi$, $\mathbf{y} \circ \xi$ are arbitrary functions and for a given $q \in M$, $(\sigma_\Gamma(q), \xi(q))$ is an arbitrary point of the interaction bundle we can conclude $A = -\mathbf{y}$, $B = \mathbf{x}$, $C_j = r(\mathbf{x}^2 + \mathbf{y}^2)p_j$; thus finishing the proof. $\square$

6.5 Corollary. *We have $(\sigma_\Gamma, \xi)^* d\alpha = 2\Im\langle \nabla\xi, \nabla\xi \rangle + r\langle \xi, \xi \rangle (\sigma_\Gamma^* \omega_2)$, where ω_2 is the symplectic 2-form defined in §4.2.*

Proof. Since ∇ is compatible with the Hermitian metric of E , for every $X, Y \in \mathfrak{X}(M)$, we have

$$\begin{aligned} ((\sigma_\Gamma, \xi)^* d\alpha)(X, Y) &= d((\sigma_\Gamma, \xi)^* \alpha)(X, Y) = d(\Im\langle \xi, \nabla\xi \rangle)(X, Y) \\ &= X\Im\langle \xi, \nabla_Y \xi \rangle - Y\Im\langle \xi, \nabla_X \xi \rangle - \Im\langle \xi, \nabla_{[X, Y]} \xi \rangle \\ &= \Im(X\langle \xi, \nabla_Y \xi \rangle - Y\langle \xi, \nabla_X \xi \rangle - \langle \xi, \nabla_{[X, Y]} \xi \rangle) \\ &= 2\Im\langle \nabla_X \xi, \nabla_Y \xi \rangle + \Im\langle \xi, R(X, Y)\xi \rangle, \end{aligned}$$

where R is the curvature tensor of ∇ . Moreover from the definition of the coordinates p_i given in §2.2 and the local expression of ω_2 given in §4.2, it follows that pulling ω_2 back along the section $\sigma_\Gamma: M \rightarrow C$ one obtains the curvature form of Γ ; that is, $\sigma_\Gamma^* \omega_2 = d\omega_\Gamma = \Omega_\Gamma$. The result thus follows from the well-known fact on the theory of connections according to which R is the image of Ω_Γ with respect to the homomorphism of Lie algebras induced by the representation under consideration; *i.e.*, in our case $(\lambda_r)_*: \mathfrak{u}(1) \rightarrow \mathfrak{gl}(2, \mathbb{R})$, $(\lambda_r)_* \circ \Omega_\Gamma = r i \Omega_\Gamma = R$. $\square$

6.6 Physical meaning of the interaction form. Let us now consider a pseudo-Riemannian metric $\langle \cdot, \cdot \rangle_M$ on the base manifold M and a Lagrangian function $\mathcal{L} \in C^\infty(J^1(C \times_M E))$. As is well known (see [4, §5.1]), for every connection Γ on P and every section ξ of E , an $\text{ad}P$ -valued 1-form on M is defined: the current $J_{\Gamma, \xi}$, which appears in the inhomogeneous part of the Euler–Lagrange equations of $\mathcal{L}$. In particular, let $\mathcal{L}_{\text{YM}}$ be the classical abelian Yang–Mills–Higgs Lagrangian, that is

$$\mathcal{L}_{\text{YM}} = \frac{1}{2} \langle \nabla\xi, \nabla\xi \rangle_{M, E} - \frac{1}{2} m^2 \langle \xi, \xi \rangle_E - \frac{1}{2} \langle \Omega_\Gamma, \Omega_\Gamma \rangle_M,$$

where $\langle \cdot, \cdot \rangle_E$ is the Hermitian pairing in E defined in §6.3 and $\langle \cdot, \cdot \rangle_{M, E}$ denotes the pairing induced by $\langle \cdot, \cdot \rangle_E$ and the metric tensor $\langle \cdot, \cdot \rangle_M$ on E -valued differential forms of M . The corresponding current is given by (*cf.* [4, §5.2])

$$J_{\Gamma, \xi} = \frac{1}{2i} \left(\langle \xi, \nabla\xi \rangle_E - \overline{\langle \xi, \nabla\xi \rangle_E} \right).$$

From the geometrical interpretation of the form α (see §6.4 above) we obtain

$$J_{\Gamma, \xi} = (\sigma_\Gamma, \xi)^* \alpha.$$

In other words, the interaction form can be understood as a “universal” current of the Yang–Mills–Higgs action in the sense that its pull-back along a section (σ_Γ, ξ) of the interaction bundle provides the corresponding current.

