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Causality, unitarity thresholds, anomalous thresholds and infrared singularities from the loop-tree duality at higher orders

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Abstract: We present the first comprehensive analysis of the unitarity thresholds and anomalous thresholds of scattering amplitudes at two loops and beyond based on the loop-tree duality, and show how non-causal unphysical thresholds are locally cancelled in an efficient way when the forest of all the dual on-shell cuts is considered as one. We also prove that soft and collinear singularities at two loops and beyond are restricted to a compact region of the loop three-momenta, which is a necessary condition for implementing a local cancellation of loop infrared singularities with the ones appearing in real emission; without relying on a subtraction formalism.

Keywords: Duality in Gauge Field Theories, Perturbative QCD, Scattering Amplitudes

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Contents

1 Introduction

It is common knowledge that scattering amplitudes in Quantum Field Theory can be reconstructed from their singularities. While tree-level amplitudes only have poles loop amplitudes develop branch cuts, corresponding to discontinuities associated with physical thresholds as well as infrared (IR) singularities in the soft and collinear limits. The singular IR behavior of QCD amplitudes is well-known through general factorization formulas [1, 2]. The physical consequences of the emergence of unitarity thresholds [3, 4], anomalous thresholds and more generally Landau singularities [4–15], for specific kinematical configurations, have also been extensively discussed in the literature. Indeed a thorough knowledge of the singular structure of scattering amplitudes is a prerequisite for obtaining theoretical predictions of physical observables.

The Loop-Tree Duality (LTD) [16–22] is a powerful framework to analyze the singular structure of scattering amplitudes directly in the loop momentum space. The LTD representation of a one-loop scattering amplitude is given by

$$\begin{aligned} \mathcal{A}^{(1)}(\{p_n\}_N) &= -\int_\ell \mathcal{N}(\ell, \{p_n\}_N) \otimes G_D(\alpha)\,, \\ G_D(\alpha) &= \sum_{i\in\alpha} \tilde{\delta}\,(q_i) \prod_{j\neq i} G_D(q_i; q_j)\,, \end{aligned} \tag{1.1}$$

where $\mathcal{N}(\ell, \{p_n\}_N)$ is the numerator of the integrand that depends on the loop momentum ℓ and the N external momenta $\{p_n\}_N$. The delta function $\tilde{\delta}\,(q_i) = \imath 2\pi\,\theta(q_{i,0})\,\delta(q_i^2 - m_i^2)$ sets on-shell the internal propagator with momentum $q_i = \ell + k_i$ and selects its positive energy mode, $q_{i,0} > 0$. At one-loop, $\alpha = \{1, \cdots, N\}$ labels all the internal momenta, and eq. (1.1) is the sum of N single-cut dual amplitudes. The dual propagators,

$$G_D(q_i; q_j) = \frac{1}{q_j^2 - m_j^2 - \imath 0\,\eta \cdot k_{ji}}\,, \tag{1.2}$$

differ from the usual Feynman propagators, $G_F(q_j) = 1/(q_j^2 - m_j^2 + \imath 0)$, only by the imaginary prescription that now depends on $\eta \cdot k_{ji}$, with $k_{ji} = q_j - q_i$. Notice that the dual propagators are implicitly linear in the loop momentum due to the on-shell conditions. Though

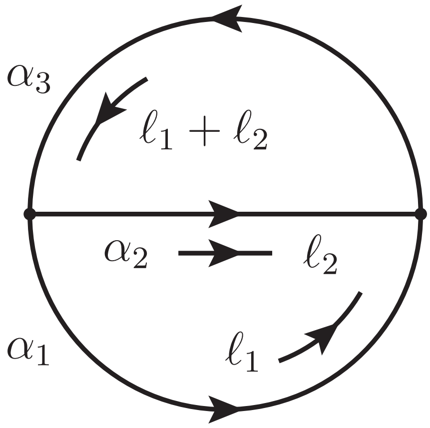

Figure 1. Momentum flow of a two-loop Feynman diagram.

the vector η is mostly arbitrary — it only has to be future-like — the most convenient choice is $\eta = (1, \mathbf{0})$. This election is equivalent to integrating out the energy component of the loop momentum, which renders the remaining integration domain Euclidean.

