k -cosymplectic manifolds and Lagrangian field theories

Manuel de León,^{a)} Eugenio Merino,^{b)} and Modesto Salgado^{c)}

Consejo Superior de Investigaciones Científicas, Instituto de Matemáticas y Física Fundamental, Serrano 123, 28006 Madrid, Spain

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A geometrical description of classical field theories of first order is given. The underlying k -cosymplectic structure permits to derive the corresponding field equations. © 2001 American Institute of Physics. [DOI: 10.1063/1.1360997]

I. INTRODUCTION

As is well-known, the natural arena to study mechanics is symplectic geometry. One interesting problem is to extend this geometric framework for the case of classical field theories. Several different approaches were developed in the past forty years, and according to them, the field equations are presented in different geometric formats: jet fields, Ehresmann connections, or multivector fields (we refer the reader to Refs. 1–5 and the references therein; see also Refs. 6–21 for recent developments).

In a recent paper²² we have constructed a geometric description of classical field theories in terms of k -cosymplectic manifolds. A k -cosymplectic manifold is a natural generalization of a cosymplectic manifold, which is the geometrical setting for nonautonomous mechanical systems (see Refs. 23–25).

In the present paper, which can be viewed as a natural extension of Ref. 22, we consider the Lagrangian description of first order classical field theories. The field equations are then derived by using the canonical almost stable k -tangent structure on the stable k -tangent bundle $\mathbf{R}^k \times T_k^1 Q$ of a manifold Q . From the Lagrangian function L , we construct in an intrinsical way the Poincaré–Cartan forms $(\omega_L)_A$, $1 \leq A \leq k$, which determine a k -cosymplectic structure if L is regular. The k -cosymplectic structure provides us a k -vector field whose integral sections are the solutions of the field equations. The integrability is not assured, but it can be checked in terms of an arbitrary solution.

If the Lagrangian is not regular, then we develop a constraint algorithm which, if the field equations are compatible, yields a final constraint submanifold where a solution exists. This algorithm is very similar to that constructed in Ref. 25.

This Lagrangian description is of course completely equivalent to the Hamiltonian one developed in our precedent paper by means of a convenient Legendre transformation.

II. k -VECTOR FIELDS

Let M be a differentiable manifold of dimension m and $\mathbf{R}^k$ the k -dimensional Euclidean space with coordinates $t = (t^A) = (t^1, \dots, t^k)$. Let $J_0^1(\mathbf{R}^k, M)$ be the $(k+1)m$ -dimensional manifold of one jet of mappings from $\mathbf{R}^k$ to M at the origin of $\mathbf{R}^k$ with elements denoted by $j_0^1 \sigma$. $J_0^1(\mathbf{R}^k, M)$ is called the tangent bundle of k^1 -velocities in Ref. 26.

Let $\{\mathbf{r}_1, \dots, \mathbf{r}_k\}$ be the canonical basis of $\mathbf{R}^k$. The manifold $J_0^1(\mathbf{R}^k, M)$ can be canonically identified with the Whitney sum $T_k^1 M = TM \oplus \dots \oplus TM$ of k copies of TM via the diffeomorphism,

$$J_0^1(\mathbf{R}^k, M) \rightarrow T_k^1 M = TM \oplus \dots \oplus TM$$

^{a)}Electronic mail: mdeleon@imaff.cfmac.csic.es

^{b)}Electronic mail: uxioomer@eps.cdf.udc.es

^{c)}Electronic mail: modesto@zmat.usc.es

$$j_0^1 \sigma \rightarrow (v_1, \dots, v_k)$$

defined by

$$v_A = \left. \frac{d\sigma_A}{ds} \right|_{s=0}, \quad s \in \mathbf{R}, \quad 1 \leq A \leq k,$$

where $\sigma_A(s) = \sigma(s \mathbf{r}_A)$.

Let $\tau_M: T_k^1 M \rightarrow M$ be the canonical projection. If (x^i) are local coordinates on $U \subseteq M$ then the induced local coordinates (x^i, x_A^i) , $1 \leq i \leq m$, $1 \leq A \leq k$, on $\tau_M^{-1}(U) \equiv T_k^1 U$ are given by

$$x^i(j_0^1 \sigma) = x^i(\sigma(0)), \quad x_A^i(j_0^1 \sigma) = \left. \frac{d}{ds} (x^i \circ \sigma^A) \right|_{s=0} = v_A(x^i). \quad (1)$$

Definition II.1: A section $\mathbf{X}: M \rightarrow T_k^1 M$ of the projection $\tau_M: T_k^1 M \rightarrow M$ will be called a k -vector field on M .

Since $T_k^1 M$ is the Whitney sum of k copies of TM we deduce that a k -vector field $\mathbf{X}$ defines k vector fields $\{X_1, \dots, X_k\}$ on M by projecting $\mathbf{X}$ onto every factor. We shall identify $\mathbf{X}$ with the k -tuple $(X_1, \dots, X_k)$.

Definition II.2: An integral section of the k -vector field $(X_1, \dots, X_k)$ passing through a point $x \in M$ is a map $\sigma: U_0 \subset \mathbf{R}^k \rightarrow M$, defined on some neighborhood U_0 of $0 \in \mathbf{R}^k$, and satisfying

$$\sigma(0) = x, \quad d\sigma(t) \left(\left. \frac{\partial}{\partial t^A} \right|_t \right) = X_A(\sigma(t)), \quad 1 \leq A \leq k, \quad \text{for all } t \in U_0.$$

We say that a k -vector field $(X_1, \dots, X_k)$ on M is integrable if there is an integral section passing through each point of M .