7 The structure of $\mathcal{I}_{\text{gau}}(E)$

7.1 Proposition. *Let $\pi_E: E \rightarrow M$ be the vector bundle associated to a $U(1)$ -principal bundle $\pi: P \rightarrow M$ by a linear representation $\lambda: U(1) \rightarrow GL(V)$. We denote by $\mathcal{A}(V)$ the algebra of differential forms on V such that $i_{A^*}\Omega = 0$, $i_{A^*}d\Omega = 0$, where $A^* \in \mathfrak{X}(V)$ is the fundamental vector field associated to the standard basis $A \in \mathfrak{u}(1)$ under the linear representation λ . We have*

- (i) *For every $\Omega \in \mathcal{A}(V)$ of degree d , there exists a unique differential d -form Ω_E on E such that for every $X_1, \dots, X_d \in T_{(u,w)}(P \times V)$,*

$$\Omega_E((\pi_V)_*X_1, \dots, (\pi_V)_*X_d) = \Omega((pr_2)_*X_1, \dots, (pr_2)_*X_d),$$

where $\pi_V: P \times V \rightarrow E = (P \times V)/U(1)$ is the canonical projection and $pr_2: P \times V \rightarrow V$ is the projection onto the second factor.

- (ii) *Furthermore, Ω_E is $\text{Aut}(P)$ -invariant; i.e., for every $\Phi \in \text{Aut}(P)$, $\Phi_E^*\Omega_E = \Omega_E$. Hence we have a homomorphism of $\mathbb{Z}$ -graded algebras $\mathcal{A}(V) \rightarrow \mathcal{I}_{\text{aut}}(E)$, $\Omega \mapsto \Omega_E$.*

Proof. (i) The formula in the statement completely determines Ω_E . Behaving as in the proof of 6.2-(i), in order to prove the existence of Ω_E we only need to check that $pr_2^*\Omega$ is π_V -projectable which follows from the hypotheses.

(ii) Every $\Phi \in \text{Aut}P$ acts on an arbitrary associated vector bundle by the same formula as in §3.3; i.e., $\Phi_E([u, w]) = [\Phi(u), w]$, $u \in P$, $w \in V$, and it is easily seen that $\Phi_E \circ \pi_V = \pi_V \circ (\Phi \times 1_V)$. Hence for every $X_1, \dots, X_d \in T_{(u,w)}(P \times V)$, we have

$$\begin{aligned} & (\Phi_E^*\Omega_E)((\pi_V)_*X_1, \dots, (\pi_V)_*X_d) \\ &= \Omega((pr_2)_*(\Phi \times 1_V)_*X_1, \dots, (pr_2)_*(\Phi \times 1_V)_*X_d) \\ &= \Omega((pr_2)_*X_1, \dots, (pr_2)_*X_d) \\ &= \Omega_E((\pi_V)_*X_1, \dots, (\pi_V)_*X_d), \end{aligned}$$

since $pr_2 \circ (\Phi \times 1_V) = pr_2$, thus finishing the proof. $\square$

7.2 Notation. Let $f: \mathbb{C} \rightarrow \mathbb{R}$ be the map $f(z) = \bar{z}z$. It is obvious that $f, df \in \mathcal{A}(\mathbb{C})$, under the representation λ_r under consideration. Moreover, as a straightforward computation shows, we have

$$\mathcal{A}(\mathbb{C}) = f^*\Omega^\bullet(\mathbb{R}),$$

that is, f and df are the generators of $\mathcal{A}(\mathbb{C})$. According to §7.1-(ii), we thus have $f_E, df_E \in \mathcal{I}_{\text{aut}}(E)$. For the sake of simplicity, we shall write f instead of f_E . Note that f is the square of the norm of the Hermitian structure on E (cf. §6.3); i.e., $f([u, w]) = \langle [u, w], [u, w] \rangle = \bar{w}w$.