The master dual representation of a two-loop scattering amplitude is [18, 22]

$$\begin{aligned}\mathcal{A}^{(2)}(\{p_n\}_N) = \int_{\ell_1}\int_{\ell_2} &\mathcal{N}(\ell_1, \ell_2, \{p_n\}_N) \\ &\otimes [G_D(\alpha_1)\, G_D(\alpha_2 \cup \alpha_3) + G_D(-\alpha_2 \cup \alpha_1) G_D(\alpha_3) \\ &- G_D(\alpha_1)\, G_F(\alpha_2)\, G_D(\alpha_3)]\,,\end{aligned} \tag{1.3}$$

where the internal momenta $q_i = \ell_1 + k_i$, $q_j = \ell_2 + k_j$ and $q_k = \ell_1 + \ell_2 + k_k$, are classified into three different sets, $i \in \alpha_1$, $j \in \alpha_2$ and $k \in \alpha_3$, as shown in figure 1. The minus sign in front of α_2 indicates that the momenta in α_2 are reversed to hold a momentum flow consistent with α_1. The dual representation in eq. (1.3) contains one contribution that depends on $G_F(\alpha) = \prod_{i\in\alpha} G_F(q_i)$, and spans over the sum of all possible double-cut contributions, with each of the two cuts belonging to a different set. At higher orders, the iterative application of LTD introduces a number of cuts equal to the number of loops [17]. The dual amplitudes are thus tree-level like objects to all orders, and can even be related to the forward limit of tree-level amplitudes [23, 24]. However, neither eq. (1.1), nor eq. (1.3), or their higher order generalization [17], has been deducted from the forward limit of tree-level amplitudes and they are therefore free of their potential spurious singularities.

A decisive feature that allows to progress from scattering amplitudes to cross-sections and other physical observables in the LTD representation is the fact that all IR and physical threshold singularities of the dual amplitudes are restricted to a compact region of the loop three-momenta [19]. This is essential to establish a mapping between the real and virtual kinematics in order to locally cancel the IR singularities without the need for subtraction counter-terms, as done in the Four-Dimensional Unsubtraction (FDU) approach [25–27]. This is remarkably the place where the dual $\imath 0$ prescription plays its main role by encoding efficiently and correctly the causal effects of the loop scattering amplitudes. A general analysis of causality [19, 28] necessarily requires studying in detail the interplay of the dual prescription among different dual propagators.

In this article, we extend explicitly to two loops the analysis of the singular structure of one-loop amplitudes in the LTD framework presented in ref. [19], and generalize it to higher orders. This allows us to present a comprehensive description of unitarity thresholds and anomalous thresholds in the loop-momentum space, and provide generalizable expressions of their singular behavior. We show how non-causal unphysical thresholds cancel locally in the forest defined by the sum of dual cuts thanks to the momentum dependent $\imath 0$ prescription of the dual propagators. Most importantly, we demonstrate that soft and collinear singularities are always restricted to a compact region of the loop three-momentum space.

2 Unitarity thresholds, anomalous thresholds and infrared singularities at one loop

To analyze the singular behavior of one-loop amplitudes in the loop momentum space, it is convenient to start by considering the integrand function

$$\mathcal{S}_{ij}^{(1)} = (2\pi\imath)^{-1}\, G_D(q_i;q_j)\, \tilde{\delta}\,(q_i) + (i \leftrightarrow j)\,, \tag{2.1}$$

representing the sum of two single-cut dual contributions. The singularities of the function $\mathcal{S}_{ij}^{(1)}$ are encoded through the set of conditions

$$\lambda_{ij}^{\pm\pm} = \pm q_{i,0}^{(+)} \pm q_{j,0}^{(+)} + k_{ji,0} \to 0\ . \tag{2.2}$$

