Precise Determination of the Branching Ratio of the Neutral-Pion Dalitz Decay

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We provide a new value for the ratio \( R = \Gamma(\pi^0 \rightarrow e^+e^-\gamma)/\Gamma(\pi^0 \rightarrow \gamma\gamma) \) = 11.978(6) \times 10^{-3}, which is by 2 orders of magnitude more precise than the current Particle Data Group average. It is obtained using the complete set of next-to-leading-order radiative corrections in the QED sector, and incorporates up-to-date values of the \( \pi^0 \)-transition-form-factor slope. The ratio \( R \) translates into the branching ratios of the two main \( \pi^0 \) decay modes: \( \mathcal{B}(\pi^0 \rightarrow \gamma\gamma) = 98.8131(6)\% \) and \( \mathcal{B}(\pi^0 \rightarrow e^+e^-\gamma) = 1.1836(6)\% \).

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Introduction.—The present 3% experimental precision on \( \mathcal{B}(\pi^0_\text{D}) \) [11] (We use the shorthand notation \( \pi^0_\text{D} := \pi^0 \rightarrow f \), with \( D := e^+e^-\gamma \) and \( DD := e^+e^-e^+e^-\gamma \)) represents a limitation for rare-\( \pi^0 \)-decay measurements, which commonly use the Dalitz decay \( \pi^0_\text{D} \) for normalization, and is also becoming a limiting factor for rare-kaon-decay measurements. An example is the \( K^+ \rightarrow \pi^+e^+e^- \) decay [2]; accurate knowledge of \( \mathcal{B}(\pi^0_\text{D}) \) would improve the precision on the rate measurement by 30%, and the precision on the low-energy parameter \( a_\pi \) [3] by 10%. The uncertainty on \( \mathcal{B}(\pi^0_\text{D}) \) also dominates the precision on the \( K^+ \rightarrow \pi^+e^+e^- \) rate measurement [4], and is among the principal contributions to the uncertainties on the measured \( K_{LS} \rightarrow \pi^+\pi^-e^+e^- \) rates [5]. In these circumstances, considering the improving precision on rare-decay measurements, and the recent progress on the \( \pi^0 \)-form-factor measurement [6] and radiative corrections for the \( \pi^0_\text{D} \) decay [7], both a precision measurement of \( \mathcal{B}(\pi^0_\text{D}) \) and an updated theoretical evaluation of this quantity are becoming more important.

Branching ratios can serve to translate lifetimes into decay widths and vice versa. There are several methods to determine the \( \pi^0 \) lifetime: a direct average-distance measurement of the highly relativistic pion, the conserved-vector-current hypothesis connecting the vector form factor (i.e., charged pions) to the \( \pi^0 \) lifetime [8], and the Primakoff effect [9]. Since 2012 its Particle Data Group (PDG) value settled to \( \tau_{\pi^0}^{\text{PDG}} = 8.52(18) \times 10^{-17} \) s [10]. Presently, the most precise \( \pi^0 \)-lifetime measurements are given by two different methods: \( \tau_{\pi^0}^{\text{PrimEx}} = 8.32(23) \times 10^{-17} \) s [11] (Primakoff effect, PrimEx experiment at JLab) and \( \tau_{\pi^0}^{CERN} = 8.97(28) \times 10^{-17} \) s [12] (direct measurement, CERN). It is clear that the situation is unsatisfactory and a new independent measurement is desirable. For the Primakoff-effect-type \( \pi^0 \)-lifetime measurements, \( \mathcal{B}(\pi^0_\text{D}) \) constitutes an essential input.