7.3 Proposition. *Assume M is connected and orientable by a volume form v_m . Then, $\mathcal{I}_{\text{gau}}(E)$ is generated over $(\pi_E, f)^*\Omega^\bullet(M \times \mathbb{R})$ by the globally defined forms $(\mathbf{x}dy - \mathbf{y}dx) \wedge \pi_E^*v_m$ and $d\mathbf{x} \wedge d\mathbf{y} \wedge \pi_E^*v_m$.*

Proof. Every differential s -form Ω_s on E can be written as follows

$$\Omega_s = h_I dq^I + h_J^x dq^J \wedge d\mathbf{x} + h_J^y dq^J \wedge d\mathbf{y} + h_K^{xy} dq^K \wedge d\mathbf{x} \wedge d\mathbf{y},$$

where $h_I, h_J^x, h_J^y, h_K^{xy} \in C^\infty(E)$, and I, J, K are multi-indices $L = (l_1, \dots, l_u)$ of degree $|L| = u$ equal to $|I| = s$, $|J| = s - 1$, and $|K| = s - 2$, and we set

$$dq^L = dq^{l_1} \wedge \dots \wedge dq^{l_u}.$$

If $X = gA^*$, then $X_E = -rg(\mathbf{y}\partial/\partial\mathbf{x} - \mathbf{x}\partial/\partial\mathbf{y})$ and by imposing the invariance condition $L_{X_E}\Omega_s = 0$ for $g = 1$ we obtain the following system of equations:

$$X_E(h_I) = 0, \quad X_E(h_J^x) - rh_J^y = 0, \quad X_E(h_J^y) + rh_J^x = 0, \quad X_E(h_K^{xy}) = 0.$$

Hence $h_I, h_K^{xy} \in (\pi_E, f)^*C^\infty(M \times \mathbb{R})$ and the second and third equations above yield

$$h_J^x = A_J \mathbf{x} + B_J \mathbf{y}, \quad h_J^y = A_J \mathbf{y} - B_J \mathbf{x},$$

for certain functions $A_J, B_J \in (\pi_E, f)^*C^\infty(M \times \mathbb{R})$. Accordingly, we have

$$\begin{aligned} \Omega_s = h_I dq^I + A_J dq^J \wedge (\mathbf{x}d\mathbf{x} + \mathbf{y}d\mathbf{y}) + B_J dq^J \wedge (\mathbf{y}d\mathbf{x} - \mathbf{x}d\mathbf{y}) \\ + h_K^{xy} dq^K \wedge d\mathbf{x} \wedge d\mathbf{y}. \end{aligned}$$

Again by imposing the invariance condition for an arbitrary coefficient g , we obtain

$$(\mathbf{x}^2 + \mathbf{y}^2)B_J dq^J \wedge dg + h_K^{xy} dq^K \wedge dg \wedge (\mathbf{x}d\mathbf{x} + \mathbf{y}d\mathbf{y}) = 0.$$

Therefore, if $|J| < m$, then $B_J = 0$, and if $|K| < m$, then $h_K^{xy} = 0$ and the result follows. $\square$

7.4 Corollary. *With the same notations as in §7.1, §7.3 we have*

$$\mathcal{I}_{\text{aut}}(E) = f^*\Omega^\bullet(\mathbb{R}) \simeq \mathcal{A}(\mathbb{C}).$$

8 Structure of $\mathcal{I}_{\text{gau}}(C \times_M E)$

8.1 Notation. Let $\mathcal{K}$ be the subalgebra of $\Omega^\bullet(C \times_M E)$ defined by

$$\mathcal{K} = (\pi_E \circ pr_2, f \circ pr_2)^*\Omega^\bullet(M \times \mathbb{R}),$$

with $pr_1: C \times_M E \rightarrow C$, $pr_2: C \times_M E \rightarrow E$ being the canonical projections onto the factors. Roughly speaking, a form ξ belongs to $\mathcal{K}$ if and only if its local expression in a coordinate system on $C \times_M E$ as in § 2.2, §2.3, is

$$\xi = h_{i_1 \dots i_s} dq^{i_1} \wedge \dots \wedge dq^{i_s} + g_{j_1 \dots j_{s-1}} dq^{j_1} \wedge \dots \wedge dq^{j_{s-1}} \wedge df$$

where

$$h_{i_1 \dots i_s} = h_{i_1 \dots i_s}(q^1, \dots, q^n, \mathbf{x}^2 + \mathbf{y}^2), \quad g_{j_1 \dots j_{s-1}} = g_{j_1 \dots j_{s-1}}(q^1, \dots, q^n, \mathbf{x}^2 + \mathbf{y}^2)$$

are differentiable mappings depending on M and the hermitian norm of E .