where $q_{r,0}^{(+)} = \sqrt{\mathbf{q}_r^2 + m_r^2}$, with $r \in \{i,j\}$, are the on-shell loop energies. There are indeed only two independent limits that determine the location of the singularities in the loop momentum space. The limit $\lambda_{ij}^{++} \to 0$ occurs in the intersection of the forward on-shell hyperboloid (positive energy mode) of one propagator with the backward on-shell hyperboloid (negative energy mode) of the other. The solution to eq. (2.2) for $\lambda_{ij}^{++} \to 0$ requires

$$k_{ji}^2 - (m_j + m_i)^2 \geq 0\,, \tag{2.3}$$

with $k_{ji,0} < 0$. This means that the q_j propagator has to be in the future of the q_i propagator with both propagators causally connected ($\lambda_{ij}^{--} \to 0$ with $k_{ji,0} > 0$ represents the complementary solution). For massless propagators, and light-like separation, $k_{ji}^2 = 0$, the singular surface pinches to a collinear singularity along a finite segment. In both cases, massive or massless,

$$\lim_{\lambda_{ij}^{++}\to 0} \mathcal{S}_{ij}^{(1)} = \frac{\theta(-k_{ji,0})\theta(k_{ji}^2 - (m_i+m_j)^2)}{x_{ij}(-\lambda_{ij}^{++} - \imath 0 k_{ji,0})} + \mathcal{O}\left((\lambda_{ij}^{++})^0\right)\,, \tag{2.4}$$

with $x_{ij} = 4\, q_{i,0}^{(+)} q_{j,0}^{(+)}$. It should be noted that the limit $\lambda_{ij}^{++} \to 0$ represents the usual unitarity threshold. Since it involves one backward and one forward on-shell hyperboloid, this situation is equivalent to two physical particles propagating in the same direction in time. Moreover, the $\imath 0$ prescription is exactly the same as in the Feynman representation, and, in particular for IR singularities, the on-shell energy is bounded by the energy of external momenta, $q_{r,0}^{(+)} \leq |k_{ji,0}|$ with $r \in \{i,j\}$.

The other potential singularities occur for $\lambda_{ij}^{+-} \to 0$, with

$$k_{ji}^2 - (m_j - m_i)^2 \leq 0 \ . \tag{2.5}$$

It generates unphysical thresholds in each of the dual components of $\mathcal{S}_{ij}^{(1)}$, but the sum over the two single-cut dual contributions is not singular

$$\lim_{\lambda_{ij}^{+-}\to 0} \mathcal{S}_{ij}^{(1)} = \mathcal{O}\left((\lambda_{ij}^{+-})^0\right) \ . \tag{2.6}$$

The cancellation of this integrand singularity is fully local due to the change of sign in the dual prescription, $q_{j,0}^{(+)} \, G_D(q_i;q_j)|_{\lambda_{ij}^{+-}\to 0} = -q_{i,0}^{(+)} \, G_D(q_j;q_i)|_{\lambda_{ij}^{+-}\to 0}$, and is not affected by other propagators because

$$\lim_{\lambda_{ij}^{+-}\to 0} G_D(q_j;q_k) = \lim_{\lambda_{ij}^{+-}\to 0} G_D(q_i;q_k) \ . \tag{2.7}$$

This second configuration corresponds to the on-shell emission and on-shell reabsorption of one virtual particle. The complete local cancellation of this integrand singularity would not occur if all the propagators were Feynman propagators. As physics cannot depend on the used representation of the loop amplitude, this mismatch in the $\imath 0$ prescription is compensated in the Feynman Tree Theorem [29] by the multiple-cut contributions.