In this work we discuss the theoretical determination of the following ratio of decay widths:

\[
R = \frac{\Gamma(\pi^0 \rightarrow e^+e^-\gamma)}{\Gamma(\pi^0 \rightarrow \gamma\gamma)} = \frac{\mathcal{B}(\pi^0_\text{D})}{\mathcal{B}(\pi^0_\text{D})},
\]

The current PDG value \( R_{\text{PDG}} = 11.88(35) \times 10^{-3} \) is an average of experimental results, the most recent of which comes from 2008 and is based on archived ALEPH data [13]. Other measurements with competitive uncertainties date back to 1981 [14] and 1961 [15]. The branching ratios \( \mathcal{B}(\pi^0_\text{D})_{\text{PDG}} = 98.823(34)\% \) and \( \mathcal{B}(\pi^0_\text{D})_{\text{PDG}} = 1.174(35)\% \) [1] are subsequently obtained from a constrained fit.

Besides the direct extraction of \( R \) from experiment, the shape of the singly virtual \( \pi^0 \) transition form factor (TFF) can be measured. This can be expanded in the transferred momentum squared, with the linear coefficient called the (TFF) slope \( a_\pi \). Since the slope embodies the most relevant input to the ratio \( R \) regarding the (nonperturbative) low-energy QCD sector [the peculiar [16] TFF normalization \( F(0) \) conveniently drops out], its knowledge from experiment is crucial to obtain a model-independent prediction of \( R \). Recently, it was measured in the NA62 experiment, which analyzed \( 1.1 \times 10^6 \) fully reconstructed \( \pi^0_\text{D} \) decays with the result \( a_N^{\text{NA62}} = 3.68(57)\% \) [6], taking into account the complete set of next-to-leading-order (NLO) radiative corrections in the QED sector [7]. The current PDG value is
dominated by two inputs: the above NA62 result and the value provided by the CELLO Collaboration \( \alpha_{\text{ELO}} \) from model-dependent extrapolation from the spacelike region.

Our calculation combines a wide range of theoretical models and available experimental results on the TFF shape and the well-established QED calculation including the complete set of NLO corrections, taking into account higher orders in the QED expansion in a conservative uncertainty estimate. As such, it represents a precise and reliable orders in the QED expansion in a conservative uncertainty estimate. Moreover, for the first time, the slope corrections were not neglected in the bremsstrahlung contribution. Finally, we present \( R \) for the full as well as partial kinematic regions.

Measurements of the TFF shape or the ratio \( R \) require significant theoretical input and depend crucially on the proper incorporation of radiative corrections. Consequently, a statement that experiment itself provides a more relevant value of \( R \) than our theoretical prediction is by its nature imprecise. However, the computation would not be possible without the experimental evidence that the TFF slope lies within a certain range of values. An example of how theoretical inputs influence the experimental values in this sector is the well-known discrepancy in the rare decay \( \pi_0^{+}e^{-}\) driven most probably by the approximate radiative corrections [18] which do not agree with the exact calculation [19]; for details and discussion see Refs. [20,21].

**Theoretical framework.**—Considering the QED expansion, the leading-order (LO) \( \pi^0 \)-Dalitz-decay differential width reads [7,22,23]

\[
d\Gamma^{\text{LO}}(x, y) = \frac{\alpha^3}{\pi} \Gamma(\pi^0 \to \gamma\gamma) \left| \frac{\mathcal{F}(M_x^2 x^2)}{\mathcal{F}(0)} \right|^2 \left( \frac{1-x}{4x} \right)^3 \left[ 1 + y^2 + \frac{4m_\pi^2}{M_x^2 x^2} \right],
\]

where the two-photon decay width is parametrized as

\[
\Gamma(\pi^0 \to \gamma\gamma) = \frac{e^4 M_\pi^2}{64\pi} |\mathcal{F}(0)|^2.
\]

Above, \( M_\pi \) and \( m_\pi \) are the neutral-pion and electron masses, respectively. The definition (3) holds to all orders in the QED and chiral perturbation theory expansions [16] and covers also possible physics from beyond the standard model, simply putting these nontrivial dynamical effects into the TFF normalization \( \mathcal{F}(0) \). As usual, kinematical variables \( x \) and \( y \) are defined as

\[
x = \frac{(p_+ + p_-)^2}{M_\pi^2}, \quad y = -\frac{2}{M_\pi^2} \frac{p_+ \cdot (p_- - p_+)}{(1-x)},
\]

with \( p \) denoting four-momenta of respective particles.