This algebra $\mathcal{K}$, together with the contact form α and the symplectic form $pr_1^* \omega_2$, allows us to state the characterization of $\mathcal{I}_{\text{gau}}(C \times_M E)$; more precisely:

8.2 Theorem. *Let $\pi: P \rightarrow M$ be a $U(1)$ -principal bundle, let $p: C \rightarrow M$ be the bundle of connections of P , and let $\pi_E: E \rightarrow M$ be the vector bundle associated to P by the linear representation λ_r , $r \in \mathbb{N}$, of $U(1)$ on $\mathbb{C}$ given by $\lambda_r(z)(w) = z^r w$, $z \in U(1)$, $w \in \mathbb{C}$. With the above hypotheses and notations the forms α , $d\alpha$, ω_2 , generate the algebra of gauge-invariant differential forms on the interaction bundle over the algebra $\mathcal{K}$, where α is the interaction 1-form defined in §6.2, ω_2 is the symplectic structure on C defined in §4.2; that is,*

$$(9) \quad \mathcal{I}_{\text{gau}}(C \times_M E) = \mathcal{K}[\alpha, d\alpha, pr_1^* \omega_2].$$

8.3 Lemma. *A differential form Ω on $C \times_M E$ is $\text{aut}P$ -invariant (resp. gauge invariant) if and only if $\varphi^* \Omega$ is $\text{aut}P$ -invariant (resp. gauge invariant) on $J^1 P \times \mathbb{C}$. Moreover, $\mathcal{I}_{\text{aut}}(C \times_M E)$ (resp. $\mathcal{I}_{\text{gau}}(C \times_M E)$) is isomorphic to the algebra of $\text{aut}P$ -invariant (resp. gauge-invariant) differential forms Ξ on $J^1 P \times \mathbb{C}$ such that:*

$$(i) \quad i_{A^\bullet} \Xi = 0,$$

$$(ii) \quad L_{A^\bullet} \Xi = 0.$$

Proof of Lemma 8.3. The first part of the statement follows from the fact that $(X^{(1)}, 0) \in \mathfrak{X}(J^1 P \times \mathbb{C})$ is projectable onto $\bar{X}$ for every $X \in \text{aut}P$, and the second part follows taking into account that the fibres of φ are connected. $\square$

8.4 Lemma. *The algebra of gauge-invariant forms on $J^1 P \times \mathbb{C}$ is given by*

$$(\pi_1 \times 1_{\mathbb{C}})^* \Omega^\bullet(M \times \mathbb{C}) = [\theta, d\theta],$$

that is, every gauge s -form Ξ on $J^1 P \times \mathbb{C}$ can be written as

$$(10) \quad \Xi = \Xi_s + \Xi_{s-1} \wedge dx + \Xi'_{s-1} \wedge dy + \Xi_{s-2} \wedge dx \wedge dy,$$

where Ξ_s , Ξ_{s-1} , Ξ'_{s-1} , Ξ_{s-2} are forms of degree s , $s-1$, $s-1$, $s-2$, respectively on $J^1 P \times \mathbb{C}$, which are polynomials in θ , $d\theta$ whose coefficients are $(\pi_1 \circ pr_1)$ -horizontal differential forms, $pr_1: J^1 P \times \mathbb{C} \rightarrow J^1 P$ being the canonical projection onto the first factor.

Proof of Lemma 8.4. First, let us study the gauge invariance on $J^1 P$. Taking into account the local expression of the structure form $\theta = dt - t_i dq^i$ in a

coordinate system (q^i, t, t_i) of J^1P , it is easy to see that every s -form Ξ on J^1P can be locally written as

$$\begin{aligned} \Xi = & \sum_{|I|+|J|=s} f_{IJ} (dq^1)^{i_1} \wedge \cdots \wedge (dq^n)^{i_n} \wedge (dt_1)^{j_1} \wedge \cdots \wedge (dt_n)^{j_n} \wedge \theta \\ & + \sum_{|K|+|L|=s} h_{KL} (dq^1)^{k_1} \wedge \cdots \wedge (dq^n)^{k_n} \wedge (dt_1)^{l_1} \wedge \cdots \wedge (dt_n)^{l_n}, \end{aligned}$$

with $f_{IJ}, h_{KL} \in C^\infty(J^1P)$, where $I = (i_1, \dots, i_n)$, $J = (j_1, \dots, j_n)$, $K = (k_1, \dots, k_n)$, $L = (l_1, \dots, l_n)$, are boolean multi-indices; *i.e.*, $I, J, K, L \in \{0, 1\}^n$, and $|I| = i_1 + \cdots + i_n$. Following the notations in §3.2, if $X = g(\partial/\partial t)$, $g \in C^\infty(M)$, is the expression of a gauge field on P , its lifting to the jet bundle is

$$X^{(1)} = g \frac{\partial}{\partial t} + \frac{\partial g}{\partial q^i} \frac{\partial}{\partial t_i}.$$