Remark II.3: Let us consider the trivial bundle $\pi: E = \mathbf{R}^k \times M \rightarrow \mathbf{R}^k$. A jet field γ on π (see Ref. 5) is a section of the projection $\pi_{1,0}: J^1 \pi \equiv \mathbf{R}^k \times T_k^1 M \rightarrow E \equiv \mathbf{R}^k \times M$. We identify each k -vector field $\mathbf{X}$ on M with the jet field $\gamma = (id_{\mathbf{R}^k}, \mathbf{X})$, that is $\gamma(t, x) = (t, X_1(x), \dots, X_k(x))$. The integral sections of the jet field γ (see Ref. 5) correspond to the solutions of the k -vector field $\mathbf{X}$ (see Ref. 27). For this reason the solutions of $\mathbf{X}$ are called its *integral sections*.

Let us remark that if σ is an integral section of a k -vector field $(X_1, \dots, X_k)$ then each curve on M defined by $\sigma_A = \sigma(s \mathbf{r}_A)$ is an integral curve of the vector field X_A on M .

However, given k integral curves of $X_1, \dots, X_k$, respectively, it is not possible in general to reconstruct an integral section of $(X_1, \dots, X_k)$.

Proposition II.4: Assume that the vector fields $\{X_1, \dots, X_k\}$ on M are linearly independent. Then, the k -vector field $(X_1, \dots, X_k)$ is integrable if and only if the distribution generated by $\{X_1, \dots, X_k\}$ is integrable.

III. THE CANONICAL STABLE k -TANGENT STRUCTURE

Let Q be a manifold of dimension n with local coordinates (q^i) and let (q^i, v_A^i) be the induced coordinates on $T_k^1 Q$. Let F be a tensor field of type $(1,1)$ on Q such that $F = \sum_{i,j} F_j^i (\partial/\partial q^i) \otimes dq^j$. Then the A -lift F^A of F to $T_k^1 Q$ is the $(1,1)$ -tensor field with local expression:

$$F^A = \sum_{i,j} F_j^i \left(\frac{\partial}{\partial v_A^i} \right) \otimes dq^j$$

(see Ref. 26 for further details about the intrinsic construction). If $I_M = \sum_i (\partial/\partial q^i) \otimes dq^i$ is the identity tensor field on M then for each $A \in \{1, 2, \dots, k\}$, its A -lifting defines the tensor field $\tilde{J}_A = I^A$ locally given by

$$\tilde{J}_A = \sum_i \frac{\partial}{\partial v_A^i} \otimes dq^i. \quad (2)$$

The family $(\tilde{J}_1, \dots, \tilde{J}_k)$ is called the *canonical k -tangent structure* of $T_k^1 M$ (see Ref. 28).

Let $J^1(\mathbf{R}^k, Q)$ be the $k + (k + 1)n$ -dimensional manifold of one jets from $\mathbf{R}^k$ to Q , with elements denoted by $j_t^1 \phi$. This manifold can be identified with the manifold $J^1 \pi$ of one jets of sections of the trivial bundle $\pi: \mathbf{R}^k \times Q \rightarrow \mathbf{R}^k$.

$J^1 \pi$ is diffeomorphic to $\mathbf{R}^k \times T_k^1 Q$ by composing the following diffeomorphisms:

$$J^1 \pi \rightarrow \mathbf{R}^k \times J_0^1(\mathbf{R}^k, Q) \rightarrow \mathbf{R}^k \times T_k^1 Q,$$

$$j_t^1 \phi \rightarrow (t, j_0^1 \phi_t) \rightarrow (t, v_1, \dots, v_k),$$

where

$$\phi_t(t') = \phi(t + t'), \quad t' \in \mathbf{R}^k, \quad v_A = \frac{d}{ds}(\phi_t^A)|_{s=0}, \quad s \in \mathbf{R}, \quad 1 \leq A \leq k,$$

and $\phi_t^A(s) = \phi_t(s \mathbf{r}_A)$.

Let $\tau: \mathbf{R}^k \times T_k^1 Q \rightarrow Q$ be the canonical projection. If (q^i) are local coordinates on $U \subseteq Q$ then the induced local coordinates (t^A, q^i, v_A^i) , $1 \leq i \leq n$, $1 \leq A \leq k$, on $\tau^{-1}(U) \equiv \mathbf{R}^k \times T_k^1 U$ are given by

$$t^A(j_t^1 \phi) = t^A, \quad q^i(j_t^1 \sigma) = q^i(\phi_t(0)), \quad v_A^i(j_t^1 \phi) = \frac{d}{ds}(q^i \circ \phi_t^A)|_{s=0} = v_A(q^i).$$

Therefore we obtain $\tau(t^A, q^i, v_A^i) = (q^i)$.

On $\mathbf{R}^k \times T_k^1 Q$ there exist a family of k tensor fields J_A of type (1,1) defined by

$$J_A = \frac{\partial}{\partial t^A} \otimes dt^A + \tilde{J}_A = \frac{\partial}{\partial t^A} \otimes dt^A + \sum_{i=1}^n \frac{\partial}{\partial v_A^i} \otimes dq^i, \quad 1 \leq A \leq k,$$

where we have transported the canonical k -tangent structure $(\tilde{J}_1, \dots, \tilde{J}_k)$ of $T_k^1 Q$ to $\mathbf{R}^k \times T_k^1 Q$.

If we set $\bar{\eta}_A = dt^A$ and $\xi_A = \partial/\partial t^A$, then the family $(J_A, \bar{\eta}_A, \xi_A; 1 \leq A \leq k)$ is called the *canonical stable k -tangent structure* on $\mathbf{R}^k \times T_k^1 Q$, and $\mathbf{R}^k \times T_k^1 Q$ is the *stable k -tangent bundle* of Q .