The slope \( a_x \) and curvature \( b_x \) of the singly virtual pion TFF are defined in terms of the Taylor expansion in the invariant mass of the vector current [24,25]:

\[
\frac{\mathcal{F}(M_x^2 x^2)}{\mathcal{F}(0)} \equiv f(x) = 1 + a_x x + b_x x^2 + \mathcal{O}(x^3).
\]

This parametrization is sufficient in the whole (small) region relevant to the \( \pi_0^+ \) decay. Having particular theoretical models at hand one can immediately explore the properties of \( f(q^2) \) and calculate \( a_x \) and \( b_x \). As examples we briefly mention the vector-meson-domiance (VMD) ansatz [25,26] together with the lowest-meson-dominance (LMD) [27,28] and two-hadron-saturation (THS) [21] models; see Ref. [29] for more details. These belong to a family of large-\( N_C \) motivated analytic resonance-saturation models and as such they can be straightforwardly used in the calculation of radiative corrections. By means of the first and second derivatives it is easy to find the analytic expressions for \( a_x \) and \( b_x \) within these models; for details see Sec. 5 of Ref. [21]. Numerical results are shown in Table I, together with other theoretical approaches and experimental results.

From Eq. (5) it follows that

\[
f^2(x) = 1 + 2 a_x x + (a_x^2 + 2 b_x) x^2 + \mathcal{O}(x^3).
\]

We can use the expansion (6) to obtain a simple formula for the LO width. Inserting Eq. (6) into Eq. (2) and taking into account that \( x \in (4m_\pi^2/M_\pi^2, 1) \) and \( y \in (-\beta(M_\pi^2 x).\beta(M_\pi^2 x)) \) with \( \beta(s) = \sqrt{1 - 4m_\pi^2/s} \), we get

\[
\frac{\mathcal{F}(M_x^2 x^2)}{\mathcal{F}(0)} \equiv f(x) = 1 + a_x x + b_x x^2 + \mathcal{O}(x^3).
\]

**TABLE I.** The slope and curvature of the singly virtual pion TFF in various approaches. The values given by the VMD, LMD, and THS models (as was the previous work [40] and supported by the low-energy timelike data) are in agreement with the recent dispersive calculation [36,37] incorporating inputs from both the space- and timelike regions (and updating Ref. [38]), the method of Padé approximants [39] mainly based on the extrapolation of the spacelike data (as was the previous work [40] and supported by the low-energy timelike data), the recent measurement performed by the NA62 experiment [6], or the PDG average [1]. Inherent model uncertainties (due to large-\( N_c \) and chiral limits) are not fully included in the THS value [21]. Additionally, a recent time-like-region measurement by the A2 Collaboration reads \( a_x^2 = 3.0(1.0)\% \) [41].

<table>
<thead>
<tr>
<th>Source</th>
<th>VMD</th>
<th>LMD</th>
<th>THS</th>
<th>Dispers.</th>
<th>Padé aps.</th>
<th>NA62</th>
<th>PDG</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_x ) [%]</td>
<td>3.00</td>
<td>2.45</td>
<td>2.92(4)</td>
<td>3.15(9)</td>
<td>3.21(19)</td>
<td>3.68(57)</td>
<td>3.35(31)</td>
</tr>
<tr>
<td>( b_x ) ( \times 10^{-3} )</td>
<td>0.90</td>
<td>0.74</td>
<td>0.87(2)</td>
<td>1.14(4)</td>
<td>1.04(22)</td>
<td>( \cdots )</td>
<td>( \cdots )</td>
</tr>
</tbody>
</table>

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\[
R^{\text{LO}} = \frac{a}{\pi} \left[ \frac{4}{3} \ln \left( \frac{M}{m_\pi} \right) - \frac{1}{3} (7 - a_x) + \frac{1}{30} (a_x^2 + 2b_x) \right] + \left( 12 - 8a_x \right) \frac{m_\pi^4}{M_x^2} + \mathcal{O} \left( \frac{m_\pi^4}{M_x^2} \right) .
\]