If we let $g = 1$, the condition of gauge invariance tells us the following:

$$\begin{aligned} 0 = & L_{X^{(1)}} \Xi = \frac{\partial f_{IJ}}{\partial t} (dq^1)^{i_1} \wedge \cdots \wedge (dq^n)^{i_n} \wedge (dq_1)^{j_1} \wedge \cdots \wedge (dt_n)^{j_n} \wedge \theta \\ & + \frac{\partial h_{KL}}{\partial t} (dq^1)^{k_1} \wedge \cdots \wedge (dq^n)^{k_n} \wedge (dq_1)^{l_1} \wedge \cdots \wedge (dt_n)^{l_n}. \end{aligned}$$

Hence $\partial f_{IJ}/\partial t = 0$, $\partial h_{KL}/\partial t = 0$. For $g = q^a$, $a = 1, \dots, n$, we obtain $\partial f_{IJ}/\partial q^a = 0$, $\partial h_{KL}/\partial q^a = 0$ and we conclude that $f_{IJ}, h_{KL} \in C^\infty(M)$, $\forall I, J, K, L$. Now, let us consider $g = \frac{1}{2}(q^1)^2$ in the definition of X . The condition of gauge invariance on the fiber $p^{-1}(q_0)$ yields

$$\begin{aligned} 0 = & L_{X^{(1)}} \Xi|_{p^{-1}(q_0)} = f_{IJ} (dq^1)^{i_1} \wedge \cdots \wedge (dq^n)^{i_n} \wedge (dq_1)^{j_1} \wedge \cdots \wedge (dt_n)^{j_n} \wedge \theta \\ & + h_{KL} (dq^1)^{k_1} \wedge \cdots \wedge (dq^n)^{k_n} \wedge (dq_1)^{l_1} \wedge \cdots \wedge (dt_n)^{l_n}. \end{aligned}$$

Hence if J is such that $j_1 = 1$, then $i_1 = 1$, and if $l_1 = 1$ then $k_1 = 1$. In general, by considering an arbitrary index $1 \leq a \leq n$ and $g = \frac{1}{2}(q^a)^2$ we conclude that $j_a = 1$ implies $i_a = 1$ and similarly, $l_a = 1$ implies $k_a = 1$. Therefore Ξ can be rewritten as

$$\begin{aligned} \Xi = & \sum_{|I|+2|J|=s} \tilde{f}_{IJ} (dq^1)^{i_1} \wedge \cdots \wedge (dq^n)^{i_n} \wedge (dq^1 \wedge dt_1)^{j_1} \wedge \cdots \wedge (dq^n \wedge dt_n)^{j_n} \wedge \theta \\ & + \sum_{|K|+2|L|=s} \tilde{h}_{KL} (dq^1)^{k_1} \wedge \cdots \wedge (dq^n)^{k_n} \wedge (dq^1 \wedge dt_1)^{l_1} \wedge \cdots \wedge (dq^n \wedge dt_n)^{l_n}, \end{aligned}$$

with $i_u + j_u \leq 1, k_u + l_u \leq 1$ for $u = 1, \dots, n$.

If we take $g = q^1 \cdot q^a$, $1 < a \leq n$, in the definition of X , the gauge invariance

condition now says

$$\begin{aligned}
0 = & L_{X^{(1)}} \Xi|_{p^{-1}(q_0)} = \tilde{f}_{IJ} (dq^1)^{i_1} \wedge \cdots \wedge (dq^n)^{i_n} \wedge (dq^1 \wedge dq^a)^{j_1} \wedge \cdots \wedge \\
& (dq^n \wedge dt_n)^{j_n} \wedge \theta + \tilde{f}_{IJ} (dq^1)^{i_1} \wedge \cdots \wedge (dq^n)^{i_n} \wedge (dq^1 \wedge dt_1)^{j_1} \wedge \cdots \wedge \\
& (dq^a \wedge dq^1)^{j_a} \wedge \cdots \wedge (dq^n \wedge dt_n)^{j_n} \wedge \theta + \tilde{h}_{KL} (dq^1)^{k_1} \wedge \cdots \wedge \\
& (dq^n)^{k_n} \wedge (dq^1 \wedge dq^a)^{l_1} \wedge \cdots \wedge (dq^n \wedge dt_n)^{l_n} + \tilde{h}_{KL} (dq^1)^{k_1} \wedge \cdots \wedge \\
& (dq^n)^{k_n} \wedge (dt_1)^{l_1} \wedge \cdots \wedge (dq^a \wedge dq^1)^{l_a} \wedge \cdots \wedge (dq^n \wedge dt_n)^{l_n}.
\end{aligned}$$