The LTD representation in eq. (1.1) is obtained by defining the momentum flow of the loop anti-clockwise and then by closing the Cauchy integration contour of the complex loop energy in the lower half-plane. Alternatively, one can close the contour in the upper half-plane and select the modes with negative energy, which is equivalent to reversing the momentum flow of the loop. The location of the causal unitarity thresholds in the loop-momentum space is invariant under these transformations because physics cannot depend on the specific dual representation. However, the emergence and location of the unphysical thresholds in individual dual contributions depends on the choice of the momentum flow. It is worth noticing that the unphysical singularities are spurious, and they cancel after adding together all the dual contributions. Explicitly, in eq. (2.5) we can distinguish between two scenarios. In the space-like case, $k_{ji}^2 < 0$, unphysical thresholds occur in the intersection of the two forward on-shell hyperboloids and in the intersection of the two backward on-shell hyperboloids. Whereas, in the time-like configuration, $0 \leq k_{ji}^2 \leq (m_i - m_j)^2$, unphysical thresholds occur only in the intersection of either the forward or the backward on-shell hyperboloids. For example, the individual contributions to the dual representation of the massive two-point function

$$\int_{\ell_1} \frac{1}{((\ell_1 - p)^2 - m_1^2 + \imath 0)(\ell_1^2 - m_2^2 + \imath 0)} \ , \tag{2.8}$$

given by eq. (1.1), with $p = (p_0, \mathbf{0})$ and p_0 positive, are free of causal and unphysical thresholds for $0 < p_0 < m_2 - m_1$. But the backward on-shell hyperboloids or the individual contributions of the dual integrand of (1.1) with inverted momentum flow ($\ell_1 \to -\ell_1$)

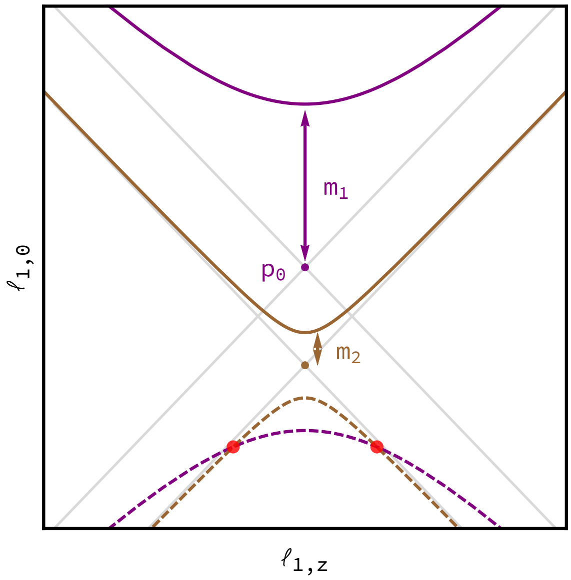

Figure 2. Forward (solid) and backward (dashed) on-shell hyperboloids of the massive two-point function in eq. (2.8). For the kinematical configuration with $0 < p_0 < m_1 - m_2$, the two forward on-shell hyperboloids do not intersect, but the two backward on-shell hyperboloids feature unphysical thresholds that cancel in the sum of individual dual contributions when the on-shell negative energy modes are also considered.

feature unphysical thresholds for $0 < p_0 < m_1 - m_2$. This case is illustrated in figure 2. Of course, the integral is invariant under this transformation. This is worth noticing because the proposal of considering together both the positive and negative modes [30] does not change the physics but inefficiently increases the number of necessary on-shell cuts to describe a given loop amplitude, and it also proliferates the number of unphysical singularities of the integrand, in particular at higher orders (A recent Erratum has been published that modifies the incorrect symmetry factors presented in ref. [30] that spoiled the expected local cancellations. We discuss later how our original dual prescription matches better the causal conditions and leads to more compact and effective representations than the alternative dual prescription introduced in ref. [30].).

We are now in the position to discuss the anomalous thresholds. For internal propagators with real masses, anomalous thresholds at one loop occur when more than two of them go on-shell simultaneously. In particular, for three propagators we should analyze the integrand function

$$\mathcal{S}^{(1)}_{ijk} = (2\pi\imath)^{-1}\, G_D(q_i; q_k)\, G_D(q_i; q_j)\, \tilde{\delta}\,(q_i) + \text{perm.}\ . \tag{2.9}$$

The potential singularity of eq. (2.9) arising in the intersection of the three forward on-shell hyperboloids cancels again locally among the dual contributions [19]. In order to generate a physical effect, we need to consider the intersection of one forward with two backward on-shell hyperboloids, or two forward with one backward.