This expression is a very good approximation with the precision \( \eta(R^{\text{LO}}) \approx 10^{-9} \) (evaluated for parameters close to the physical ones). As is a common practice, e.g., in Ref. [6], further on the prescription \( f^2(x) \big|_{\text{lin}} = (1 + a_x x)^2 \) is used, which is accurate to first order in \( x \). The effect of the missing \( 2b_x x^2 \) term is negligible, since the region where such a difference arises (i.e., when \( x \approx 1 \)) is suppressed; cf. Eq. (2) and also Table III later on. If we consider \( b_x \approx a_x^2 \), as suggested by the models (cf. Table I), we introduce an error \( \sigma(a_x) \approx 2a_x^2/5 \), i.e., a relative error \( \sigma(a_x) \lesssim 1\% \), on the estimate of \( a_x \) being well under the current experimental precision. The previous discussion also implies that the effect of the \( a_x x \) term on the Dalitz-decay rate is limited, letting us provide a very precise determination of \( R \) while allowing for 20% uncertainty on \( a_x \). Finally, let us note that dropping \( b_x \) out of Eq. (7) decreases its precision down to \( \eta(R^{\text{LO}}) \big|_{b_x=0} \approx 10^{-3} \), being still a good approximation for our purpose in view of the above discussion.

In the rest of the section we address the NLO QED-sector radiative corrections; see Fig. 1 for Feynman diagrams. It is convenient to introduce the NLO correction \( \delta \) to the LO differential width (and thus to write schematically \( d\Gamma = (1 + \delta + \ldots) d\Gamma^{\text{LO}} \)), which can be in general (for the two- and one-fold differential case, respectively) defined as

\[
\delta(x, y) = \frac{d\Gamma^{\text{NLO}}}{dxdy} / \frac{d\Gamma^{\text{LO}}}{dxdy}, \quad \delta(x) = \frac{d\Gamma^{\text{NLO}}}{dx} / \frac{d\Gamma^{\text{LO}}}{dx} .
\]

One can obtain \( \delta(x) \) from \( \delta(x, y) \) using the following prescription:

\[
R^{\text{NLO}} = \frac{a}{\pi} \left[ \frac{4}{3} \ln \left( \frac{M}{m_\pi} \right) - \frac{1}{3} (7 - a_x) + \frac{1}{30} (a_x^2 + 2b_x) \right] + \left( 12 - 8a_x \right) \frac{m_\pi^4}{M_x^2} + \mathcal{O} \left( \frac{m_\pi^4}{M_x^2} \right) .
\]

To calculate the NLO radiative corrections we use the approach documented in Refs. [7,42], which reviewed and extended the classical work of Mikaelian and Smith [22]. Hence, together with the bremsstrahlung (BS) beyond the soft-photon approximation, we take into account in the following calculations the one-photon-irreducible (1γIR) contribution; see Figs. 1(b) and 1(c).

For historical reasons [22,23,43–45], let us discuss the case when the 1γIR contribution to the NLO radiative corrections is not considered in the analysis to extract the TFF slope from the data. If we start with the equation among the measured spectral shapes (onefold differential widths) and eliminate \( d\Gamma^{\text{LO}} / dx \big|_{(x)=1} \) from both sides, take the expansion (6) to the linear order, and neglect the corrections of order \( aa_x \), we find

\[
\Delta a_x x \equiv (a_x - a_x^{\text{1γIR}}) x \approx - \frac{1}{2} \delta^{\text{NLO}}(x), \quad x \ll 1.
\]