That is, $\tilde{f}_{IJ} - \tilde{f}_{IJ'} = 0$ whenever

$$J = (1, j_2, \dots, j_{a-1}, 0, j_{a+1}, \dots, j_n), \quad J' = (0, j_2, \dots, j_{a-1}, 1, j_{a+1}, \dots, j_n),$$

and $\tilde{h}_{KL} - \tilde{h}_{KL'} = 0$, whenever

$$L = (1, l_2, \dots, l_{a-1}, 0, l_{a+1}, \dots, l_n), \quad L' = (0, l_2, \dots, l_{a-1}, 1, l_{a+1}, \dots, l_n).$$

Accordingly, if Ξ contains a summand of the form $\omega_{s-2} \wedge dq^a \wedge dt_a$, where $a = 1, \dots, n$ is an arbitrary fixed index, then Ξ contains the summand $\omega_{s-2} \wedge dq^1 \wedge dt_1$ and conversely. Recalling that $d\theta = dq^i \wedge dt_i$, we have that Ξ is a polynomial of θ and $d\theta$; i.e., $\mathcal{I}_{\text{gau}}(J^1 P) = \pi_1^* \Omega^\bullet(M)[\theta, d\theta]$.

Finally, we note that the gauge group $\text{Gau} P$ acts trivially on $\mathbb{C}$, that is, the action on $J^1 P \times \mathbb{C}$ is only defined on the jet bundle. Hence, the result follows. $\square$

Proof of Theorem 8.2. According to the previous lemmas we are led to study the conditions of φ -projectability $i_{A^\bullet} \Xi = 0$, $L_{A^\bullet} \Xi = 0$, of a form

$$(11) \quad \Xi = \Xi_s + \Xi_{s-1} \wedge dx + \Xi'_{s-1} \wedge dy + \Xi_{s-2} \wedge dx \wedge dy,$$

with $\Xi_s, \Xi_{s-1}, \Xi'_{s-1}, \Xi_{s-2}$ as in Lemma 8.4. The vector field $A^\bullet$ is as follows:

$$A^\bullet = \frac{\partial}{\partial t} + r \left(y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right).$$

Hence

$$\begin{aligned}
i_{A^\bullet} \Xi = & i_{\partial/\partial t} \Xi_s + (i_{\partial/\partial t} \Xi_{s-1}) \wedge dx + (-1)^{s-1} r y \Xi'_{s-1} \\
& (i_{\partial/\partial t} \Xi'_{s-1}) \wedge dy - (-1)^{s-1} r x \Xi_{s-1} + (i_{\partial/\partial t} \Xi_{s-2}) \wedge dx \wedge dy \\
& + (-1)^s r \Xi_{s-2} \wedge (y dy + x dx)
\end{aligned}$$

vanishes if and only if

$$\begin{aligned}
i_{\partial/\partial t} \Xi_{s-2} &= 0, \\
i_{\partial/\partial t} \Xi_{s-1} + (-1)^s r x \Xi_{s-2} &= 0, \\
i_{\partial/\partial t} \Xi'_{s-1} + (-1)^s r y \Xi_{s-2} &= 0, \\
i_{\partial/\partial t} \Xi_s + (-1)^{s-1} r (y \Xi_{s-1} - x \Xi'_{s-1}) &= 0.
\end{aligned}$$