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Explicitly, in the double limit λ_{ij}^{++} and $\lambda_{ik}^{++} \to 0$ with $k_{ji,0}$ and $k_{ki,0}$ negative,

$$\lim_{\lambda_{ij}^{++},\lambda_{ik}^{++}\to 0} \mathcal{S}_{ijk}^{(1)} = \frac{1}{x_{ijk}} \prod_{r=j,k} \frac{\theta(-k_{ri,0})\,\theta(k_{ri}^2-(m_i+m_r)^2)}{(-\lambda_{ir}^{++}-\imath 0 k_{ri,0})} + \mathcal{O}\left((\lambda_{ij}^{++})^{-1},(\lambda_{ik}^{++})^{-1}\right) , \tag{2.10}$$

with $x_{ijk} = 8\, q_{i,0}^{(+)} q_{j,0}^{(+)} q_{k,0}^{(+)}$. As expected from the discussion of the cancellation of unphysical thresholds, eqs. (2.6) and (2.7), the leading term is free of singularities in λ_{jk}^{-+}. This is also true for the next terms in the expansion, even though $\lambda_{jk}^{-+} = \lambda_{ik}^{++} - \lambda_{ij}^{++}$.

Again, the local cancellation of the λ_{jk}^{-+} singularity occurs thanks to the momentum dependent $\imath 0$ dual prescription, and the remaining singularities are described by causal $+\imath 0$ contributions. If all the uncut propagators were Feynman propagators, a mismatch would be generated. Conversely, the anomalous threshold generated in the intersection of one backward with two forward on-shell hyperboloids, λ_{ik}^{++} and $\lambda_{jk}^{++} \to 0$ with $\lambda_{ij}^{+-} = \lambda_{ik}^{++} - \lambda_{jk}^{++}$, is free of singularities in λ_{ij}^{+-}. This configuration also describes a soft singularity in the limiting case of massless partons, with e.g. $q_{i,0}^{(+)} \to 0$. Notice that for a soft singularity to be generated the participation of three propagators is necessary due to the soft suppression of the integration measure. The extension of the discussion to anomalous box singularities is straightforward from the results presented here. This analysis also allows us to move on now to the discussion of the two-loop case.

3 Unitarity thresholds, anomalous thresholds and infrared singularities at two loops

The characterization of singularities arising at two loops from two or more propagators of the same subset α_r is a replica of the one-loop case and does not need further discussion. The genuine two-loop IR and threshold singularities arise when the propagator that eventually goes on shell and the two other on-shell propagators belong to a different subset each. In this case, we have to consider the forest defined by all the possible permutations. Inspired by the dual representation in eq. (1.3), we define the following integrand function

$$\begin{aligned}\mathcal{S}_{ijk}^{(2)} &= (2\pi\imath)^{-2} \Big[G_D(q_j;q_k)\,\tilde\delta(q_i,q_j) + G_D(-q_j;q_i)\,\tilde\delta(-q_j,q_k) \\ &\quad + \left[G_D(q_k;q_j) + G_D(q_i;-q_j) - G_F(q_j)\right] \tilde\delta(q_i,q_k) \Big] ,\end{aligned} \tag{3.1}$$

with the shorthand notation $\tilde\delta(q_r,q_s) = \tilde\delta(q_r)\,\tilde\delta(q_s)$, and $i\in\alpha_1$, $j\in\alpha_2$ and $k\in\alpha_3$. The location of the singularities of $\mathcal{S}_{ijk}^{(2)}$ in the loop momentum space are determined by the set of conditions [22]

$$\lambda_{ijk}^{\pm\pm\pm} = \pm q_{i,0}^{(+)} \pm q_{j,0}^{(+)} \pm q_{k,0}^{(+)} + k_{k(ij),0} \to 0 , \tag{3.2}$$

where $k_{k(ij),0} = q_k - q_i - q_j$ depends on external momenta only, with our choice of the momentum flow (see figure 1). Now, the unitarity threshold is defined by the limit

$$\lim_{\lambda_{ijk}^{+++}\to 0} \mathcal{S}_{ijk}^{(2)} = \frac{\theta(-k_{k(ij),0})\,\theta(k_{k(ij)}^2 - (m_i+m_j+m_k)^2)}{x_{ijk}(-\lambda_{ijk}^{+++} - \imath 0 k_{kj,0})} + \mathcal{O}\left((\lambda_{ijk}^{+++})^0\right) . \tag{3.3}$$