Numerically, \( \Delta a_x \approx 0.5\% \) [23]. This is the value to be added to the experimental value \( a_x^{\text{1γIR}} \) (extracted neglecting the 1γIR contribution) in order to find an estimate of the pure-low-energy-QCD-sector parameter \( a_x \) with all the QED radiative corrections subtracted. Above, \( \delta^{\text{NLO}}(x) \) is calculated from \( \delta^{\text{1γIR}}(x, y) \) stated in Sec. IV of Ref. [7] using the prescription (9). Note that the 1γIR contribution was already taken into account in the NA62 analysis [6] and \( a_x^{\text{NA62}} \) does not need to be corrected by \( \Delta a_x \)

Finally, taking the prescription \( f^2(x) \big|_{\text{lin}} \) and the NLO QED radiative corrections to approximate the exact Dalitz-decay differential width beyond LO (and consequently \( \Delta R \equiv R - R^{\text{LO}} \)) we arrive at

\[
\Delta R \approx R^{\text{NLO}} \equiv \frac{a}{\pi} \int (1 + a_x x)^2 \delta(x, y) \times \frac{1}{4x} \left( 1 + x^2 + \frac{4m_\pi^2}{M_x^2} \right) dxdy .
\]

**Calculation and uncertainty estimation.**—Our aim now is to precisely and reliably (using conservative error estimates) determine \( R \). In the following we choose \( a_x^{\text{50\%}} = 3.55(70)\% \) by stretching the uncertainty band over the whole interval of values suggested by different approaches; cf. Table I. From Eq. (7) we get

\[
R^{\text{LO}} = 11.879(5) \times 10^{-3} ,
\]

and based on Eq. (11) we arrive at
TABLE II. Individual contributions of the NLO radiative corrections for $R$ in \([10^{-3}]\). The virt label stands for virtual corrections [Figs. 1(a) and 1(d)] and div (conv) for the divergent (convergent) parts of the bremsstrahlung contribution [Fig. 1(e)] [46]. The listed uncertainties stem from the uncertainty of $a_s^{\text{inv}}$. In the case of the 1IR correction [Figs. 1(b) and 1(c)], a particular model for the doubly virtual TFF (LMD, etc.) is necessary to introduce. The resulting model dependence is suppressed [7] and related uncertainty included.

<table>
<thead>
<tr>
<th>$x$ (cut)</th>
<th>$R(x_{\text{cut}})$ [$10^{-3}$]</th>
<th>$x$ (cut)</th>
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<th>$x$ (cut)</th>
<th>$R(x_{\text{cut}})$ [$10^{-6}$]</th>
<th>$x$ (cut)</th>
<th>$R(x_{\text{cut}})$ [$10^{-8}$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.2037(2)</td>
<td>0.30</td>
<td>0.2213(13)</td>
<td>0.55</td>
<td>0.2420(22)</td>
<td>0.80</td>
<td>67.28(85)</td>
</tr>
<tr>
<td>0.10</td>
<td>117.03(36)</td>
<td>0.35</td>
<td>14.787(97)</td>
<td>0.60</td>
<td>14.07(14)</td>
<td>0.85</td>
<td>19.81(27)</td>
</tr>
<tr>
<td>0.15</td>
<td>74.38(29)</td>
<td>0.40</td>
<td>9.756(71)</td>
<td>0.65</td>
<td>7.703(80)</td>
<td>0.90</td>
<td>3.594(54)</td>
</tr>
<tr>
<td>0.20</td>
<td>49.11(23)</td>
<td>0.45</td>
<td>6.313(50)</td>
<td>0.70</td>
<td>3.890(43)</td>
<td>0.95</td>
<td>0.1967(35)</td>
</tr>
<tr>
<td>0.25</td>
<td>32.92(17)</td>
<td>0.50</td>
<td>3.978(34)</td>
<td>0.75</td>
<td>1.757(21)</td>
<td>1.00</td>
<td>0</td>
</tr>
</tbody>
</table>