As $\theta(\partial/\partial t) = 1$, from the first equation above we conclude that Ξ_{s-2} depends only on $d\theta$. From the last three equations we obtain

$$(12) \quad \begin{aligned} \Xi_{s-1} &= (-1)^{s-1} r x \theta \wedge \Xi_{s-2} + \xi_{s-1}, \\ \Xi'_{s-1} &= (-1)^{s-1} r y \theta \wedge \Xi_{s-2} + \xi'_{s-1}, \\ \Xi_s &= (-1)^s r \theta \wedge (y \xi_{s-1} - x \xi'_{s-1}) + \xi_s, \end{aligned}$$

where ξ_{s-1} , ξ'_{s-1} , ξ_s are polynomials in $d\theta$ whose coefficients are $(\pi_1 \circ pr_1)$ -horizontal forms. Moreover, substituting the expressions above for Ξ_{s-1} , Ξ'_{s-1} , Ξ_s into the formula (11) and simplifying it, we have

$$\begin{aligned} L_{A\bullet} \Xi &= L_{A\bullet} \xi_s + (-1)^{s-1} r \theta \wedge (r y \xi'_{s-1} + x L_{A\bullet} \xi'_{s-1} + r x \xi_{s-1} - y L_{A\bullet} \xi_{s-1}) \\ &\quad - (-1)^s r \theta \wedge L_{A\bullet} \Xi_{s-2} \wedge (x dx + y dy) + L_{A\bullet} \xi_{s-1} \wedge dx \\ &\quad + L_{A\bullet} \xi'_{s-1} \wedge dy + r (\xi_{s-1} \wedge dy - \xi'_{s-1} \wedge dx) + L_{A\bullet} \Xi_{s-2} \wedge dx \wedge dy. \end{aligned}$$

Hence $L_{A\bullet} \Xi = 0$ if and only if,

$$\begin{aligned} L_{A\bullet} \Xi_{s-2} &= 0, \\ L_{A\bullet} \xi_s &= 0, \\ L_{A\bullet} \xi_{s-1} - r \xi'_{s-1} &= 0, \\ L_{A\bullet} \xi'_{s-1} + r \xi_{s-1} &= 0. \end{aligned}$$

As $d\theta$ does not depend on the variable t , the first two equations above tell us that the coefficients of the differential forms Ξ_{s-2} , ξ_s are invariant under rotations around the origin of the plane $\mathbb{C}$, that is, their dependence on x, y is via the mapping $f = x^2 + y^2$. In the other hand, the last two equations can be seen as a system of partial differential equations and is not difficult to check that this system is completely integrable and its solution is

$$(13) \quad \begin{cases} \xi_{s-1} = x \zeta_{s-1} + y \zeta'_{s-1} \\ \xi'_{s-1} = y \zeta_{s-1} - x \zeta'_{s-1} \end{cases}$$

ζ_{s-1} , ζ'_{s-1} being polynomic $s-1$ forms on $dq's$ and $d\theta$ and with coefficients mappings on $q^1, \dots, q^n, x^2 + y^2$.

Taking into account formulas (11), (12) and (13), we finally obtain

$$\begin{aligned} \Xi &= \xi_s - \zeta'_{s-1} \wedge (r(y^2 + x^2)\theta - y \wedge dx + x dy) + \zeta_{s-1} \wedge (x dx + y dy) \\ &\quad \Xi_{s-2} \wedge (r x dx \wedge \theta + r y dy \wedge \theta + dx \wedge dy), \end{aligned}$$

which projects, by virtue of the local expression of the contact form α (cf. §6.2), onto the form of $C \times_M E$,

$$\xi_s - \frac{1}{2} r (\mathbf{x}^2 + \mathbf{y}^2) \Xi_{s-2} \wedge \omega_2 - \zeta'_{s-1} \wedge \alpha + \frac{1}{2} \zeta_{s-1} \wedge df + \frac{1}{2} \Xi_{s-2} \wedge d\alpha,$$

thus finishing the proof. $\square$

8.5 Corollary. *The algebra of autP-invariant forms on $C \times_M E$ is given by*

$$\mathcal{I}_{\text{aut}}(C \times_M E) = f^* \Omega^\bullet(\mathbb{R})[\alpha, d\alpha, pr_1^* \omega_2].$$

Proof. By virtue of 6.2–(ii) and 7.1–(ii), respectively the form α and the function f are autP-invariant. From Theorem 8.2 and [8, Theorem 3.1] the result thus follows. $\square$

9 Concluding remarks

9.1 Remark. The fundamental relation among α , f and the symplectic form ω_2 on $C \times_M E$, is

$$df \wedge \alpha = f(d\alpha - r f \omega_2).$$

This follows from 7.1 and formula (7) in 6.2, taking into account the local expression of $\omega_2 = dp_i \wedge dq^i$.