IV. SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

In this section we characterize the k -vector fields on $\mathbf{R}^k \times T_k^1 Q$ such that their integral sections are canonical prolongations of maps from $\mathbf{R}^k$ to Q .

Let C be the *canonical vector field* of the vector bundle $\pi_{1,0}: \mathbf{R}^k \times T_k^1 Q \rightarrow \mathbf{R}^k \times Q$. This vector field C is the infinitesimal generator of the following flow:

$$\mathbf{R} \times (\mathbf{R}^k \times T_k^1 Q) \rightarrow \mathbf{R}^k \times T_k^1 Q$$

$$(s, (t, q^i, v_A^i)) \rightarrow (t, q^i, e^s v_A^i),$$

and in local coordinates it has the form

$$C = \sum_{i,A} v_A^i \frac{\partial}{\partial v_A^i}. \quad (3)$$

C is a sum of vector fields,

$$C = \sum_{A=1}^k C_A,$$

where each C_A is the infinitesimal generator of the following flow:

$$\begin{aligned} \mathbf{R} \times (\mathbf{R}^k \times T_k^1 Q) &\rightarrow \mathbf{R}^k \times T_k^1 Q \\ (s, (t, q^i, v_A^i)) &\rightarrow (t, q^i, v_1^i, \dots, v_{A-1}^i, e^s v_A^i, v_{A+1}^i, \dots, v_k^i). \end{aligned}$$

In local coordinates they have the form

$$C_A = \sum_{i=1}^n v_A^i \frac{\partial}{\partial v_A^i}, \quad 1 \leq A \leq k. \quad (4)$$

Let us consider on $\mathbf{R}^k \times T_k^1 Q$ the tensor fields $\hat{J}_1, \dots, \hat{J}_k$ of type (1,1), defined as follows:

$$\hat{J}_A = \tilde{J}_A - C_A \otimes dt^A, \quad 1 \leq A \leq k.$$

Definition IV.1: A k -vector field $\zeta = (\zeta_1, \dots, \zeta_k)$ on $\mathbf{R}^k \times T_k^1 Q$ is said to be a second order partial differential equation (SOPDE for short) if

$$\hat{J}_A(\zeta_A) = 0, \quad \bar{\eta}_A(\zeta_B) = \delta_{AB},$$

for all $1 \leq A, B \leq k$.

From a direct computation in local coordinates we obtain that the local expression of a SOPDE $\zeta = (\zeta_1, \dots, \zeta_k)$ on $\mathbf{R}^k \times T_k^1 Q$ is

$$\zeta_A = \frac{\partial}{\partial t^A} + v_A^i \frac{\partial}{\partial q^i} + (\zeta_A)^i_B \frac{\partial}{\partial v_B^i}, \quad 1 \leq A \leq k, \quad (5)$$

where $(\zeta_A)^i_B$ are functions on $\mathbf{R}^k \times T_k^1 Q$.

As a direct consequence of the above local expressions, we deduce that the family of vector fields $\{\zeta_1, \dots, \zeta_k\}$ are linearly independent.

Definition IV.2: Let $\phi: \mathbf{R}^k \rightarrow Q$ be a map, we define the first prolongation $\phi^{(1)}$ of ϕ as the map

$$\phi^{(1)}: \mathbf{R}^k \rightarrow J^1 \pi \equiv \mathbf{R}^k \times T_k^1 Q,$$

$$t \mapsto j_t^1 \phi \equiv (t, j_0^1 \phi_t).$$

In local coordinates:

$$\phi^{(1)}(t^1, \dots, t^k) = \left(t^1, \dots, t^k, \phi^i(t^1, \dots, t^k), \frac{\partial \phi^i}{\partial t^A}(t^1, \dots, t^k) \right), \quad 1 \leq A \leq k, 1 \leq i \leq n. \quad (6)$$

From (5) it follows that an integral section σ of a SOPDE ζ is the first prolongation $\phi^{(1)}$ of a map ϕ from $\mathbf{R}^k$ to Q .

Definition IV.3: $\phi: \mathbf{R}^k \rightarrow Q$ is called a solution of the SOPDE ζ on $\mathbf{R}^k \times T_k^1 Q$ if the first prolongation $\phi^{(1)}$ of ϕ is an integral section of ζ .

From (5) and (6) we have the following.

Proposition IV.4: $\phi: \mathbf{R}^k \rightarrow Q$ is a solution of the SOPDE $\zeta = (\zeta_1, \dots, \zeta_k)$, locally given by (5), if and only if

$$\frac{\partial \phi^i}{\partial t^A} = v_A^i, \quad \frac{\partial^2 \phi^i}{\partial t^A \partial t^B} = (\zeta_A)^i_B.$$

Remark IV.5: Let us consider the trivial bundles $\pi: E = R^k \times Q \rightarrow R^k$ and $\pi_1: \mathbf{R}^k \times T_k^1 Q \rightarrow \mathbf{R}^k$. We identify each SOPDE $(\zeta_1, \dots, \zeta_k)$ with the following *semi-holonomic second order jet field*:

$$J^1 \pi \equiv \mathbf{R}^k \times T_k^1 Q \rightarrow J^1 \pi_1 \equiv \mathbf{R}^k \times T_k^1(T_k^1 Q),$$

$$(t^A, q^i, v_A^i) \rightarrow (t^A, q^i, v_A^i, v_A^i, (\zeta_A)_B^i).$$

If the SOPDE ζ on $\mathbf{R}^k \times T_k^1 Q$ is integrable, then its integral sections are canonical prolongations of maps from $\mathbf{R}^k$ to Q and then ζ defines a second order jet field $\tilde{\Gamma}$ on π whose coordinate representation of the corresponding connection $\tilde{\Gamma}$ is

$$\tilde{\Gamma} = dt^A \otimes \left(\frac{\partial}{\partial t^A} + v_A^i \frac{\partial}{\partial q^i} + (\zeta_A)_B^i \frac{\partial}{\partial v_B^i} \right),$$

since $(\zeta_A)_B^i = (\zeta_B)_A^i$ (see Ref. 5).