This calculation also includes all contributions from the decays where additional photon(s) with arbitrarily high (kinematically allowed) energies are radiated. Indeed, the bremsstrahlung correction at NLO (calculated à la Refs. [7,22]) takes into account an additional final-state photon and integrates over its energy and emission angle without any additional cuts. The results are thus meant to be used for the inclusive process. However, quantities for exclusive processes can be obtained in a similar way while introducing some specific cutoff in the bremsstrahlung correction $\delta^\text{BS}(x,y)$. A combined approach was used in the analysis of the recent NA62 measurement [6], when an additional photon was simulated above the cutoff given by the detector sensitivity. To conclude, for each experimental setup the specific approach for including radiative corrections must be used. When it applies, we explicitly state (as in the abstract) that the results include an additional final-state photon, denoting it as $\gamma$. We also take this tacitly into account in the results for $R$, e.g., in Eq. (14).

In experiments, specific kinematic regions might be considered. The sample values for

$$R(x_{\text{cut}}) = \frac{B(\pi^0 \to e^+e^-\gamma(\gamma), x > x_{\text{cut}})}{B(\pi^0_{2\gamma})}$$

are listed in Table III, using which one can also obtain values for any intermediate region. As an example, in the $\pi^0$-rare-decay measurement performed by KTeV [47] the region $x > x_{\text{cut}} = 0.232$ was used for the Dalitz decay, which served as the normalization channel in this search. The direct calculation based on this work leads to \(R(0.232) = 0.380(2) \times 10^{-3}\) and the interpolation based on Table III gives $R(0.232)|_{\text{impol}} = 0.379(2) \times 10^{-3}$, which is compatible within uncertainties. In Ref. [48] the value $[R(0.2319)/R]_{KTeV} = 0.0319$ was used, which is compatible with our calculation: $R(0.232)/R = 0.0317(2)$.

Based on $R$, we can predict $B(\pi^0_{2\gamma})$ and $B(\pi^0_{D(\gamma)})$. Considering the uncertainty of $R$, we can write

$$1 - B(\pi^0_{DD}) \simeq B(\pi^0_{2\gamma}) + B(\pi^0_{D(\gamma)})$$

During the estimation of the above uncertainty, higher orders in the QED expansion were considered, surpassing in size the uncertainty stemming from the TFF dependence. In this regard, we can take the absolute value of the dominant correction ($R_{NLO}^{BS,\text{div}}$, see Table II) as the typical expected maximum value appearing in NLO and anticipate that the NNLO correction is suppressed compared to the NLO one in the similar manner as NLO is suppressed with respect to LO: circa on the level of 3%. This uncertainty is already conservative: the total NLO correction is on the level of 1%. Summing Eqs. (12) and (13) we finally obtain

$$R = 11.978(5) \times 10^{-3}.$$ (14)
since the branching ratios of other decay modes are smaller than $10^{-6}$. Using $B(\pi_0^{(a)}) = 3.3(2) \times 10^{-5}$ [1,49] (doubleDalitz decay), we find

$$B(\pi_0^{(a)}) \approx 1 - B(\pi_0^{(DD)}) = 98.8131(6)\%.$$  \hfill (17)

Note that taking tacitly into account inclusive Dalitz decays in Eq. (16) is justified and contributes to the relevant decay modes. Finally, the Dalitz-decay branching ratio reads

$$B(\pi_0^{(D)}) \approx \frac{R}{1 + R} [1 - B(\pi_0^{(DD)})] = 1.1836(6)\%.$$ \hfill (18)

The above results are compatible with the PDG averages, exhibiting much higher precision.

Let us see how the new result on the Dalitz-decay branching ratio (18) influences a completely different family of processes on a simple example of the $K^+ \to \pi^+ e^+ e^-$ decay measurements. The low-energy parameters $a_+ \text{ and } b_+$ were measured by the NA48/2 collaboration to be $a_+ = -0.578(16)$ and $b_+ = -0.779(66)$, leading to the model-dependent branching ratio $B(K^+ \to \pi^+ e^+ e^-) = 3.11(12) \times 10^{-7}$, using the 2008 PDG average $B(\pi_0^{(D)}) = 1.198(32)\%$ [50] for normalization [2]. The central value of our result (18) is 1.2% lower than the quoted PDG average and has a negligible error. The remaining external uncertainty on the measurement [2] related to the normalization comes from $B(K^+ \to \pi^+ e^0)$ known to 0.4% precision. The corrected values are $a_+ = -0.575(14)$, $b_+ = -0.771(64)$, and $B(K^+ \to \pi^+ e^+ e^-) = 3.07(10) \times 10^{-7}$. Note that considering the external errors on $a_+$ and $b_+$ quoted in Ref. [2], further experimental progress on $K^+ \to \pi^+ e^+ e^-$ measurement would be impossible without improvement on $B(\pi_0^{(D)})$.