Notice that $k_{kj,0}$ depends on the energy of the loop momentum ℓ_1, however,

$$k_{kj,0}|_{\lambda_{ijk}^{+++}\to 0} = q_{i,0}^{(+)} + k_{k(ij),0} < 0\,, \tag{3.4}$$

therefore $-\imath 0 k_{kj,0} = +\imath 0$ on the physical threshold. We can interpret this singularity as the intersection of the backward on-shell hyperboloid of q_k with the forward on-shell hyperboloids of q_i and q_j. The latter are independent of each other as each of them depends on a different loop momentum. In other words, the three internal physical momenta flow in the same direction in time, which is equivalent to the unitarity cut. Further, this configuration generates a triple collinear singularity for massless partons and light-like separation, $k_{k(ij)}^2 = 0$. Likewise, the complementary solution with $k_{k(ij),0} > 0$ would generate IR or threshold singularities in the limit $\lambda_{ijk}^{---} \to 0$. In any of the two cases discussed so far, the on-shell energies are limited by the energy of the external partons, $q_{r,0}^{(+)} \leq |k_{k(ij),0}|$ with $r \in \{i,j,k\}$. Since this is an essential condition for the implementation of FDU [25–27] at higher orders this observation constitutes one of the main results of this work.

Similarly to the one-loop case, there are other potential singularities at $\lambda_{ijk}^{++-} \to 0$ and $\lambda_{ijk}^{+--} \to 0$. However, those singularities cancel locally in the sum of all the dual components of $\mathcal{S}_{ijk}^{(2)}$. This is again possible because, even though the dual prescriptions depend on the loop momenta, the following conditions are fulfilled in each of the cases

$$\begin{aligned} \lambda_{ijk}^{++-} &= 0\,, &\quad \to \quad& q_{k,0}^{(+)} - k_{k(ij),0} > 0\,, \\ \lambda_{ijk}^{+--} &= 0\,, &\quad \to \quad& q_{i,0}^{(+)} + k_{k(ij),0} > 0\,, \end{aligned} \tag{3.5}$$

in such a way that all the contributions conspire to match their $\imath 0$ prescriptions on the singularity.

Let us stress that the on-shell conditions are determined by a set of equations that are linear in the on-shell loop energies to all orders (e.g. eq. (3.2) and its obvious generalization). Alternative dual prescriptions have been proposed recently [30, 31], which involve a different dependence on the loop energies. Even if the final result has to be equivalent independently of the prescription applied, we would like to highlight that the linear dependence of our original LTD prescription straightforwardly exhibits the analytic cross-cancellation of non-causal singularities among dual contributions. These dual cancellations have been confirmed numerically in ref. [31].

Concerning anomalous thresholds, we will consider any configuration that involves more on-shell propagators than those appearing in the unitarity cuts. Again, we can distinguish the case where the propagators involved belong to the same subset α_r, from the genuine two-loop case where it is necessary to consider propagators going simultaneously on shell from the three different sets. Prototype configurations are illustrated in figure 3. Anomalous thresholds are generated in either of the two benchmarks shown in figure 3 with the participation of the two propagators adjacent to the external momentum p_2, with $p_2 = q_{i_2} - q_{i_1}$, $\{i_1, i_2\} \in \alpha_1$, satisfying eq. (2.5). Specifically, for the right diagram the anomalous threshold appears when four propagators become singular (i.e. there are two