The integrability of the SOPDE is equivalent to the condition given by $\mathcal{R} = 0$, where $\mathcal{R}$ is the curvature tensor of the above connection (see Ref. 13 and 5).

V. k-COSYMPLECTIC STRUCTURES

The keystone of the nonautonomous Lagrangian formalism of classical mechanics is the cosymplectic structure on the phase space $\mathbf{R} \times TQ$. For field theory we shall show that this statement remains true, provided the two-forms that define the cosymplectic structure are replaced by two vector valued forms.

Let $J^1(Q, \mathbf{R}^k)_0$ be the manifold of one jets of maps from Q to $\mathbf{R}^k$ with target at $0 \in \mathbf{R}^k$.

The manifold $J^1(Q, \mathbf{R}^k)_0$ can be canonically identified with the Whitney sum $(T_k^1)^* Q$ of k copies of T^*Q , that is

$$J^1(Q, \mathbf{R}^k)_0 \rightarrow (T_k^1)^* Q = T^*Q \oplus \dots \oplus T^*Q,$$

$$j_{q,0}^1 \sigma \equiv (p_q^1, \dots, p_q^k),$$

where $p_q^A = d(\pi_A \circ \sigma)(q)$ being $\pi_A: \mathbf{R}^k \rightarrow \mathbf{R}$ the canonical projection.

Let $J^1(Q, \mathbf{R}^k)$ be the $k + (k+1)n$ -dimensional manifold of one jets from Q to $\mathbf{R}^k$, with elements denoted by $j_{q,t}^1 \sigma$. We recall that one jets of mappings from Q to $\mathbf{R}^k$ can be identified with the manifold $J^1 \rho$ of one jets of sections of the trivial bundle $\rho: \mathbf{R}^k \times Q \rightarrow Q$.

$J^1 \rho$ is diffeomorphic to $\mathbf{R}^k \times (T_k^1)^* Q$, via the diffeomorphism given by

$$j_q^1 \alpha \in J^1 \rho \rightarrow (\sigma(q), j_{q,0}^1 \sigma_q) \in \mathbf{R}^k \times (T_k^1)^* Q,$$

where $\sigma_q(\tilde{q}) = \sigma(\tilde{q}) - \sigma(q)$ and $\tilde{q}$ denotes an arbitrary point in Q .

Let us denote by $\tau^*: \mathbf{R}^k \times (T_k^1)^* Q \rightarrow Q$ the canonical projection. If (q^i) are local coordinates on $U \subseteq Q$ then the induced local coordinates $(t^A, q^i, p_i^A, 1 \leq i \leq n, 1 \leq A \leq k)$ on $(\tau^*)^{-1}(U) \equiv \mathbf{R}^k \times (T_k^1)^* U$ are given by

$$t^A(j_q^1 \sigma) = t^A(\sigma(q)), \quad q^i(j_q^1 \sigma) = q^i(q), \quad p_i^A(j_q^1 \sigma) = d(\sigma_q^A)(q) \left(\frac{\partial}{\partial q^i} \right)_q,$$

where $\sigma_q^A = \pi_A \circ \sigma_q$.

An $\mathbf{R}^k$ -valued one-form η_0 and a $\mathbf{R}^k$ -valued two-form ω_0 on $\mathbf{R}^k \times (T_k^1)^* Q$ are defined by

$$\eta_0 = \sum_{A=1}^m (\eta_0)_A \mathbf{r}_A = \sum_{A=1}^k ((\pi_A^1)^* dt) \mathbf{r}_A, \quad \omega_0 = \sum_{A=1}^k (\omega_0)_A \mathbf{r}_A = \sum_{A=1}^m (\pi_A^2)^* (\omega_Q) \mathbf{r}_A, \quad (7)$$

where $\pi_A^1: \mathbf{R}^k \times (T_k^1)^*Q \rightarrow \mathbf{R}$ and $\pi_A^2: \mathbf{R}^k \times (T_k^1)^*Q \rightarrow T^*Q$ are the projections defined by

$$\pi_A^1((t^B), (p^B)) = t^A, \quad \pi_A^2((t^B), (p^B)) = p^A,$$

and ω_Q is the canonical symplectic form on T^*Q .

In local coordinates we have

$$(\eta_0)_A = dt^A, \quad (\omega_0)_A = \sum_{i=1}^n dq^i \wedge dp_i^A \quad 1 \leq A \leq k. \quad (8)$$

Moreover, let be $V = \ker T\mu^*$, where $\mu^*: \mathbf{R}^k \times (T_k^1)^*Q \rightarrow \mathbf{R}^k \times Q$. A simple inspection in local coordinates shows that the forms $(\eta_0)_A$ and $(\omega_0)_A$ are closed and the following relations hold:

- (i) $(\eta_0)_1 \wedge \dots \wedge (\eta_0)_k \neq 0, (\eta_0)_{A|V} = 0, \omega_{A|V \times V} = 0,$
- (ii) $(\cap_{A=1}^n \ker(\eta_0)_A) \cap (\cap_{A=1}^n \ker(\omega_0)_A) = \{0\}, \dim(\cap_{A=1}^n \ker(\omega_0)_A) = k.$

Inspired in the above geometrical model we have introduced in Ref. 22 the following definition.