9.2 Notations. Let Z be the zero section of E ; i.e., $Z = f^{-1}(0)$. We set $Z_C = C \times_M Z$, $\mathcal{O} = C \times_M E - Z_C$. It follows that $\mathcal{O}$ is a dense open subset of the interaction bundle. We denote by $(\pi_E \circ pr_2, f \circ pr_2)_{\mathcal{O}} : \mathcal{O} \rightarrow M \times \mathbb{R}$ the restriction of $(\pi_E \circ pr_2, f \circ pr_2)$ to $\mathcal{O}$.

9.3 Remark. From 9.1 it follows that $d\alpha|_{\mathcal{O}} = (f^{-1}df \wedge \alpha + r f \omega_2)|_{\mathcal{O}}$. Hence,

$$\mathcal{I}_{\text{gau}}(\mathcal{O}) = \mathcal{K}[\alpha, pr_1^* \omega_2].$$

9.4 Remark. Also, in $\mathcal{K}[\alpha, d\alpha, \omega_2]$, we only need to take one factor for $d\alpha$, since

$$d\alpha \wedge d\alpha = r \omega_2 \wedge (r f \omega_2 + 2df \wedge \alpha),$$

and the factor $\alpha \wedge d\alpha$ does not appear either since $\alpha \wedge d\alpha = r f \alpha \wedge \omega_2$. For the sake of simplicity we shall usually identify $\Omega^\bullet(C)$ with $pr_1^* \Omega^\bullet(C)$, and $\Omega^\bullet(E)$ with $pr_2^* \Omega^\bullet(E)$. Accordingly the general expression for a gauge-invariant n -form Ω_n on the interaction bundle is

$$\Omega_n = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \eta_{n-2j} \wedge (\omega_2)^j + \sum_{C=0}^{\lfloor \frac{n-1}{2} \rfloor} \eta'_{n-1-2j} \wedge (\omega_2)^j \wedge \alpha + \sum_{j=0}^{\lfloor \frac{n-2}{2} \rfloor} \eta''_{n-2-2j} \wedge (\omega_2)^j \wedge d\alpha,$$

where $\eta, \eta', \eta'' \in (\pi_E, f)^* \Omega^\bullet(M \times \mathbb{R})$. Also note that for $n > 2m$, $\Omega_n = 0$, necessarily.

9.5 Remark. On $\mathcal{O}$, a proof of Corollary 6.5 can be also given by using the formula of §9.1. In fact, if ξ is a non-vanishing section of E on an open subset

$U \subset M$, on U we can define an ordinary 1-form by setting $\nabla_X \xi = \eta(X)\xi$, and taking into account that $\xi^*(df) = d\langle \xi, \xi \rangle$, we have

$$\begin{aligned}
(\sigma_\Gamma, \xi)^*(df \wedge \alpha)(X, Y) &= ((d\langle \xi, \xi \rangle) \wedge (\Im \langle \xi, \nabla \xi \rangle))(X, Y) \\
&= X\langle \xi, \xi \rangle \cdot \Im \langle \xi, \nabla_Y \xi \rangle - Y\langle \xi, \xi \rangle \cdot \Im \langle \xi, \nabla_X \xi \rangle \\
&= \Im (X\langle \xi, \xi \rangle \cdot \Im \langle \xi, \nabla_Y \xi \rangle - Y\langle \xi, \xi \rangle \cdot \Im \langle \xi, \nabla_X \xi \rangle) \\
&= \Im (\langle \nabla_X \xi, \xi \rangle \langle \xi, \nabla_Y \xi \rangle - \langle \nabla_Y \xi, \xi \rangle \langle \xi, \nabla_X \xi \rangle) \\
&= \Im \left(\left(\overline{\eta(X)}\eta(Y) - \eta(X)\overline{\eta(Y)} \right) \langle \xi, \xi \rangle^2 \right) \\
&= 2\Im \left(\overline{\eta(X)}\eta(Y) \right) \langle \xi, \xi \rangle^2 \\
&= 2(\Im \langle \nabla_X \xi, \nabla_Y \xi \rangle) \langle \xi, \xi \rangle \\
&= \langle \xi, \xi \rangle [((\sigma_\Gamma, \xi)^* d\alpha)(X, Y) - r \langle \xi, \xi \rangle (\sigma_\Gamma^* \omega_2)(X, Y)].
\end{aligned}$$

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