Comparison and conclusion.—Radiative corrections for the integral decay width were first addressed by Joseph [51], who numerically arrived to $\Delta R|_{\text{ph}} = 0.105 \times 10^{-5}$ neglecting, among others, the pion TFF slope. A simple analytical prescription in the limit of the vanishing electron mass was later found by Lautrup and Smith [52]:

$$\Delta R|_{\text{L&S}} = \left[ \frac{\alpha}{\pi} \right]^2 \left[ \frac{8}{9} \ln^2 \frac{M_\pi}{m_e} - \frac{1}{9} (19 - 4a_\pi) \ln \frac{M_\pi}{m_e} \right] + 2\zeta(3) - \frac{2}{27} \pi^2 + \frac{137}{81} - \frac{63}{108} a_\pi + O\left(\frac{m_e}{M_\pi}\right).$$ \hfill (19)

Numerically, $\Delta R|_{\text{L&S}}^{m_e=0} = 0.10378 \times 10^{-3}$ and $\Delta R|_{\text{L&S}}^{m_e=0}$ is $0.10414(7) \times 10^{-3}$. The two approaches are compatible and should be compared with our result (13). However, the $1\gamma$ IR contribution was, due to inappropriate assumptions and arguments based on Low’s theorem [53–55], considered negligible and left out; see also Refs. [22,44]. The exact calculation shows its significance [7,23,43,45] and it thus embodies the main source of the difference between our result and the previous works. Moreover, the symmetrization with respect to the two photons in the bremsstrahlung contribution was neglected in Refs. [51,52]. This interference of the diagrams from Fig. 1(e) is indeed negligible and corresponds (for $a_\pi = 0$) to $\Delta R|_{\text{interf}}^{BS} = 0.000360 \times 10^{-3}$. Let us stress again that the prediction (14) is based on the complete calculation which includes the entire bremsstrahlung and 1\gamma contributions. Here, TFF effects were taken into account also in the bremsstrahlung correction [46] and the mass of the final-state leptons was not neglected.

Our main result (14) together with the value (17) should be considered as an alternative to the current PDG averages which opens the way to a new level of precision for a whole class of other processes, for instance, for the already mentioned kaon decays. Similarly, the current situation, when the precision on $B(\pi_0^{(D)})$ [56] dominates the uncertainty on $B(\pi_0^{(D)})$ and is the largest source of uncertainty on $B(\pi_0^{(D)})$ [47], is improved.

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[31] The simplest models do not, for instance, satisfy some of the major high-energy constraints: VMD violates operator-product-expansion (OPE) constraints [21,30] and the LMD model cannot satisfy the Brodsky-Lepage scaling limit [31,32]. Although the THS model by construction satisfies the mentioned constraints [21] and surpasses in this sense the VMD and LMD models, it inevitably suffers from shortcomings common to the whole family: for instance, a direct consequence of satisfying OPE [35].
[47] The value corresponding to the correction \( \Delta R_{\text{conv}}^{\text{THS}} \) includes not only the part with neglected slope \( \Delta q_x \) but newly the form-factor correction was taken into account: \( \Delta R_{\text{conv}}^{\text{THS}} = -0.1582(1) \times 10^{-3} \). This was calculated based on the final part of Section V of [7] and is (due to the smallness of \( \Delta q_x \)) compatible with the approach described in Section V of [42].