Definition V.1: Let M be a differentiable manifold of dimension $(k+1)n+k$. A family $(\eta_A, \omega_A, V; 1 \leq A \leq k)$, where each η_A is a closed 1-form, each ω_A is a closed 2-form and V is an nk -dimensional integrable distribution on M , such that

- (i) $\eta_1 \wedge \dots \wedge \eta_k \neq 0, \eta_{A|V} = 0, \omega_{A|V \times V} = 0,$
- (ii) $(\cap_{A=1}^k \ker \eta_A) \cap (\cap_{A=1}^k \ker \omega_A) = \{0\}, \dim(\cap_{A=1}^k \ker \omega_A) = k,$

is called a k -cosymplectic structure, and the manifold M a k -cosymplectic manifold.

The canonical model for these geometrical structures is $(\mathbf{R}^k \times (T_k^1)^*Q, \eta_0, \omega_0, V)$.

For any k -cosymplectic structure (η_A, ω_A, V) on M , there exists a family of k vector fields $(\xi_1, \dots, \xi_k)$ characterized by the conditions

$$\eta_A(\xi_B) = \delta_{AB}, \quad \iota_{\xi_B} \omega_A = 0,$$

for all $1 \leq A, B \leq k$. These vector fields are called the Reeb vector fields associated to the k -cosymplectic structure.

If (M, η, ω, V) is a k -cosymplectic manifold we can define the vector bundle morphism,

$$\begin{aligned} \Omega^\#: T_k^1 M &\rightarrow T^*M \\ (X_1, \dots, X_k) &\rightarrow \Omega^\#(X_1, \dots, X_k) = \sum_{A=1}^k \iota_{X_A} \omega_A + \eta_A(X_A) \eta_A. \end{aligned} \quad (9)$$

Remark V.2: If $k=1$ then $\Omega^\#$ is defined from TM onto T^*M and it is in fact the isomorphism $\chi_{\eta, \omega}$ defined on the cosymplectic manifold (M, η, ω) by (see Ref. 23, 24)

$$\chi_{\eta, \omega}(X) = \iota_X \omega + \eta(X) \eta.$$

Let (M, η_A, ω_A, V) be a k -cosymplectic manifold, $H: M \rightarrow \mathbf{R}$ a Hamiltonian function and ξ_A are the Reeb vector fields determined by (η_A, ω_A, V) . In Ref. 22 we have proved that if an integrable k -vector field $\mathbf{X} = (X_1, \dots, X_k)$ satisfies the equations

$$\begin{aligned} \eta_A(X_B) &= \delta_{AB}, \quad \forall A, B, \\ \Omega^\#(X_1, \dots, X_k) &= dH + \sum_{A=1}^k (1 - \xi_A(H)) \eta_A, \end{aligned} \quad (10)$$

then its integral sections ϕ are solutions of the Hamiltonian field equations corresponding to H :

$$\frac{\partial H}{\partial x^i} = - \sum_{A=1}^k \frac{\partial \phi_i^A}{\partial s^A}, \quad \frac{\partial H}{\partial x_i^A} = \frac{\partial \phi^i}{\partial s^A}.$$

The existence of canonical coordinates (s^A, x^i, x_i^A) is ensured by the corresponding Darboux theorem (see Ref. 22).

VI. THE LAGRANGIAN FORMALISM

Given a Lagrangian function of the form $L = L(t^A, q^i, v_A^i)$ one obtains, by using a variational principle, the *Euler–Lagrange equations* for L :

$$\sum_{A=1}^k \frac{d}{dt^A} \left(\frac{\partial L}{\partial v_A^i} \right) - \frac{\partial L}{\partial q^i} = 0, \quad v_A^i = \frac{\partial q^i}{\partial t^A}. \quad (11)$$

First of all, one realizes that such a L can be considered as a function $L: \mathbf{R}^k \times T_k^1 Q \rightarrow \mathbf{R}$.

In this section we shall give a geometrical description of the above equations (11) using a k -cosymplectic structure on $\mathbf{R}^k \times T_k^1 Q$ associated to the regular Lagrangian L . This k -cosymplectic structure shall be constructed using the canonical stable k -tangent structure of $\mathbf{R}^k \times T_k^1 Q$.

For each $1 \leq A \leq k$, we define the following.

- The vertical derivation ι_{J_A} of forms on $\mathbf{R}^k \times T_k^1 Q$ by

$$\iota_{J_A} f = 0, \quad (\iota_{J_A} \alpha)(X_1, \dots, X_p) = \sum_{j=1}^p \alpha(X_1, \dots, J_A X_j, \dots, X_p),$$

for any function f and any p -form α on $\mathbf{R}^k \times T_k^1 Q$;

- the vertical differentiation d_{J_A} of forms on $\mathbf{R}^k \times T_k^1 Q$ by

$$d_{J_A} = [\iota_{J_A}, d] = \iota_{J_A} \circ d - d \circ \iota_{J_A},$$

where d denotes the usual exterior differentiation.

Let us consider the 1-forms,

$$(\beta_L)_A = d_{J_A} L - \xi_A(L) \bar{\eta}_A, \quad 1 \leq A \leq k.$$

In bundle coordinates (t^A, q^i, v_A^i) we have

$$(\beta_L)_A = \sum_{i=1}^n \frac{\partial L}{\partial v_A^i} dq^i, \quad 1 \leq i \leq k. \quad (12)$$

Definition VI.1: A Lagrangian L is called regular if and only if the Hessian matrix,

$$\left(\frac{\partial^2 L}{\partial v_A^i \partial v_B^j} \right), \quad (13)$$

is nonsingular.