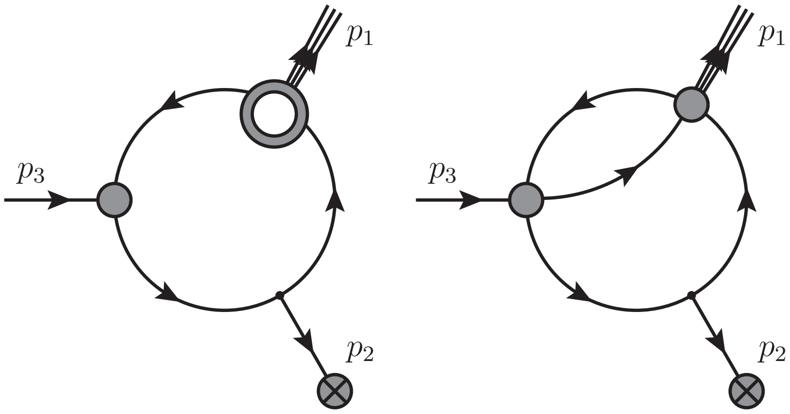

Figure 3. Prototype two-loop Feynman diagrams with anomalous thresholds.

dual cuts and two additional internal momenta become on-shell due to kinematics) which we see in the limit

$$\lim_{\lambda^{+++}_{i_1jk},\lambda^{+++}_{i_2jk}\to 0} \mathcal{S}^{(2)}_{i_1i_2jk} = \frac{1}{x_{i_1i_2jk}} \prod_{i=i_1i_2} \frac{\theta(-k_{k(ij),0})\,\theta(k^2_{k(ij)}-(m_i+m_j+m_k)^2)}{(-\lambda^{+++}_{ijk}-\imath 0 k_{kj,0})} + \mathcal{O}\left((\lambda^{+++}_{i_1jk})^{-1},(\lambda^{+++}_{i_2jk})^{-1}\right)\,, \tag{3.6}$$

with the obvious generalization of the notation. Other anomalous threshold configurations can easily be inferred from these results. At higher orders, the threshold and singular structure can be deducted from e.g. the explicit dual representations reported in ref. [17].

Finally, let us comment that the most general dual representation of any planar diagram at two loops (these are diagrams where one of the sets α_i is composed by one single propagator, e.g. assuming that it is α_2) can be rewritten as

$$\mathcal{A}^{(2)}(\{p_n\}_N) = \int_{\ell_1}\int_{\ell_2} \mathcal{N}(\ell_1,\ell_2,\{p_n\}_N) \otimes[G_D(\alpha_1)\,G_D(\alpha_2)\,G_F(\alpha_3)+G_F(\alpha_1)\,G_D(-\alpha_2)\,G_D(\alpha_3) +G_D(\alpha_1)\,G^*_F(\alpha_2)\,G_D(\alpha_3)]\,, \tag{3.7}$$

with

$$G^*_F(\alpha_2) = G^*_F(q_j) = \frac{1}{q_j^2-m_j^2-\imath 0}\,. \tag{3.8}$$

The threshold and singular solutions to eq. (3.7) are in full agreement with the equivalent representation in eq. (1.3) due to a similar local matching of the dual prescriptions. Another great advantage of eq. (3.7) is that the dual prescriptions are independent of the loop momenta, and therefore are simply fixed by the energy components of the external particle momenta. In the specific case of the planar sunrise diagram, eq. (3.7) generates only three double-cut contributions.

4 Conclusions

In summary, we have presented the first comprehensive description of the singular structure of scattering amplitudes in the LTD formalism at higher orders. The LTD representation allows for a detailed discussion directly in the loop momentum space. Contrary to the Landau equations [5], it does not rely on the Feynman parametrization and keeps track of the imaginary prescription of internal propagators in a consistent way. This consistent deployment of the imaginary prescriptions is needed to proof that there is a perfect local cancellation of non-causal or unphysical thresholds in the forest defined by the sum of all the on-shell dual contributions. The remaining causal thresholds, and in particular the soft and collinear singularities, which are described as the limiting case of threshold singularities, are restricted to a compact region of the loop three-momenta space. This feature is essential to enable the establishment of momenta mappings between the kinematics of the virtual and real contributions to physical observables in such a way that the cancellation of IR singularities occurs locally, as defined in the FDU approach.

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