Now, we introduce the following 2-forms:

$$(\omega_L)_A = -d(\beta_L)_A, \quad 1 \leq A \leq k.$$

Proposition VI.2: Let $L: \mathbf{R}^k \times T_k^1 Q \rightarrow \mathbf{R}$ be a regular Lagrangian, and V_0 the vertical distribution of the bundle $\pi_{1,0}: \mathbf{R}^k \times T_k^1 Q \rightarrow \mathbf{R}^k \times Q$. Then, L is regular if and only if $(\mathbf{R}^k \times T_k^1 Q, \bar{\eta}_A, (\omega_L)_A, V_0)$ is a k -cosymplectic manifold.

Proof: Assume that L is a regular Lagrangian. We shall prove that $(\mathbf{R}^k \times T_k^1 Q, \bar{\eta}_A, (\omega_L)_A, V_0)$ is a k -cosymplectic manifold.

Conditions (i) are rather obvious. Now, we shall prove condition (ii) in Definition V.1. Let X be a vector field with the local expression

$$X = X^A \frac{\partial}{\partial t^A} + X^i \frac{\partial}{\partial q^i} + X_A^i \frac{\partial}{\partial v_A^i},$$

such that

$$\iota_X(\omega_L)_A = 0, \text{ and } \bar{\eta}_A(X) = 0, \text{ for all } 1 \leq A \leq k.$$

The second condition implies that $X^A = 0$, for all A , and then the first condition yields

$$X^i \frac{\partial^2 L}{\partial v_A^i \partial v_B^j} = 0, \quad (14)$$

and

$$X^i \frac{\partial^2 L}{\partial v_A^i \partial q^j} - X^i \frac{\partial^2 L}{\partial v_A^j \partial q^i} - X_B^i \frac{\partial^2 L}{\partial v_A^j \partial v_B^i} = 0. \quad (15)$$

Since L is regular, from (14) we deduce that $X^i = 0$, for all i , so that (15) implies $X_A^i = 0$, for all i and A . Therefore, we conclude that $X = 0$.

Next, we shall prove the second part of condition (ii) in Definition V.1. Let Y be a vector field with the local expression

$$Y = Y^A \frac{\partial}{\partial t^A} + Y^i \frac{\partial}{\partial q^i} + Y_A^i \frac{\partial}{\partial v_A^i},$$

such that

$$\iota_X(\omega_L)_A = 0, \text{ for all } 1 \leq A \leq k.$$

We then have

$$Y^j \frac{\partial^2 L}{\partial v_A^i \partial v_B^j} = 0, \quad (16)$$

and

$$-Y^B \frac{\partial^2 L}{\partial v_A^i \partial t^B} + Y^j \frac{\partial^2 L}{\partial v_A^j \partial q^i} - Y^j \frac{\partial^2 L}{\partial v_A^i \partial q^j} - Y_B^j \frac{\partial^2 L}{\partial v_A^i \partial v_B^j} = 0. \quad (17)$$

From (16) and the regularity of L we deduce that $Y^j = 0$, for all j , so that (17) reduces to

$$-Y^B \frac{\partial^2 L}{\partial v_A^i \partial t^B} - Y_B^j \frac{\partial^2 L}{\partial v_A^i \partial v_B^j} = 0. \quad (18)$$

Since the matrix $(\partial^2 L / \partial v_A^i \partial v_B^j)$ is regular, (18) implies that

$$Y_B^j = \mathcal{A}_{BD}^j Y^D,$$

for some matrix $\mathcal{A}_{BD}^j$. Therefore, we get

$$Y = Y^B \left(\frac{\partial}{\partial t^B} + \mathcal{A}_{CB}^j \frac{\partial}{\partial v_C^j} \right).$$

Thus, $\cap_{A=1}^n \ker(\omega_L)_A$ is locally spanned by the family of k independent local vector fields $\{ \partial/\partial t^B + \mathcal{A}_{CB}^j (\partial/\partial v_C^j) \}$.

Then we have proved that $(\mathbf{R}^k \times T_k^1 Q, (\eta_0)_A, (\omega_L)_A, V_0)$ is a k -cosymplectic manifold.

The converse is proved by reversing the above arguments. ■

Let $L: \mathbf{R}^k \times T_k^1 Q \rightarrow \mathbf{R}$ be a regular Lagrangian and $(\bar{\eta}_A, (\omega_L)_A, V_0)$ the associated k -cosymplectic structure on $\mathbf{R}^k \times T_k^1 Q$.

The equations,

$$\bar{\eta}_A((\xi_L)_B) = \delta_B^A, \quad \iota_{(\xi_L)_A}(\omega_L)_B = 0, \quad 1 \leq A, B \leq k, \quad (19)$$

define the Reeb vector fields $\{(\xi_L)_1, \dots, (\xi_L)_k\}$ on $\mathbf{R}^k \times T_k^1 Q$ which are locally given by

$$(\xi_L)_A = \frac{\partial}{\partial t^A} + ((\xi_L)_A)_B^i \frac{\partial}{\partial v_B^i}, \quad (20)$$

where the functions $((\xi_L)_A)_B^i$ satisfy

$$\frac{\partial^2 L}{\partial t^A \partial v_C^j} + \frac{\partial^2 L}{\partial v_B^i \partial v_C^j} ((\xi_L)_A)_B^i = 0, \quad (21)$$

for all $1 \leq A, B, C \leq k$ and $1 \leq i, j \leq n$.

Since L is regular, from the local conditions (21) we can define, in a neighborhood of each point of $\mathbf{R}^k \times T_k^1 Q$, a k -vector field that satisfies (19). Next, one can construct a global k -vector field ξ_L , which is a solution of (19), by using a partition of unity.

Let L be a regular Lagrangian and let $\Omega_L^\#$ be the $\#$ -morphism defined by the k -cosymplectic structure $(\bar{\eta}_A, (\omega_L)_A, V_0)$, as in (9):

$$\Omega_L^\#: T_k^1(\mathbf{R}^k \times T_k^1 Q) \rightarrow T^*(\mathbf{R}^k \times T_k^1 Q) \quad (22)$$

$$(X_1, \dots, X_k) \rightarrow \Omega_L^\#(X_1, \dots, X_k) = \sum_{A=1}^k \iota_{X_A}(\omega_L)_A + \bar{\eta}_A(X_A) \bar{\eta}_A.$$

Theorem VI.3: Let L be a regular Lagrangian and let $\mathbf{X} = (X_1, \dots, X_k)$ be a k -vector field such that

$$\bar{\eta}_A(X_B) = \delta_{AB}, \quad 1 \leq A, B \leq k, \quad (23)$$

$$\Omega_L^\#(X_1, \dots, X_k) = dE_L + \sum_{A=1}^k (1 - (\xi_L)_A(E_L)) \bar{\eta}_A,$$

where $E_L = C(L) - L$. Then $\mathbf{X} = (X_1, \dots, X_k)$ is a SOPDE. In addition, if $\mathbf{X} = (X_1, \dots, X_k)$ is integrable then its solutions satisfy the Euler–Lagrange equations (11).

Proof: It should be noticed that in general Eqs. (23) have not a unique solution. In fact, the solutions of (23) are given by $(X_1, \dots, X_k) + (\ker \Omega^\#)$, where $(X_1, \dots, X_k)$ is a particular solution. Nevertheless, we shall show now that there exist always solutions of equations (23) when L is assumed to be regular. In this case, from (3) and (20) we obtain

$$(\xi_L)_A(E_L) = - \frac{\partial L}{\partial t^A}. \quad (24)$$

Let $(X_1, \dots, X_k)$ be a solution of (23) locally given by

$$X_A(t, q^i, v_A^i) = \frac{\partial}{\partial t^A} + (X_A)^i \frac{\partial}{\partial q^i} + (X_A)^j_B \frac{\partial}{\partial v_B^j}.$$

Then, from (24) we obtain

$$\sum_{A,B,j} \left(-\frac{\partial^2 L}{\partial t^A \partial v_A^i} + (X_A)^j \left(\frac{\partial^2 L}{\partial q^i \partial v_A^j} - \frac{\partial^2 L}{\partial q^j \partial v_A^i} \right) - (X_A)^j_B \frac{\partial^2 L}{\partial v_B^j \partial v_A^i} \right) = \sum_{A,j} \left(v_A^j \frac{\partial^2 L}{\partial q^i \partial v_A^j} - \frac{\partial L}{\partial q^i} \right), \quad (25)$$

$$\sum_{A,i} (X_A)^i \frac{\partial^2 L}{\partial v_B^j \partial v_A^i} = \sum_{A,i} v_A^i \frac{\partial^2 L}{\partial v_B^j \partial v_A^i}, \quad (26)$$

and since L is regular, from (26) we deduce that the solutions of Eqs. (23) are SOPDE's, that is, each X_A has the form

$$X_A(t, q^i, v_A^i) = \frac{\partial}{\partial t^A} + v_A^i \frac{\partial}{\partial q^i} + (X_A)^j_B \frac{\partial}{\partial v_B^j}. \quad (27)$$

From (25) we deduce that the functions $(X_A)^j_B$ satisfy the equations

$$\sum_A \frac{\partial^2 L}{\partial t^A \partial v_A^i} + \sum_{j,B} \left(v_B^j \frac{\partial^2 L}{\partial q^j \partial v_B^i} + \sum_A (X_A)^j_B \frac{\partial^2 L}{\partial v_B^j \partial v_A^i} \right) = \frac{\partial L}{\partial q^i}, \quad 1 \leq i \leq n. \quad (28)$$

Since L is regular, (28) leads us to define local solutions of (23) in a neighborhood of each point of $\mathbf{R}^k \times T_k^1 Q$. Using a partition of unity one can easily obtain a global solution of (23).

Now, let

$$\phi: \mathbf{R}^k \rightarrow Q,$$

$$(t^B) \rightarrow (\phi^i(t^1, \dots, t^k)),$$

be a solution of $(X_1, \dots, X_k)$. From Proposition IV.4 and Eq. (27) we deduce that

$$\frac{\partial \phi^i}{\partial t^A} = v_A^i, \quad \frac{\partial^2 \phi_A^i}{\partial t^A \partial t^B} = (X_A)^i_B.$$

Replacing in (28) we get

$$\sum_A \frac{\partial^2 L}{\partial t^A \partial v_A^i} + \sum_{j,B} \left(\frac{\partial \phi^j}{\partial t^B} \frac{\partial^2 L}{\partial q^j \partial v_B^i} + \sum_A \frac{\partial^2 \phi^j}{\partial t^A \partial t^B} \frac{\partial^2 L}{\partial v_B^j \partial v_A^i} \right) = \frac{\partial L}{\partial q^i}, \quad 1 \leq i \leq n, \quad (29)$$

which shows that ϕ is a solution of the Euler–Lagrange equations (11). ■

In conclusion, we can consider Eqs. (23) as a *geometric version* of the Euler–Lagrange field equations.

Remark VI.4: We have given a geometric version of the Euler–Lagrange equations for a nonautonomous Lagrangian constructing a k -cosymplectic structure on $\mathbf{R}^k \times T_k^1 Q$ defined from the Lagrangian and the canonical stable k -tangent structure on $\mathbf{R}^k \times T_k^1 Q$. We can also construct this k -cosymplectic structure using the *Legendre transformation* $\mathcal{FL}$ of L which is the map

$$\mathcal{FL}: \mathbf{R}^k \times T_k^1 Q \rightarrow \mathbf{R}^k \times (T_k^1)^* Q,$$

defined as follows: if $(t, \mathbf{v}) = (t^1, \dots, t^k, v_1, \dots, v_k) \in \mathbf{R}^k \times (T_k^1 Q)_q$ with $q \in Q$ and $v_A \in T_q Q$, then

$$\mathcal{FL}(t, y) = (t^1, \dots, t^k, p^1, \dots, p^k) \in \mathbf{R}^k \times (T_k^1 Q)_q^*, \quad p^A \in T_q^* Q$$

is given by

$$p^A(v_q) = (\beta_L)_A(\overline{v_q}), \quad 1 \leq A \leq k,$$

for any $v_q \in T_q Q$, where $\overline{v_q} \in T_v(T_k^1 Q)$ is any tangent vector such that $d\tau_Q(v)(\overline{v_q}) = v_q$, with $\tau_Q: T_k^1 Q \rightarrow Q$ the canonical projection. In induced coordinates we have

$$\mathcal{FL}: (t^A, q^i, v_A^i) \rightarrow \left(t^A, q^i, \frac{\partial L}{\partial v_A^i} \right). \quad (30)$$

The Jacobian matrix of $\mathcal{FL}$ is

$$\begin{pmatrix} I_k & 0 & 0 & \dots & 0 \\ 0 & I_n & 0 & \dots & 0 \\ & \mathcal{B} & & \mathcal{C} & \end{pmatrix}, \quad (31)$$

where $I_k \in I_n$ are the identity matrices of order k and n , respectively, $\mathcal{B}$ is a matrix $nk \times (k+n)$ and $\mathcal{C}$ is the matrix

$$\left(\frac{\partial^2 L}{\partial v_A^i \partial v_B^j} \right).$$

Now, from (12) and (30) we deduce the following.

Lemma VI.5: We have

$$(\omega_L)_A = \mathcal{FL}^*((\omega_0)_A), \quad \bar{\eta}_A = \mathcal{FL}^*((\eta_0)_A),$$

for all A .

Next, from (31) we obtain the following proposition.

Proposition VI.6: The following conditions are equivalent:

- (1) L is regular.
- (2) $\mathcal{FL}$ is a local diffeomorphism.
- (3) $(\bar{\eta}_1, \dots, \bar{\eta}_k, (\omega_L)_1, \dots, (\omega_L)_k, V_0)$ is a k -cosymplectic structure on $\mathbf{R}^k \times T_k^1 Q$.

VII. SINGULAR CASE

When the Lagrangian function L is not regular, the family $(\bar{\eta}_A, (\omega_L)_A, V_0, 1 \leq A \leq k)$ is no longer a k -cosymplectic structure. Even in this case, from (24), we deduce that if $\mathbf{X} = (X_1, \dots, X_k)$ is an integrable SOPDE such that

$$\bar{\eta}_A(X_B) = \delta_{AB}, \quad (32)$$

$$\Omega_L^\#(X_1, \dots, X_k) = dE_L + \sum_{A=1}^k \left(1 + \frac{\partial}{\partial t^A}(L) \right) \bar{\eta}_A,$$

then its solutions satisfy the Euler–Lagrange equations.

But we cannot assure that such a solution exists. Therefore we shall develop a constraint algorithm inspired in the well-known one for singular Lagrangians in mechanics in order to obtain a final constraint submanifold where such a solution exists.

We put $P_1 = \mathbf{R}^k \times T_k^1 Q$. Next, let P_2 be the subset of P_1 which consists of those points where there exists solution of (32), that is,

$P_2 = \{z \in P_1 / \exists \mathbf{X} \in (T_k^1(\mathbf{R}^k \times (T_k^1)^* Q))_z \text{ satisfying the SOPDE condition and being a solution of (32)}\}.$

If P_2 is a submanifold of P_1 , then there exists a section of $\tau_{\mathbf{R}^k \times T_k^1 Q}$ over P_2 . This section is not in general a k -vector field on P_2 . To find solutions taking values into $T_k^1 P_2$ we construct a new subset P_3 of P_2 as follows:

$P_3 = \{z \in P_2 / \exists \mathbf{X} \in (T_k^1 P_2)_z \text{ satisfying the SOPDE condition and being a solution of (32)}\}.$

If P_3 is a submanifold of P_2 , there exists a section of τ_{P_2} over P_3 which is solution of (23), but that not defines, in general, a k -vector field on P_3 .

Proceeding further, we get a family of constraint manifolds,

$$\cdots \rightarrow P_3 \rightarrow P_2 \rightarrow P_1 = \mathbf{R}^k \times T_k^1 Q.$$

In the most favorable case, this constraint algorithm stabilizes at some step, say $P_{h+1} = P_h$ and $\dim P_h > 0$. In this case, we call $P_f = P_h$ the final constraint manifold. On P_f there exists a SOPDE solution of (32).

Of course, this solution would not be unique as in the regular case. It should be noticed that, in addition, the solutions on P_f shall not be in principle integrable. To guarantee the existence of an integrable solution one has to develop an additional constraint algorithm taking the brackets of the vector fields which compose a SOPDE solution. Doing that we shall obtain an integrable SOPDE solution on a (smaller, in general) submanifold S of P_f ; its solutions shall satisfy the Euler–Lagrange equations (11) (see Ref. 13).

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